Linear optimization
A furniture company produces two types of chairs from beech and oak wood.

The first type (basic) requires 9 boards of beech wood and 2 boards of oak wood. A basic chair is easy to construct and requires one worker’s hour.

The second type (classic) requires 7 boards of beech wood and 5 boards of oak wood. Due to the more important completion time, the classic chair requires three worker’s hours.

The price of a basic chair is 30 euros whereas a classic chair’s price is 70 euros.

The stock of boards of the entreprise is 800 boards of beech wood and 200 boards of oak wood. There are 4 workers working each 40 hours per week.

What is the number of basic and classic chairs that the entreprise must produce for this week?
Production problem

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Modeling of the problem

**Choice of the decision variables**

\[ x_B = \text{Number of basic chairs to construct this week} \]
\[ x_C = \text{Number of classic chairs to construct this week} \]

**Objective to optimize**

\[
\max 30x_B + 70x_C
\]

**Constraints**

- **Beech**: \[ 9x_B + 7x_C \leq 800 \]
- **Oak**: \[ 2x_B + 5x_C \leq 200 \]
- **Work**: \[ x_B + 3x_C \leq 160 \]
- **Other**: \[ x_B \geq 0, x_C \geq 0 \] (and \( x_B, x_C \) integer).

\[
\max 30x_B + 70x_C \\
\text{s.t.} \quad 9x_B + 7x_C \leq 1000 \\
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Alloy production

The company Steel has received an order for 500 tons of steel to be used in shipbuilding. This steel must have the following characteristics

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<thead>
<tr>
<th>Chemical element</th>
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<tbody>
<tr>
<td>Carbon (C)</td>
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The company has seven different raw materials in stock that may be used for the production of this steel. The following Table lists the grades, available amounts and prices for all raw materials.

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Formulation

Decision Variables

\( use_i \): Quantity of alloy \( i \) used \((i \in I)\)

Objective to optimize

\[
\min \sum_{i \in I} price_i \cdot use_i
\]

Constraints

- Carbon: \( LB_C \leq \sum_{i \in I} C_i \cdot use_i \leq UB_C \)
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Different forms of linear programming

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\text{max } & 2x_1 + 3x_2 \\
\text{s.t. } & 3x_1 + x_2 \leq 3 \\
& x_1 - x_2 = 2 \\
& x_1, x_2 \geq 0
\end{align*}

\begin{align*}
\text{min } & -2x_1 + 3x_2 \\
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& x_1 \geq 0, \\
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\text{min } & 2x_1 - 3x_2 \\
\text{s.t. } & 7x_1 - x_2 \leq 3 \\
& x_1 + 2x_2 = 5 \\
& x_1 \geq 0, x_2 \in \mathbb{R}
\end{align*}

Objective : \( \min \) ou \( \max \)
Constraints : \( \geq, \leq, = \)
Bounds : \( \geq 0, \leq 0, [l, u], \mathbb{R} \)
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Different forms of linear programming

We can go equivalently from one form to the other

Objective:

\[
\max f(x) \equiv -\min -f(x)
\]

\[
\max 2x_1 - 7x_2 \equiv -\min -2x_1 + 7x_2
\]

Constraints:

\[
f(x) \leq b \equiv -f(x) \geq -b
\]

\[
2x_1 - x_2 \leq 1 \equiv -2x_1 + x_2 \geq -1
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\[
f(x) = b \equiv f(x) \leq b \text{ et } f(x) \geq b
\]

\[
3x_1 - x_2 = 3 \equiv 3x_1 - x_2 \leq 3 \text{ et } 3x_1 - x_2 \geq 3
\]

\[
f(x) \leq b \equiv f(x) + s = b, \text{ with } s \geq 0
\]

\[
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Bounds:

\[
x \leq 0 \equiv \hat{x} := -x \text{ et } \hat{x} \geq 0
\]

\[
y \in \mathbb{R} \rightarrow y = y^+ - y^- \text{ and } y^+, y^- \geq 0 ! \text{ Not equivalent !}
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x \leq 0 \equiv \hat{x} := -x \text{ et } \hat{x} \geq 0
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\[
y \in \mathbb{R} \rightarrow y = y^+ - y^- \text{ and } y^+, y^- \geq 0! \text{Not equivalent!}
\]
Different forms of linear programming

We can go equivalently from one form to the other

**Objective:**

\[
\max f(x) \equiv - \min -f(x)
\]

\[
\max 2x_1 - 7x_2 \equiv - \min -2x_1 + 7x_2
\]

