Logic - Tutorial 4

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Reminder

Decision procedure

Let U be a formula set (i.e., a set of formulas). An algorithm is a decision procedure for U if, given A, the computation stops with the answer 'yes' if $A \in U$ and the answer 'no' if $A \notin U$.

Formal logic : often U will be the set of valid formulas (or consistent formulas, or inconsistent formulas)

Example of decision procedure for satisfiability (consistency): Truth Tables

Today, we will see another example: Semantic Tableaux

Reminder

Differences between the two procedures

1. Semantic tableaux **faster** than the truth table.
2.
    - Truth Tables: **from smaller** subformulas **to bigger** subformulas: the truth value of a formula is function of the truth values of its (immediate) components.
    - Semantic Tableaux: **from bigger** subformulas **to smaller** subformulas: the truth value of an (immediate) component is related to the truth value of a formula.

Reminder

Underlying concepts and goal

A **literal** is an atom or a negated atom. If p is an atomic proposition, $\{p, \neg p\}$ is a complementary pair of literals.
A **literal set** (set of literals) is consistent if and only if it includes no complementary pair (which is determined by inspection).

The principle of the tableau method is to reduce the question
Is formula A consistent?
to the easier question
Are all members of the (finite) set $\mathcal{A}$ consistent literal sets ?

The **goal** is therefore to transform formula A into set $\mathcal{A}$. To achieve that, a semantic tableau takes the form of a **tree** ; its root is formula A and its leaves are the elements of $\mathcal{A}$.

Reminder

How to build a tableau from the root to the leaves?

Each intermediate nodes will contain a set of formulas.
Those formulas can be of three types:

- literals
- conjunctive formulas or α-formulas;
- disjunctive formulas or β-formulas.

Remarks:

- $\neg\neg X \leftrightarrow X$.
- $X \Rightarrow Y \leftrightarrow X \vee Y$ is thus disjunctive
- $\neg(X \Rightarrow Y) \leftrightarrow X \wedge \neg Y$ and is thus conjunctive
- $X \equiv Y \leftrightarrow (X \Rightarrow Y) \wedge (Y \Rightarrow X)$ and is thus conjunctive.

Reminder

To create children from non-literals, one can apply α-rules to break conjunctive formulas and β-rules to break disjunctive formulas.

α-rule

Conjunctive, α-formulas give rise to a single child; $v(\alpha) = T$ if and only if $v(\alpha_1) = v(\alpha_2) = T$.

α	α_1	α_2
$A_1 \wedge A_2$	A_1	A_2
$\neg(A_1 \vee A_2)$	$\neg A_1$	$\neg A_2$
$\neg(A_1 \Rightarrow A_2)$	A_1	$\neg A_2$
$\neg(A_1 \Leftarrow A_2)$	$\neg A_1$	A_2

Reminder

β-rule

Disjunctive, β-formulas give rise to two children; $v(\beta) = T$ if and only if $v(\beta_1) = T$ or $v(\beta_2) = T$.

β	β_1	β_2
$B_1 \vee B_2$	B_1	B_2
$\neg(B_1 \wedge B_2)$	$\neg B_1$	$\neg B_2$
$B_1 \Rightarrow B_2$	B_1	$\neg B_2$
$B_1 \Leftarrow B_2$	$\neg B_1$	B_2

Reminder - Algorithm

Each node is labelled with a formula set.
Init: root is labelled $\{C\}$; it is an unmarked leaf.
Induction step: select an unmarked leaf l labelled $U(l)$.

- If $U(l)$ is a literal set :
  - if $U(l)$ contains a complementary pair, then mark l as closed 'X' ;
  - else mark l as open 'O'.
- If $U(l)$ is not a literal set, select a non-literal formula in $U(l)$:
  - If it is an α-formula A, generate a child node l' and label it with

$$U(l') = (U(l) - \{A\}) \cup \{\alpha_1, \alpha_2\};$$

  - if it is a β-formula B, generate two child nodes l' and l'' ; their labels respectively are

$$U(l') = (U(l) - \{B\}) \cup \{\beta_1\}$$

$$U(l'') = (U(l) - \{B\}) \cup \{\beta_2\}.$$

Termination : when all leaves are marked 'X' or 'O'.

Reminder

How to interpret a semantic tableaux

- Formula A is inconsistent if and only if $T(A)$ is closed.
- Formula B is valid if and only if $T(\neg B)$ is closed.
- Formula C is simply consistent (contingent) if and only if both $T(C)$ and $T(\neg C)$ are open.

The tableau method is a decision algorithm for validity, consistency, contingency, inconsistency.

