Logic - Tutorial 5

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Reminder

Craig's interpolation theorem

If $A \models B$ (or $\models A \Rightarrow B$),
then there exists a formula C,
containing only atoms occurring in both A and B,
such that $A \models C$ (or $\models A \Rightarrow C$) and $C \models B$ (or $\models C \Rightarrow B$).

Search for an interpolant by induction

Induction on the set Π of atoms occurring in both A and B.

- *Base case.* If $\Pi = \emptyset$, $\models A \Rightarrow B$ implies either A is inconsistent (and $C \triangleq \mathrm{false}$ is an appropriate choice) or B is valid (and $C \triangleq \mathrm{true}$ is an appropriate choice).
- *Induction step.* If $p \in \Pi$, induction hypothesis applies to formulas $A(p/\mathrm{true})$, $B(p/\mathrm{true})$ and also to formulas $A(p/\mathrm{false})$, $B(p/\mathrm{false})$. If C_T and C_F are corresponding interpolants, then formula $(p \wedge C_T) \vee (\neg p C_F)$ is an interpolant for A and B.

Exercise 1

Consider the following two formulas:

$$A \triangleq p \wedge (r \vee q) \wedge t \qquad \text{and} \qquad B \triangleq (p \vee r) \wedge (q \vee t)$$

Do we have that

- $A \models B$ ($\models A \Rightarrow B$)?
- $B \models A$ ($\models B \Rightarrow A$)?

If one of those is true, give an interpolation formula.

Exercise 1 - Solution

$\underline{A \models B}$
If $v(A) = T$, then $v(p) = v(t) = T$ and thus $v(B) = T$

$\underline{B \not\models A}$
With $v(p) = v(q) = T$ and $v(r) = v(t) = F$, we have $v(B) = T$ and $v(A) = F$

Exercise 1 - Solution

Interpolation formula

By Craig's theorem, as $A \models B$, there exist C s.t. $A \models C$ and $C \models B$ containing only atoms in both A and B.
We can easily find that $C \triangleq p \wedge t$ works.

Note:

$C \triangleq A$ and $C \triangleq B$ also work but they are trivial and bring no new information. Moreover, it works in this case because A and B have all their atoms in common.

Exercise 2

Consider the following two formulas:

$$A \triangleq [p \vee (q \wedge r)] \wedge (q \vee t) \qquad \text{and} \qquad B \triangleq (s \vee r) \wedge q \wedge t \wedge p$$

Do we have that $A \models B$ or $B \models A$?
If one of those is true, give an interpolation formula.
To construct the interpolation formula, use the method presented in the proof of the theorem. Is this interpolation formula unique?

Exercise 2 - Solution

<u>$A \not\models B$</u>
If $v(t) = v(p) = T$ and $v(q) = F$, $v(A) = T$ and $v(B) = F$

<u>$B \models A$</u>
If $v(B) = T$, $v(q) = v(t) = v(p) = T$, then $v(A) = T$

<u>Interpolation formula</u>
The set of common atoms is $E = \{p, q, t, r\}$.
We will find the interpolant using the **induction method**.

Exercise 2 - Solution

Let's select atom p, and built $C \triangleq (p \wedge C_T) \vee (\neg p \wedge C_F)$ where

- C_T is an interpolant of $A(p/\mathrm{true})$ and $B(p/\mathrm{true})$
- and C_F is an interpolant of $A(p/\mathrm{false})$ and $B(p/\mathrm{false})$

$\underline{C_T}$
$A(p/\mathrm{true}) \triangleq q \vee t$
$B(p/\mathrm{true}) \triangleq (s \vee r) \wedge q \wedge t$

To find C_T, we apply the same method taking out q out of the common set of atoms $E_T = \{q, t\}$ and building $C_T \triangleq (q \wedge C_{TT}) \vee (\neg q \wedge C_{TF})$

Exercise 2 - Solution

$\underline{C_{TT}}$
$A(p/\text{true}, q/\text{true}) \triangleq \text{true}$
$B(p/\text{true}, q/\text{true}) \triangleq (s \vee r) \wedge t$

We have $C_{TT} \triangleq \text{true}.$

$\underline{C_{TF}}$
$A(p/\text{true}, q/\text{false}) = t$
$B(p/\text{true}, q/\text{false}) = \text{false}$

We have $C_{TF} \triangleq \text{false}.$

Thus $C_T \triangleq (q \wedge \text{true}) \vee (\neg q \wedge \text{false}) \triangleq q$

Exercise 2 - Solution

Now let's do C_F.

