Logic - Tutorial 1

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About me

- Graduated in 2018 from the master : **computer engineer, specialized in 'intelligent systems'**

  ⇒ Master thesis on **facial recognition** with L. Wehenkel and RAGI

- Left one year to travel in Australia and Asia

- PhD thesis on **intercontinental electricity connections** with D. Ernst

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About the tutorial

- Once a week, after theory
- Max 2h
- Slides in English but given in French
- Structure
  - A reminder if need be
  - One exercise together
  - For other exercise, you try then we discuss the solution

Reminder

Propositional calculus: formal language to determine the truth values of propositions.

Syntax: Define the structure of propositions

Propositions:

- **Atoms** or atomic propositions.
  Expl:
  - s: the sun is shinning
  - r: the rain is falling
- **Formulas** or compound propositions = atoms + boolean connectives.
  Expl:
  - A: the sun is shinning or the rain is falling → $A \triangleq s \vee r$

Reminder

More formally, a formula of propositional calculus is a symbol string generated by the grammar

$$
\begin{aligned}
\textit{formula} &::= p, \forall p \in P \text{ (i.e a set of atoms)} \\
\textit{formula} &::= \textit{true} | \textit{false} \\
\textit{formula} &::= \neg \textit{formula} \\
\textit{formula} &::= (\textit{formula op formula}) \\
\textit{op} &::= \vee | \wedge | \Rightarrow | \equiv | \Leftarrow
\end{aligned}
$$

Reminder

Semantics: Assigning truth values to propositions

Interpretation (/Valuation)
An interpretation or valuation v is a function assigning a truth value, T or F, to a proposition.

Remark: 'true' vs 'T'

- 'true' $\rightarrow$ syntactic
- 'T' $\rightarrow$ semantic

Reminder

For a formula A built from the atoms $\{p_1, ..., p_n\}$, v assigns a truth value to each atom and the truth value of A is then assigned according to the following inductive rules:

A	$v(A_1)$	$v(A_2)$	$v(A)$
true			T
false			F
$\neg A_1$	T		F
$\neg A_1$	F		T
$A_1 \vee A_2$	F	F	F
$A_1 \vee A_2$	else		T
$A_1 \wedge A_2$	T	T	T
$A_1 \wedge A_2$	else		F
$A_1 \Rightarrow A_2$	T	F	F
$A_1 \Rightarrow A_2$	else		T
$A_1 \Leftarrow A_2$	F	T	F
$A_1 \Leftarrow A_2$	else		T
$A_1 \equiv A_2$	$v(A_1) = v(A_2)$		T
$A_1 \equiv A_2$	$v(A_1) \neq v(A_2)$		F

Truth tables allows to test different valuations in a structured way.

Reminder

Satisfiability (consistency)

- A valuation v of formula A is a **model** of A if $v(A) = T$
- A is **satisfiable** or **consistent** if A has at least one model.
- A is **unsatisfiable** or **inconsistent** if there exist no valuation v that is a model of A.
  Expl: Joe is strong and Joe is not strong.

Validity

- A is **valid**, or a **tautology**, if $v(A) = T$ for all possible valuations v.
  Expl: Joe is strong or Joe is not strong.
- Notation: $\models A$
- A is valid if and only if its negation $\neg A$ is unsatisfiable.

Reminder

Formula sets

Let S be a set of formulas $\{A_1, ..., A_n\}$.

- A valuation v of S is a model of S if it is a model of all formulas in S
- S is **satisfiable** or **consistent** if S has at least one model.
- The models of the finite set $S = \{A_1, ..., A_n\}$ are the models of the conjunction $A_1 \wedge ... \wedge A_n$

Expl: $S = \{$Joe is strong, Joe is intelligent, Joe is funny$\}$

Logical consequence

- A formula A is a **logical consequence** of a formula set S if every $S - model$ is an $A - model$
- Notation: $S \models A$
  Expl: $\{$Joe is strong, Joe is intelligent, Joe is funny$\} \models$ I don't like Joe.
- Remark on $\models A$: A formula is valid iff it is a logical consequence of the empty set.