**Constraints:**

\[
f(x) \leq b \equiv -f(x) \geq -b
\]

\[
2x_1 - x_2 \leq 1 \equiv -2x_1 + x_2 \geq -1
\]

\[
f(x) = b \equiv f(x) \leq b \text{ et } f(x) \geq b
\]

\[
3x_1 - x_2 = 3 \equiv 3x_1 - x_2 \leq 3 \text{ et } 3x_1 - x_2 \geq 3
\]

\[
f(x) \leq b \equiv f(x) + s = b, \text{ with } s \geq 0
\]

\[
3x_1 - 2x_2 \geq 0 \equiv 3x_1 - 2x_2 - s \geq 0 \text{ with } s \geq 0
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The standard form consists in

- **Objective**: minimization
- **Constraints**: equalities
- **Bounds**: Nonnegative variables

\[
\begin{align*}
\min & \quad c^T x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \in \mathbb{R}^n_+
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**Exercise**: Reduce a given problem into standard form.
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**Exercise**: Reduce a given problem into standard form
Graphic representation
We can represent a problem in two dimensions graphically.

Example:

\[
\begin{align*}
\text{max} & \quad x_1 + 2x_2 \\
-x_1 + 2x_2 & \leq 1 \\
-x_1 + x_2 & \leq 0 \\
4x_1 + 3x_2 & \leq 12 \\
x_1, x_2 & \geq 0
\end{align*}
\]
Graphic representation

\begin{align}
\max \ x_1 + 2x_2 & \quad (1) \\
-x_1 + 2x_2 & \leq 1 \quad (2) \\
-x_1 + x_2 & \leq 0 \quad (3) \\
4x_1 + 3x_2 & \leq 12 \quad (4) \\
x_1, \ x_2 & \geq 0 \quad (5)
\end{align}
max $x_1 + 2x_2$  
$-x_1 + 2x_2 \leq 1$  
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Graphic representation

\[
\begin{align*}
\max & \quad x_1 + 2x_2 & (1) \\
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max \( x_1 + 2x_2 \)
\[-x_1 + 2x_2 \leq 1 \]  \hspace{1cm} (1)
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Degenerate cases

In the example we had a unique solution at a vertex of the polyhedron. Some degenerate cases can lead to different solutions.
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Polyhedra

Definition

A **polyhedron** is a set \( \{ x \in \mathbb{R}^n | Ax \geq b \} \)

A set of the form \( Ax \leq b \) is also a polyhedron.
A set \( \{ x \in \mathbb{R}^n | Ax = b, x \geq 0 \} \) is a polyhedron in **standard form**.

Definition

Let \( a \in \mathbb{R}^n \setminus \{0\} \).

(a) The set \( \{ x \in \mathbb{R}^n | a^T x = b \} \) is a hyperplane

(b) The set \( \{ x \in \mathbb{R}^n | a^T x \geq b \} \) is a halfspace
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Convex Sets

**Definition**

A set \( S \subseteq \mathbb{R}^n \) is **convex** if for all \( x, y \in S \) and all \( \lambda \in [0, 1] \), \( \lambda x + (1 - \lambda)y \in S \).

**Definition**

Let \( x^1, \ldots, x^k \) be vectors of \( \mathbb{R}^n \).

1. \( \lambda_1 x^1 + \cdots + \lambda_k x^k \) is a conic combination if \( \lambda_1, \ldots, \lambda_k \geq 0 \)
2. \( \lambda_1 x^1 + \cdots + \lambda_k x^k \) is a convex combination if \( \lambda_1, \ldots, \lambda_k \geq 0 \) and \( \lambda_1 + \cdots + \lambda_k = 1 \)
3. The convex hull of \( x^1, \ldots, x^k \) is the set of all convex combinations of \( x^1, \ldots, x^k \).

**Theorem**

(a) The intersection of two convex sets is convex
(b) Every polyhedron is convex
(c) The convex hull of a finite number of points is a polyhedron.
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(ii) $\lambda_1 x^1 + \cdots + \lambda_k x^k$ is a **convex combination** if $\lambda_1, \ldots, \lambda_k \geq 0$ and $\lambda_1 + \cdots + \lambda_k = 1$

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**Convex Sets**

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Extreme points and vertices

**Definition**

Let $P$ be a polyhedron. A point $x \in P$ is an extreme point of $P$ if there do not exist two points $y, z \in P$ such that $x$ is a convex combination of $y$ and $z$.