Reminder

Tips

- If we suspect inconsistency for formula X, $T(X)$ will be considered first ;
- If we suspect validity, $T(\neg X)$ will be considered first.
- Simplification : a branch can be closed as soon as a complementary pair $A, \neg A$ occurs, even if A is not an atom.
- Heuristics : use α-rules first (if you have the choice)

Exercise 1

Using the semantic tableaux method, determine whether the following formula is valid, consistent or inconsistent.

$$(p \Rightarrow q) \Rightarrow [(\neg p \Rightarrow q) \Rightarrow q]$$

Exercise 1 - Solution

$$(p \Rightarrow q) \Rightarrow [(\neg p \Rightarrow q) \Rightarrow q]$$

Exercise 1 - Solution

This a disjunctive formula so we apply a β-rule.

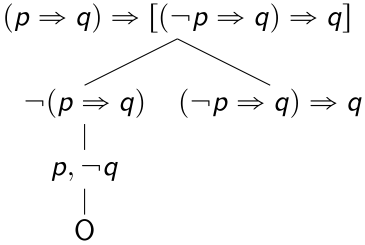
$$(p \Rightarrow q) \Rightarrow [(\neg p \Rightarrow q) \Rightarrow q]$$

$$\neg(p \Rightarrow q) \qquad (\neg p \Rightarrow q) \Rightarrow q$$

We will start by analyzing the left child.

Exercise 1 - Solution

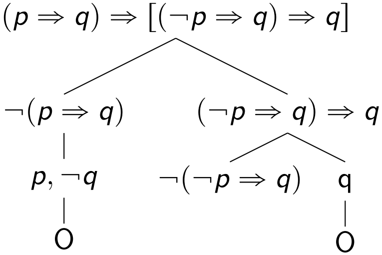
The left child is a conjunctive formula so we apply an α-rule



As there are only literals and no complementary pairs in this node, we set it as open.

Exercise 1 - Solution

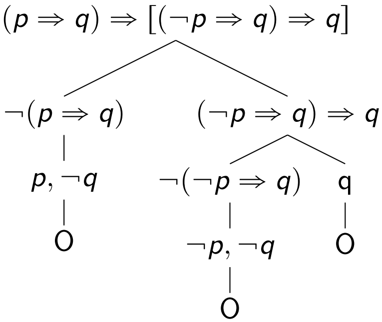
This a disjunctive formula so we apply a β-rule.



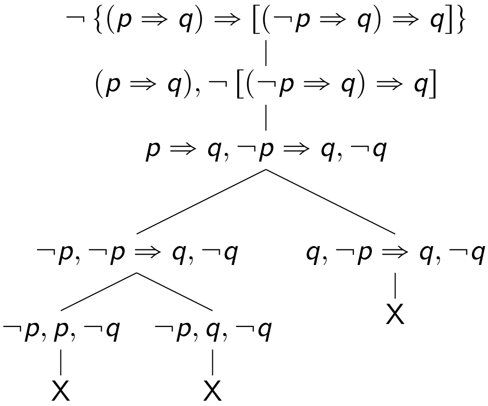
The right node is open.

Exercise 1 - Solution

It is a conjunctive formula so we apply an α-rule



Some of the leafs are open so the formula is consistent. Actually, we could have stop as soon as a leaf was open. Let's see if it the formula valid now.



The formula is valid as its negation is inconsistent.

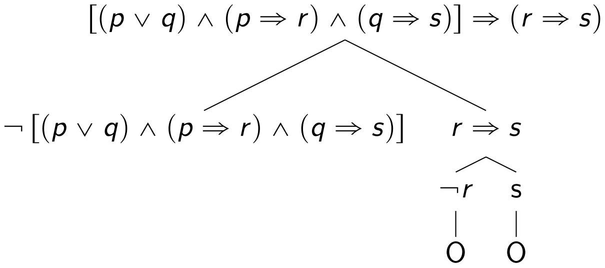
Exercise 2

Using the semantic tableaux method, determine whether the following formula is valid, consistent or inconsistent.

$$[(p \vee q) \wedge (p \Rightarrow r) \wedge (q \Rightarrow s)] \Rightarrow (r \Rightarrow s)$$

Give a model of the formula if possible.

Exercise 2 - Solution



The formula is consistent. A model is $v(s) = T$, the root can be anything. Or $v(r) = F$.

