Logic - Exam Exercices

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Reminder

- $\forall x\, A \models \exists x\, A$
- $\neg\forall x\, A \leftrightarrow \exists x\, \neg A$
- $\neg\exists x\, A \leftrightarrow \forall x\, \neg A$

Reminder

- $(\forall x\, A \wedge \forall x\, B) \leftrightarrow \forall x\, (A \wedge B)$
- $(\forall x\, A \vee \forall x\, B) \models \forall x\, (A \vee B)$

  $\forall x\, (A \vee B) \not\models (\forall x\, A \vee \forall x\, B)$
  Counter-example:
  $D = \{a, b\}, I[A(a)] = T, I[A(b)] = F, I[B(a)] = F, I[B(b)] = T$

- $\exists x (A \vee B) \leftrightarrow (\exists x\, A \vee \exists x\, B)$
- $\exists x (A \wedge B) \models (\exists x\, A \wedge \exists x\, B)$

  $(\exists x\, A \wedge \exists x\, B) \not\models \exists x (A \wedge B)$
  Counter-example:
  $D = \{a, b\}, I[A(a)] = T, I[A(b)] = F, I[B(a)] = T, I[B(b)] = F$

Reminder

- $\exists x(P(x) \land Q(y)) \leftrightarrow \exists x P(x) \land Q(y)$
- $\forall x(P(x) \lor Q(y)) \leftrightarrow \forall x P(x) \lor Q(y)$

Reminder

- $\exists x(A \Rightarrow B) \leftrightarrow (\forall x\, A \Rightarrow \exists x\, B)$

  $\exists x(A \Rightarrow B) \leftrightarrow \exists x(\neg A \vee B) \leftrightarrow (\exists x \neg A \vee \exists x B) \leftrightarrow (\neg \forall x A \vee \exists x B) \leftrightarrow (\forall x\, A \Rightarrow \exists x\, B)$

- $\forall x(A \Rightarrow B) \models (\forall x\, A \Rightarrow \forall x\, B)$

  For any interpretation I such that $\mathcal{I}[\forall x(A \Rightarrow B)] = T$, we have $\mathcal{I}_{x/d}[A(x)] = F$ or $\mathcal{I}_{x/d}[B(x)] = T$ for every $d \in D$.
  Then either, $\mathcal{I}_{x/d}[A(x)] = F$ for some d and then $\mathcal{I}[\forall x A(x)] = F$ and the second formula is true. Or $\mathcal{I}[\forall x A(x)] = F$ for all d which means $\mathcal{I}[\forall x A(x)] = T$ but then $\mathcal{I}_{x/d}[B(x)] = T$ for all d. Thus $\mathcal{I}[\forall x B(x)] = T$ and the second formula is T.

  $(\forall x\, A \Rightarrow \forall x\, B), \not\models \forall x(A \Rightarrow B)$
  Counter-example:
  $D = \{a, b\}, I[A(a)] = F, I[A(b)] = T, I[B(a)] = F, I[B(b)] = F$

Reminder

- $\forall x(A \equiv B) \models (\forall x\, A \equiv \forall x\, B)$

  $\forall x(A \equiv B) \leftrightarrow \forall x[(A \Rightarrow B) \wedge (B \Rightarrow A) \models [\forall x(A \Rightarrow B) \wedge \forall x(B \Rightarrow A)] \models [(\forall xA \Rightarrow \forall xB) \wedge (\forall xB \Rightarrow \forall xA)] \leftrightarrow (\forall x\, A \equiv \forall x\, B)$

Reminder

- $\forall x \forall y\, A \leftrightarrow \forall y \forall x\, A$
- $\exists x \exists y\, A \leftrightarrow \exists y \exists x\, A$
- $\exists x \forall y\, A \models \forall y \exists x\, A$

August 2019 - Exercise 3

Exercise 1

$$\frac{\exists x \forall y [C(x) \land \neg A(x,y)],\ \forall x \exists y [A(x,y) \lor B(x)]}{\exists x [B(x) \land C(x)]}$$

1. What is this object?
2. What can you find about it?
3. Can it be investigated?
   1. With a semantic tableau?
   2. Using Hilbert system?
   3. Using syllogism theory?
   4. Using prenex and/or Skolem forms?
4. Choose a method to establish the relevant properties.

Exercise 1 - Solution

1) and 2) This is an inference rule and we can find out if it is correct (i.e. valid).