**Definition**

Let $P$ be a polyhedron. A point $x \in P$ is a vertex of $P$ if there exists $c \in \mathbb{R}^n$ such that $c^T x < c^T y$ for all $y \in P$ and $y \neq x$. 
Extreme points and vertices

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**Definition**

Let $P$ be a polyhedron. A point $x \in P$ is a **vertex** of $P$ if there exists $c \in \mathbb{R}^n$ such that $c^T x < c^T y$ for all $y \in P$ and $y \neq x$. 
Bases of a polyhedron

We subdivide the equalities and inequalities into three categories:

\[ a_i^T x \geq b_i \quad i \in M_\geq \]
\[ a_i^T x \leq b_i \quad i \in M_\leq \]
\[ a_i^T x = b_i \quad i \in M_= \]

Definition

Let \( \bar{x} \) be a point satisfying \( a_i^T \bar{x} = b_i \) for some \( i \in M_\geq, M_\leq \) or \( M_= \). The constraint \( i \) is said to be active or tight.

Theorem

Let \( \bar{x} \in \mathbb{R}^n \) and let \( I \) be the set of active constraints for \( \bar{x} \). The three following statements are equivalent.

(i) There exist \( n \) linearly independent vectors in \( \{ a_i | i \in I \} \).
(ii) \( \text{span}\{ a_i | i \in I \} = \mathbb{R}^n \).
(iii) The system \( a_i^T x = b_i, i \in I \) has a unique solution.
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Bases of a polyhedron

**Definition**

Let $P$ be a polyhedron and let $\bar{x} \in \mathbb{R}^n$.

(a) $\bar{x}$ is a **basic solution** if
   - all equalities ($i \in M_=$) are **active**
   - among the active constraints, there are $n$ linearly independent

(b) if $\bar{x}$ is a basic solution **that satisfies all constraints**, then $\bar{x}$ is a **feasible basic solution**.

**Theorem**

Let $P$ be a polyhedron and let $\bar{x} \in P$. The three following statements are equivalent.

(i) $\bar{x}$ is a vertex

(ii) $\bar{x}$ is an extreme point

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Bases of a polyhedron

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Polyhedra in standard form

Consider \( P = \{ x \in \mathbb{R}^n | A x = b, x \geq 0 \} \).
We assume that the rows of \( A \) are linearly independent.

**Theorem**

A point \( \bar{x} \) is a basic feasible solution if \( A \bar{x} = b \) and if there exist \( m \) indices \( B(1), \ldots, B(m) \) such that

(i) The columns \( A_{B(1)}, \ldots, A_{B(m)} \) are linearly independent

(ii) If \( i \neq B(1), \ldots, B(m) \), then \( x_i = 0 \)

**Explanation:**

\[
\begin{pmatrix}
A \\
l
\end{pmatrix} x \geq \begin{pmatrix} b \\ 0 \end{pmatrix}
\]

We have \( n + m \) constraints and \( n \) variables.
A basic solution \( \Rightarrow \) \( n \) constraints satisfied with equality.
The \( m \) equalities are automatically satisfied.
There are \( n - m \) inequalities \( x_i \geq 0 \) that are active (the nonbasic variables).
There are \( m \) inequalities \( x_i \geq 0 \) that are possibly not active (basic variables).
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Construction of a basis

Procedure (≠ Algorithm)

(i) Choose \( m \) linearly independent columns \( A_{B(1)}, \ldots, A_{B(m)} \)

(ii) \( x_i = 0 \) for all \( i \neq B(1), \ldots, B(m) \)

(iii) Solve \( Ax = b \) for the unknowns \( x_{B(1)}, \ldots, x_{B(m)} \)

If the solution \( x \geq 0 \), then \( x \) is a basic feasible solution.

We construct the basic matrix as

\[
A_B = \begin{pmatrix} A_{B(1)} & A_{B(2)} & \cdots & A_{B(m)} \end{pmatrix}
\]

The nonbasic matrix \( N \) corresponds to nonbasic indices.

The basic vector is \( x_B = (x_{B(1)}, \ldots, x_{B(m)}) \) and the nonbasic vector corresponds to the other indices.

We have

\[
A_B x_B = b
\]

\[
x_N = 0
\]

\[
A_B x_B + A_N x_N = b
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Example
Construction of a basis

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**Example**
Some important remarks

Correspondence between the base and the basic solution

Two different bases could lead to the same solution $x$.