Reminder

Logical consequence (bis)

Let A be a formula and $S = \{A_1, ..., A_n\}$ be a formula set, the followings are equivalent:

1. $S \models A$
2. $S \cup \{\neg A\}$ is inconsistent
3. $A_1 \wedge ... \wedge A_n \Rightarrow A$ is valid
4. $A_1 \wedge ... \wedge A_n \wedge \neg A$ is inconsistent

Logical equivalence

- Two formulas A_1 and A_2 are **logically equivalent** if they have the same models.
- Notation: $A_1 \longleftrightarrow A_2$

Exercise 1

Give the truth table of the following formula:

$$G \triangleq (p \Rightarrow q) \Rightarrow [(\neg p \Rightarrow q) \Rightarrow q]$$

What conclusions can you make?

Exercise 1 - Solution

$$G \triangleq (p \Rightarrow q) \Rightarrow [(\neg p \Rightarrow q) \Rightarrow q]$$

First step: decompose the formula into columns

p	q	$p \Rightarrow q$	$\neg p$	$\neg p \Rightarrow q$	$(\neg p \Rightarrow q) \Rightarrow q$	G

Exercise 1 - Solution

$$G \triangleq (p \Rightarrow q) \Rightarrow [(\neg p \Rightarrow q) \Rightarrow q]$$

p	q	$p \Rightarrow q$	$\neg p$	$\neg p \Rightarrow q$	$(\neg p \Rightarrow q) \Rightarrow q$	G
T	T	T	F	T	T	T
T	F	F	F	T	F	T
F	T	T	T	T	T	T
F	F	T	T	F	T	T

This formula is **valid**, as it is always true (i.e. true for every possible model).

Exercise 2

Give the truth table of the following formula:

$$G \triangleq (p \equiv \mathrm{true}) \Rightarrow [(\neg p \wedge q) \Rightarrow \mathrm{true}]$$

What can you say about the formula $(\neg p \wedge q) \Rightarrow \mathrm{true}$?
Is G valid, inconsistent or consistent?

Exercise 2 - Solution

$$G \triangleq (p \equiv \text{true}) \Rightarrow [(\neg p \wedge q) \Rightarrow \text{true}]$$

p	q	$p \equiv \text{true}$	$\neg p \wedge q$	$(\neg p \wedge q) \Rightarrow \text{true}$	G
T	T	T	F	T	T
T	F	F	F	T	T
F	T	F	T	T	T
F	F	T	F	T	T

The formula G is **valid** as it is true for every model.
The formula $(\neg p \wedge q) \Rightarrow \text{true}$ is also valid. An implication evaluates to true if either the antecedent is F or if the consequent is T. In this case, the consequent, 'true', is always T, hence the formula is valid.

Exercise 3

Giving a truth table of a formula consists in enumerating all possible interpretations over the atoms of said formula.

1. How many lines are in a truth table?
2. How many non-logically equivalent formulas can be constructed using a set of n atoms?

Exercise 3 - Solution

1. If there are n atoms in the formula, the truth table will have 2^n lines.

   Indeed, each atom can be evaluated at either T or F, so we have $\underbrace{2 * ... * 2}_{n} = 2^n$ lines.

2. There are 2^{2^n} non-logically equivalent formulas.

   We have 2^n lines (/valuations) and each of them can lead to a truth value, either T or F (i.e. value in the last column of the truth table). As soon as one line leads to a different truth values for two different formulas, these two formulas are not logically equivalent. Therefore, the number of non-logically equivalent formulas is $\underbrace{2 * ... * 2}_{2^n} = 2^{2^n}$ lines.

Exercise 4

<u>Exercise 4</u>

Give the truth table of the following formula:

$$G \triangleq (q \Rightarrow r) \Rightarrow [(p \Rightarrow q) \Rightarrow (p \Rightarrow r)]$$

What can you say about G?

Exercise 4 - Solution

$$G \triangleq (q \Rightarrow r) \Rightarrow [(p \Rightarrow q) \Rightarrow (p \Rightarrow r)]$$

p	q	r	$q \Rightarrow r$	$p \Rightarrow q$	$p \Rightarrow r$	$(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$	G
T	T	T	T	T	T	T	T
T	T	F	F	T	F	F	T
T	F	T	T	F	T	T	T
T	F	F	T	F	F	T	T
F	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

This formula is **valid**.

Exercise 5

Give the truth table of the following formula:

$$G \triangleq (p \vee q) \wedge \neg p \wedge \neg q$$

What can you say about G?

