Logic - Tutorial 9

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Reminder

A formula is in *prenex form* if it has the form

$$\underbrace{Q_1 x_1 \times \cdots \times Q_n x_n}_{\text{prefix}} \underbrace{M}_{\text{matrix}}$$

where $Q_i \in \{\forall, \exists\} \forall i$ and M is a quantification-free formula.

The scope of the prefix must be the whole matrix.

<u>Theorem</u>: For every predicate formula, some logically equivalent prenex form exists.

Reminder

Reduction to the prenex form

1. Eliminate all boolean connectives except $\neg$, $\vee$, $\wedge$
2. Rename bound variables (if necessary) so that no variable has both free and bound occurences in any subformula
3. Eliminate spurious quantifications
4. Propagate $\neg$ downwards and eliminate double negations
5. Propagate quantifications upwards

Reminder

A *Skolem form* is a prenex form with only universal quantifications

From prenex to Skolem form

For each existential quantification $\exists x$ in the scope of $k \geqslant 0$ universal quantifications $(\forall x_1 \dots \forall x_k))$

1. replace each occurence of x in the matrix by $f(x_1, \dots, x_k)$ where f is a fresh k-ary function symbol ($k = 0$: replace x by a fresh constant)
2. delete the quantification $\exists x$.

Theorem: The Skolem form S_A associated with the prenex form A is consistent if and only if A is consistent.

A formula is in *clausal form* if it is in Skolem form and if its matrix is in conjunctive normal form.

Exercise 1

Give the prenex, Skolem and clausal form of the following formulas:

1. $p(a) \wedge \exists x \neg p(x)$
2. $\forall x \; [p(x) \Rightarrow \forall y \, [\forall z \, q(x, y) \Rightarrow \neg \forall z \, r(y, x)]]$
3. $\forall x \, p(x) \Rightarrow \exists x \, [\forall z \, q(x, z) \vee \forall z \, r(x, y, z)]$
4. $\exists x \, p(x, z) \Rightarrow \forall z \, [\exists y \, p(x, z) \Rightarrow \neg \forall x \exists y \, p(x, y)]$
5. $[\exists x \, p(x) \vee \exists x \, q(x)] \Rightarrow \exists x \, [p(x) \vee q(x)]$

Exercise 1

1)

$$p(a) \land \exists x \neg p(x) \leftrightarrow \exists x (p(a) \land \neg p(x)) \quad (1)$$
$$\leftrightarrow p(a) \land \neg p(b) \quad (2)$$

(1) Prenex form
(2) Skolem and clausal form.

Exercise

2)

$$\begin{aligned}
\forall x\,[p(x) \Rightarrow \forall y[\forall z\, q(x,y) \Rightarrow \neg\forall z\, r(y,x)]] & \quad (1)\\
\leftrightarrow \forall x\,[p(x) \Rightarrow \forall y[q(x,y) \Rightarrow \neg r(y,x)]] & \quad (2)\\
\leftrightarrow \forall x\,[\neg p(x) \vee \forall y[\neg q(x,y) \vee \neg r(y,x)]] & \quad (3)\\
\leftrightarrow \forall x\,\forall y\,[\neg p(x) \vee \neg q(x,y) \vee \neg r(y,x)] & \quad (4)
\end{aligned}$$

(4) Prenex, Skolem and clausal form.

Exercise 1

3)

$$
\begin{aligned}
\forall x\, p(x) \Rightarrow \exists x[\forall z\, q(x,z) \vee \forall z\, r(x,y,z)] & & (1)\\
\leftrightarrow \exists x\,[p(x) \Rightarrow [\forall z\, q(x,z) \vee \forall z\, r(x,y,z)] & & (2)\\
\leftrightarrow \exists x\,[\neg p(x) \vee \forall z\, q(x,z) \vee \forall z\, r(x,y,z)] & & (3)\\
\leftrightarrow \exists x\,[\neg p(x) \vee \forall z\, q(x,z) \vee \forall t\, r(x,y,t)] & & (4)\\
\leftrightarrow \exists x \forall z \forall t\,[\neg p(x) \vee q(x,z) \vee r(x,y,t)] & & (5)\\
\leftrightarrow \forall z \forall t\,[\neg p(a) \vee q(a,z) \vee r(a,y,t)] & & (6)
\end{aligned}
$$