3.1) We can prove that the rule is correct by showing that the set $\{\exists x \forall y[C(x) \wedge \neg A(x,y)], \forall x \exists y[A(x,y) \vee B(x)], \neg \exists x[B(x) \wedge C(x)]\}$ is inconsistent.

The inconsistency of this set can be investigated through the semantic tableaux method with this set as the root.

If the tree is closed, the set is inconsistent and the rule is correct while if it's open we can find a model of the set which means that the rule is incorrect.

The development of this set via the semantic tableaux method being quite long, it will not be done here.

Exercise 1 - Solution

3.2) The Hilbert system is complete and therefore if we can proof that the rule is valid using this technique, it is indeed valid.

Exercise 1 - Solution

3.3) The syllogism theory can be used but we need to modify the rule before applying it.
Let's first make the $\exists y$ enter the brackets in the two premises, we obtain:

$$\frac{\exists x[C(x) \wedge \forall y \neg A(x,y)],\ \forall x[\exists y\, A(x,y) \vee B(x)]}{\exists x\,[B(x) \wedge C(x)]}$$

We then pass the quantifier inside the negation in the first premise and define $A'(x) = \neg\exists y\, A(x,y)$.

$$\frac{\exists x[C(x) \wedge A'(x)],\ \forall x[\neg A'(x) \vee B(x)]}{\exists x\,[B(x) \wedge C(x)]}$$

Exercise 1 - Solution

Finally we obtain:

$$\frac{\exists x[C(x) \wedge A'(x)],\ \forall x[A'(x) \Rightarrow B(x)]}{\exists x\,[B(x) \wedge C(x)]}$$

We can now analyze the syllogism.
Let's consider that the first premise is the major, and the second the minor.
The mode is IAI.
C is the major predicate, A' the midterm and B the minor.
The figure is the fourth.

Doing the Venn diagram, we can easily show that the syllogism is valid and thus the rule is correct.

Exercise 1 - Solution

3.4)
To work with prenex and Skolem forms, we could try to transform the following formula:

$$\exists x \forall y [C(x) \wedge \neg A(x, y)] \wedge \forall x \exists y [A(x, y) \vee B(x)] \wedge \neg \exists x [B(x) \wedge C(x)]$$

The prenex form being logically equivalent to the original formula and the Skolem form being a logical consequence of the Prenex form, if we manage to show that the Skolem form is inconsistent then we will have shown that the original formula is inconsistent and therefore the rule is correct.

August 2019 - Exercise 4

Exercise 2

Compare the formulas:

A: $\forall x\,[P(x)\,\mathbf{W}\,\forall y\,Q(x,y)]$
B: $\forall x\,\forall y\,[P(x)\,\mathbf{W}\,Q(x,y)]$

Note: **W** being the exclusive or

Exercise 2 - Solution

Let's define $A'(x) \triangleq [P(x) \, \mathbf{W} \, \forall y \, Q(x,y)]$ and
$B'(x) \triangleq \forall y \, [P(x) \, \mathbf{W} \, Q(x,y)]$ and compare these two formulas.
Let's first define a valuation $\mathcal{I}$ such that $\mathcal{I}[P(x)] = T$.
In that case $\mathcal{I}[A'] = \mathcal{I}[\neg \forall y Q(x,y)] = \mathcal{I}[\exists y \neg Q(x,y)]$ and
$\mathcal{I}[B'] = \mathcal{I}[\forall y \neg Q(x,y)]$. As $\forall y \, C \models \exists y \, C$, we have $B' \models A'$ and $A' \not\models B'$)

Let's take another valuation $\mathcal{I}'$ such that $\mathcal{I}'[P(x)] = F$. In that case
$\mathcal{I}'[A'] = \mathcal{I}'[\forall y Q(x,y)]$ and $\mathcal{I}'[B'] = \mathcal{I}'[\forall y Q(x,y)]$. Therefore, $A' \leftrightarrow B'$.

Therefore, we have for any valuation $B' \models A'$

Exercise 2 - Solution

What can we say about our original formulas?

Let I be an interpretation such that $\mathcal{I}[B] = T$, this implies that $\mathcal{I}_{x/d}[B'(x)] = T$ for any $d \in D$. As A' is a logical consequence of B', we necessarily have $\mathcal{I}_{x/d}[A'(x)] = T$ for any $d \in D$ and $\mathcal{I}[A] = T$. Thus $B \models A$.

Given that $A' \not\models B'$, we can similarly come to the conclusion that $A \not\models B$.