Exercise 5 - Solution

$$G \triangleq (p \lor q) \land \neg p \land \neg q$$

p	q	$\neg p$	$\neg q$	$p \lor q$	G
T	T	F	F	T	F
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	F	F

This formula is **inconsistent** as it admits no model.

Exercise 5 - Remarks

Easy to show that G is inconsistent without truth table. Indeed, by De Morgan algebraic law,

$$\neg p \wedge \neg q \longleftrightarrow \neg (p \vee q)$$

So no there is no valuation that is a model of both $\neg p \wedge \neg q$ and $(p \vee q)$ and therefore G is inconsistent.

Exercise 6

If Robinson is elected president, then Smith will be designated vice-president. If Thompson is elected president, then Smith will designated be vice-president. Either Thompson or Robinson will be elected president. Therefore Smith will be designated vice-president.

Is this text correct?

Exercise 6 - Solution

Solution method:

1) Define atoms

- r : "Robinson is elected president"
- s : "Smith is designated vice-president"
- t : "Thompson is elected president"

2) Transform sentences in formulas:

- $H_1 \triangleq r \Rightarrow s$
- $H_2 \triangleq t \Rightarrow s$
- $H_3 \triangleq t \vee r$
- $C \triangleq s$

Exercise 6 - Solution

3) Prove that the sentence is true, i.e. C is a logical consequence of $\{H_1, H_2, H_3\}$

Reminder, 3 possibilities:

1. Prove $\{H_1, H_2, H_3\} \models C$
2. Prove $\{H_1, H_2, H_3, \neg C\}$ is inconsistent
3. Prove $H_1 \wedge H_2 \wedge H_3 \Rightarrow C$ is valid
4. Prove $H_1 \wedge H_2 \wedge H_3 \wedge \neg C$ is inconsistent

We will try case 3 and case 1.

Exercise 6 - Solution

Method 3:

Show $G \triangleq H_1 \wedge H_2 \wedge H_3 \Rightarrow C \triangleq (r \Rightarrow s) \wedge (t \Rightarrow s) \wedge (t \vee r) \Rightarrow s$ is valid.

A. Using a truth table.

r	s	t	$r \Rightarrow s$	$t \Rightarrow s$	$t \vee r$	$H_1 \wedge H_2 \wedge H_3$	G
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	F	F	T	F	T
T	F	F	F	T	T	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	T
F	F	T	T	F	T	F	T
F	F	F	T	T	F	F	T

Exercise 6 - Solution

B. Using valuation.

Ad absurdum
We need to show that there exist no v such that $v(H_1 \wedge H_2 \wedge H_3 \Rightarrow C) = F$. Let's consider that such a valuation exists. Then:

1. $v((r \Rightarrow s) \wedge (t \Rightarrow s) \wedge (t \vee r) \Rightarrow s) = F$
2. $v((r \Rightarrow s) \wedge (t \Rightarrow s) \wedge (t \vee r)) = T$ and $v(s) = F$
3. $v(r \Rightarrow s) = T$ and $v(s) = F$ implies $v(r) = F$
4. $v(t \vee r)$ and $v(r) = F$ implies $v(t) = T$

But then we have simultaneously that $v(t \Rightarrow s)$ must be T through 2 and F as $v(s) = F$ and $v(t) = T \rightarrow$ **Contradiction!**

There exist no such valuation and therefore the proposition is valid.

Method 1

Show $\{H_1, H_2, H_3\} \models C$.

We need to show that $v(C) = T$ when $v(H_1 \wedge H_2 \wedge H_3) = T$ for every possible valuation v.

Let v be a valuation such that $v(H_1 \wedge H_2 \wedge H_3) = T$. We therefore have that $v(H_1) = v(H_2) = v(H_3) = T$.

If $v(H_3) = T$, then $v(t \vee r) = T$ and we end up with two cases:

- $v(t) = T$
- $v(r) = T$

Exercise 6 - Solution

Case 1: $v(t) = T$

As we have $v(H_2) = T$, $v(t \Rightarrow s) = v(\text{true} \Rightarrow s) = T$. Therefore, $v(s)$ must be T and $v(C)$ also.

Case 2: $v(r) = T$

As we have $v(H_1) = T$, $v(r \Rightarrow s) = v(\text{true} \Rightarrow s) = T$. Therefore, $v(s)$ must be T and $v(C)$ also.

Conclusion

$v(C) = T$ in all cases so the sentence is true.