<u>C_F</u>
$A(p/\text{false}) \triangleq (q \vee r) \wedge (q \vee t)$
$B(p/\text{false}) \triangleq \text{false}$

Thus $C_F \triangleq \text{false}$.

Therefore, $C \triangleq (p \wedge q) \vee (\neg p \wedge \text{false}) \triangleq p \wedge q$

<u>Note</u>
This is not a unique solution. For example, we can easily find that $C \triangleq p \wedge q \wedge t$ works as well.

<u>Exercise 3</u>

Consider the following two formulas:

$$A \triangleq [(q \Rightarrow r) \wedge s] \qquad \text{and} \qquad B \triangleq (p \Rightarrow q) \Rightarrow (p \Rightarrow r)$$

Do we have that $A \models B$ or $B \models A$?
If one of those is true, give an interpolation formula.

Exercise 3 - Solution

$A \models B$

If $v(A) = T$, $v(s) = T$ and $v(q) = F$ or $v(r) = T$.

- Case 1: $v(r) = T$, then $v(B) = T \rightarrow$ **OK**
- Case 2: $v(q) = F$.
  - Case 2.1: $v(p) = T$, then $v(B) = v(\text{false} \Rightarrow (p \Rightarrow r)) = T \rightarrow$ **OK**
  - Case 2.2: $v(p) = F$, then $v(B) = v(\text{true} \Rightarrow \text{true}) = T \rightarrow$ **OK**

$B \not\models A$

If $v(s) = F$ and $v(r) = T$, then $v(B) = T$ but $v(A) = F$.

Interpolation formula

We can directly infer that $C \triangleq q \Rightarrow r$ works. Indeed, $A \models C$ is trivial and $C \models B$ by following exactly the same proof as $A \models B$.

Exercise 4

Consider the following two formulas:

$$A \triangleq (p \vee q) \wedge (q \Rightarrow r) \qquad \text{and} \qquad B \triangleq \neg p \Rightarrow r$$

Do we have that $A \models B$ or $B \models A$?
If one of those is true, give an interpolation formula.

Exercise 4 - Solution

$A \models B$

If $v(A) = T$, $v(p \vee q) = T$ and $v(q \Rightarrow r) = T$.

We then have two interesting cases:

- Case 1: $v(p) = T$ then $v(B) = T \rightarrow$ **OK**
- Case 1: $v(r) = T$ then $v(B) = T \rightarrow$ **OK**

$B \not\models A$

If $v(r) = T$ and $v(p) = v(q) = F$, then $v(B) = T$ but $v(A) = F$.

Interpolation formula

Following the proof of $A \models B$, we see that $C \triangleq p \vee r$.

Exercise 5

For

$$A \triangleq a \wedge (b \vee c) \qquad \text{and} \qquad B \triangleq a \vee (b \wedge c)$$

$$A \triangleq (p \Rightarrow q) \Rightarrow r \qquad \text{and} \qquad B \triangleq p \Rightarrow (q \Rightarrow r)$$

$$A \triangleq (a \vee b \vee c) \wedge d \qquad \text{and} \qquad B \triangleq (a \vee b) \wedge c \wedge d$$

Do we have that $A \models B$ or $B \models A$?
If one of those is true, give an interpolation formula.

Exercise 5 - Solution

1. $A \models B,\ C \triangleq a$
2. $A \models B,\ C \triangleq q \Rightarrow r$
3. $B \models A,\ C \triangleq c \land d$