Exercise 2 - Solution

$$\neg\{[(p \vee q) \wedge (p \Rightarrow r) \wedge (q \Rightarrow s)] \Rightarrow (r \Rightarrow s)\}$$

$$|$$

$$(p \vee q) \wedge (p \Rightarrow r) \wedge (q \Rightarrow s), \neg(r \Rightarrow s)$$

$$|$$

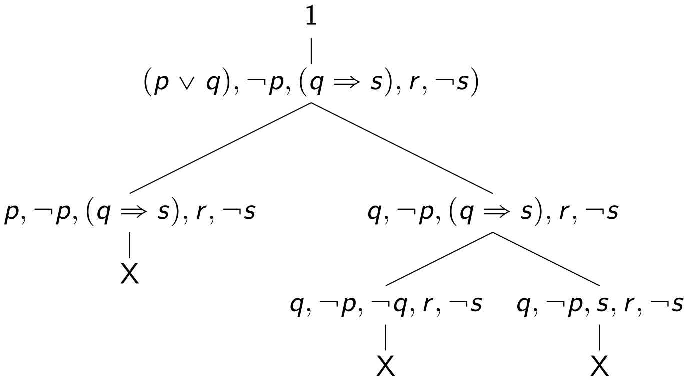
$$(p \vee q), (p \Rightarrow r), (q \Rightarrow s), \neg(r \Rightarrow s)$$

$$|$$

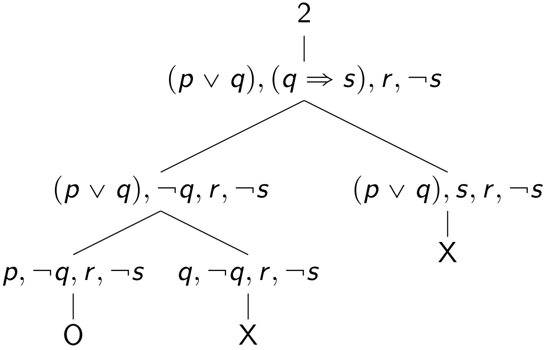
$$(p \vee q), (p \Rightarrow r), (q \Rightarrow s), r, \neg s$$

1 2

Exercise 2 - Solution



Exercise 2 - Solution



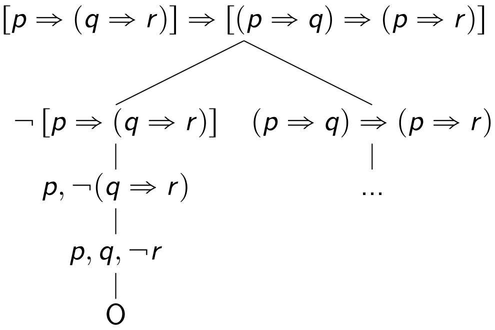
The formula is consistent but not valid as its negation is also consistent.

Exercise 3

Using the semantic tableaux method, determine whether the following formula is valid, consistent or inconsistent.

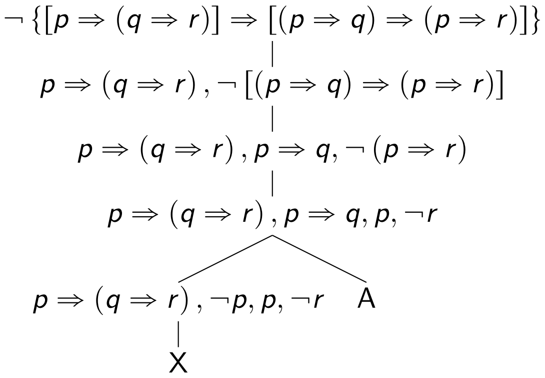
$$[p \Rightarrow (q \Rightarrow r)] \Rightarrow [(p \Rightarrow q) \Rightarrow (p \Rightarrow r)]$$

Exercise 3 - Solution

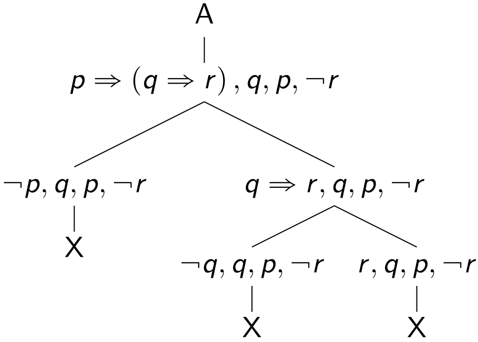


The formula is consistent.

Exercise 3 - Solution



Exercise 3 - Solution



The formula is valid as its negation is inconsistent.

Exercise 4

Determine whether the following formulas are valid, consistent or inconsistent using three different methods.

1. $(\neg p \Rightarrow q) \vee (p \Rightarrow \neg q)$
2. $(p \wedge q) \vee (q \wedge r) \vee (r \wedge p)$
3. $[(p \wedge q) \vee (\neg p \wedge \neg q)] \vee [(\neg p \wedge q) \vee (p \wedge \neg q)]$
4. $[(p \wedge q) \Rightarrow (r \wedge s)] \Rightarrow [(p \wedge q) \Rightarrow (r \wedge s)]$
5. $(a \equiv (b \Rightarrow c)) \equiv [(a \wedge c) \vee (\neg (a \equiv b) \wedge \neg c)]$

Exercise 4 - Solution

1. Valid
2. Consistent
3. Valid
4. Valid
5. Valid