(1) $\rightarrow$ (2) $\forall x\, A \Rightarrow \exists x\, B \leftrightarrow \exists x\,(A \Rightarrow B)$

(5) Prenex form

(6) Skolem and clausal form

Exercise 1

4)

$$
\begin{aligned}
\exists x\, p(x,z) \Rightarrow \forall z[\exists y\, p(x,z) \Rightarrow \neg\forall x \exists y\, p(x,y)] & \qquad (1)\\
\leftrightarrow \exists x\, p(x,u) \Rightarrow \forall z[p(t,z) \Rightarrow \neg\forall v \exists y\, p(v,y)] & \qquad (2)\\
\leftrightarrow \neg\exists x\, p(x,u) \vee \forall z[\neg p(t,z) \vee \neg\forall v \exists y\, p(v,y)] & \qquad (3)\\
\leftrightarrow \forall x\, \neg p(x,u) \vee \forall z[\neg p(t,z) \vee \exists v \forall y\, \neg p(v,y)] & \qquad (4)\\
\leftrightarrow \forall x\, \neg p(x,u) \vee \exists v \forall y \forall z[\neg p(t,z) \vee \neg p(v,y)] & \qquad (5)\\
\leftrightarrow \exists v \forall y \forall z \forall x\, [\neg p(x,u) \vee \neg p(t,z) \vee \neg p(v,y)] & \qquad (6)\\
\leftrightarrow \forall y \forall z \forall x\, [\neg p(x,u) \vee \neg p(t,z) \vee \neg p(a,y)] & \qquad (7)
\end{aligned}
$$

(6) Prenex form
(7) Skolem and clausal form

Exercise 1

5)

$$[\exists x\, p(x) \vee \exists x\, q(x)] \Rightarrow \exists x[p(x) \vee q(x)] \tag{1}$$

$$[\exists x\, p(x) \vee \exists y\, q(y)] \Rightarrow \exists z[p(z) \vee q(z)] \tag{2}$$

$$\leftrightarrow \neg[\exists x\, p(x) \vee \exists y\, q(y)] \vee \exists z[p(z) \vee q(z)] \tag{3}$$

$$\leftrightarrow [\forall x\, \neg p(x) \wedge \forall y\, \neg q(y)] \vee \exists z[p(z) \vee q(z)] \tag{4}$$

$$\leftrightarrow \forall x \forall y\, [\neg p(x) \wedge \neg q(y)] \vee \exists z[p(z) \vee q(z)] \tag{5}$$

$$\leftrightarrow \exists z \forall x \forall y\, [(\neg p(x) \wedge \neg q(y)) \vee p(z) \vee q(z)] \tag{6}$$

$$\leftrightarrow \forall x \forall y\, [(\neg p(x) \wedge \neg q(y)) \vee p(a) \vee q(a)] \tag{7}$$

$$\leftrightarrow \forall x \forall y\, [(\neg p(x) \vee p(a) \vee q(a)) \wedge (\neg q(y) \vee p(a) \vee q(a))] \tag{8}$$

(6) Prenex form
(7) Skolem form
(8) Clausal form

Reminder

- 4 basic formulas

$$
\begin{array}{lr}
A : \forall x(P(x) \Rightarrow Q(x)) & \text{universal afffirmative} \\
E : \forall x(P(x) \Rightarrow \neg Q(x)) & \text{universal negative} \\
I : \exists x(P(x) \wedge Q(x)) & \text{particluar affirmative} \\
O : \exists x(P(x) \wedge \neg Q(x)) & \text{particular negative}
\end{array}
$$

- The *(categorical) syllogism* is the inference rule

$$
\frac{\text{Major } \{Q, R\} \qquad \text{Minor } \{P, Q\}}{\text{Conclusion } \{P, R\}}
$$

  Each of the formula in the inference rule must be either an A-,E-,I- or O-formula.

- Based on the type of the formulas we can define 64 modes of the form XYZ where X is the major, Y is the minor and Z is the conclusion

Reminder

- Depending on their positions in the formula, the predicates have different names:
  - The predicate not appearing in the conclusion formula is the *midterm*, e.g. $Q(x)$
  - The predicate not appearing in the minor formula is the *major*, e.g. $R(x)$
  - The predicate not appearing in the major formula is the *minor*, e.g. $P(x)$
- Finally, based on the order of predicates in the 3 premises, we define 4 figures.For example, considering P has the minor, Q as the midterm and R has the major.

Figure	1	2	3	4
Major	QR	RQ	QR	RQ
Minor	PQ	PQ	QP	QP
Conclusion	PR	PR	PR	PR

- A syllogism is *quasi-valid* if it is a syllogism that is not valid, but that becomes valid by adding $\exists x P(x)$ or $\exists x Q(x)$ or $\exists x R(x)$

Exercise 2

Determine the predicates and the formulas of the following syllogisms, state their mode and figure. Using a Venn diagram, determine whether these syllogisms are valid, quasi-valid, ...

$$\frac{\begin{array}{c}\forall x\,(Q(x) \Rightarrow R(x))\\ \forall x\,(P(x) \Rightarrow Q(x))\end{array}}{\forall x\,(P(x) \Rightarrow R(x))}$$

$$\frac{\begin{array}{c}\forall x\,(A(x) \Rightarrow B(x))\\ \exists x\,(B(x) \wedge C(x))\end{array}}{\exists x\,(C(x) \wedge \neg A(x))}$$

Exercise 2

$$\begin{array}{cc} \forall x\,(Q(x) \Rightarrow R(x)) & \text{Major} \\ \forall x\,(P(x) \Rightarrow Q(x)) & \text{Minor} \\ \hline \forall x\,(P(x) \Rightarrow R(x)) & \text{Conclusion} \end{array}$$

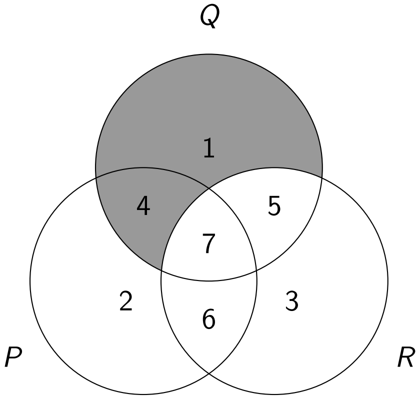
Predicates

- $Q(x)$ - midterm (does not appear in the conclusion)
- $P(x)$ - minor (appears in the minor)
- $R(x)$ - major (appears in the major)

Nature of the formulas: We have an A-major, A-minor and A-conclusion.
Mode: So the mode is AAA.
Figure. We are in figure 1.
Syllogism: Syllogism AAA-1

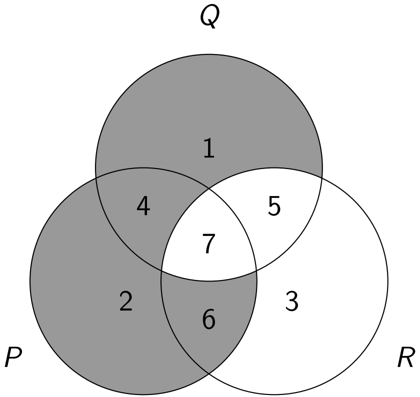
Exercise 2

Major: $\forall x \, (Q(x) \Rightarrow R(x))$
This means that all x that are in Q must also be in R.
Therefore, $1 \cup 4 = \varnothing$



Exercise 2

Minor: $\forall x\,(P(x) \Rightarrow Q(x)) \rightarrow 2 \cup 6 = \varnothing$



Conclusion: $\forall x\,(P(x) \Rightarrow R(x))$?

$\rightarrow$ Are all x which are in P also in R?

$\rightarrow 2 \cup 4 = \varnothing$? Correct!

Exercise 2

$$\begin{array}{cc}
\forall x\,(A(x) \Rightarrow B(x)) & \text{Major} \\
\exists x\,(B(x) \wedge C(x)) & \text{Minor} \\
\hline
\exists x\,(C(x) \wedge \neg A(x)) & \text{Conclusion}
\end{array}$$

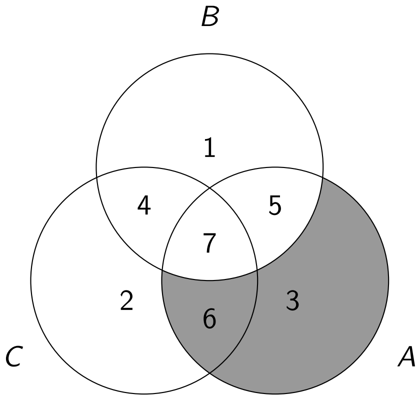
Predicates

- $B(x)$ - midterm (does not appear in the conclusion)
- $C(x)$ - minor (appears in the minor)
- $A(x)$ - major (appears in the major)

Determining the formulas. We have an A-major, I-minor and O-conclusion.
Determining the mode. So the mode is AIO.
Determining the figure. We are in figure 4.
Determining the syllogism: Syllogism AIO-4.

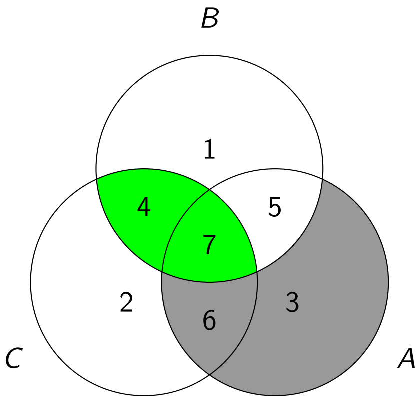
Exercise 2

Major: $\forall x\,(A(x) \Rightarrow B(x))$
All x in A must be in B also.
Therefore, $3 \cup 6 = \varnothing$



Exercise 2

Minor: $\exists x\,(B(x) \wedge C(x))$.
Therefore, there exists some x that is both in B and C.
Thus $\rightarrow 4 \cup 7 \neq \varnothing$



Exercise 2

Conclusion: $\exists x\,(C(x) \wedge \neg A(x)) \to 2 \cup 4 \neq \emptyset$ Not valid.

Counter-example: $1 = 5 = 2 = 3 = 4 = 6 = \emptyset$ and $7 \neq \emptyset$

To show that it is partially valid, add $\exists x\, A(x), \exists x\, B(x), \exists x\, C(x)$ but the counter-example still holds.

<u>Exercise 3</u>

Is the following rule a syllogism? Can it be transformed into a syllogism? Is it correct?

$$\frac{\begin{array}{c}\forall x \exists y\, [\neg Q(x,y) \vee R(x)] \\ \exists x \forall y\, [P(x) \wedge Q(x,y)]\end{array}}{\exists y\, [R(y) \wedge P(y)]}$$

Exercise 3

It is not a syllogism because it contains several variables. Let's try to see if we can make one of the variable 'disappear'. First, we make the y quantification enter the brackets

$$\frac{\begin{array}{c}\forall x\,[\neg\forall y\, Q(x,y) \vee R(x)] \\ \exists x\,[P(x) \wedge \forall y Q(x,y)]\end{array}}{\exists y\,[R(y) \wedge P(y)]}$$

Let's set $Q_2(x) = \forall y\, Q(x,y)$ and we can change y to x in the conclusion. Then we have:

$$\frac{\begin{array}{c}\forall x\,[\neg Q_2(x) \vee R(x)] \\ \exists x\,[P(x) \wedge Q_2(x)]\end{array}}{\exists x\,[R(x) \wedge P(x)]}$$

We obtain a IAI-4 syllogism.