Adjacent Bases

Two bases are adjacent they differ by only one index. Differently stated they have $n - 1$ indices in common!
Some important remarks

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Two bases are adjacent they differ by only one index.
Differently stated they have $n - 1$ indices in common!
Some important remarks

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Degenerescence

**Definition**

A basic solution \( x \in \mathbb{R}^n \) is **degenerate** if more than \( n \) constraints are active at the solution.

**Degenerescence for a standard form**

Let \( P = \{ x \in \mathbb{R}^n | Ax = b, x \geq 0 \} \), with \( A \in \mathbb{R}^{m \times n} \). A basic solution \( x \) is degenerate if \( x \) has more than \( n - m \) zero elements.

**Remark:** Degenerescence may be representation-dependent. A non degenerate basis can be degenerate in another representation of the problem and conversely.

**Example:**
Degenerescence

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Example:
The simplex algorithm

**Principle**

- Start at a **feasible basic solution**
- Check whether the current basis is **optimal**
- If not, find a **direction of improvement**
- The direction of improvement leads to either ... a better **feasible basic solution**
- or ... proving that the problem is **unbounded**

**Fundamental question**

Let \( x \) be a feasible basic solution.
Find a direction \( d \) such that \( x + \theta d \) is feasible for some \( \theta \geq 0 \).
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Finding a feasible direction from a basic feasible solution

Given: \( x = (x_B \ x_N) \in P \)
Find \( d \) such that \( x + d\theta \in P \)

- We want to change at least one nonbasic solution.
- Select one index \( j \in N \)
  \[
  d_j = 1 \quad d_i = 0 \text{ for all } i \neq j
  \]
- We have \( Ax = b \) and \( A(x + \theta d) = b \Rightarrow Ad = 0 \)

\[
0 = Ad = \sum_{i=1}^{n} A_{B(i)}d_{B(i)} + A_j
\]
\[
Bd_B + A_j
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\( d_B = -B^{-1}A_j \) is the \( j^{th} \) basic direction
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Taking care of the nonnegativity constraints

- **Nonbasic variables**: $x_i \ (i \neq j) : = 0 : \Rightarrow \text{OK} !$
  $x_j : \text{goes in the positive direction} \Rightarrow \text{OK} !$

- **Basic Variables**
  Nondegenerate case: $x_B > 0$
  Therefore $x_B + \theta d_B \geq 0$ for $\theta$ sufficiently small

- **Degenerate case**: $x_B(i) = 0$ for some $i$
  2 cases
  (1) $(d_B)_i = (-B^{-1}A_j)_i \geq 0 \Rightarrow \text{feasible for } \theta \text{ small enough}$
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Checking optimality

Essential question: Is the current basis optimal or can we find a basic direction that improves the objective function?

Definition: $c_B = (c_{B(1)}, \ldots, c_{B(m)})$

Let us compute the rate of objective change for the $j^{th}$ basic direction.

$d = (d_B \ d_N)$ with $d_B = -B^{-1}A_j$ \hspace{1em} $d_N = e_j$

$$c^T d = c^T d_B + c^T d_N$$

$$= -c_B^T B^{-1} A_j + c_j$$

Definition

The reduced cost of the $j^{th}$ nonbasic variable is

$$\bar{c}_j = c_j - c_B^T B^{-1} A_j.$$
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Optimality Criterion

Theorem

Consider a basic feasible solution $x$

(i) If $\bar{c} \geq 0$ then $x$ is optimal

(ii) If $x$ is optimal and nondegenerate then $\bar{c} \geq 0$.

Definition

A basis matrix $B$ is said to be optimal if

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Iterative step : From a basis $B$ do

If $\bar{c}_j \geq 0$ for all $j$

then the current basis is optimal

else select a nonbasic variable $j$ with $\bar{c}_j < 0$

We bring $j$ into the basis
We now look for $\theta$ maximal such that $x + \theta d \in P$

If $d_i \geq 0$ for all $i \in B$

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The problem is unbounded and $OPT = -\infty$

If $d_i < 0$ for some $i \in B$

then $\theta^* = \min\{i \in B | d_B(i) < 0\} \left( -\frac{x_B(i)}{d_B(i)} \right)$

Variable $i$ achieving the minimum goes out of the basis
We move to the next vertex (basic feasible solution)

$B \leftarrow B \cup \{j\} \setminus \{i\}$
New point := $x + \theta^* d$
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The operation of moving to another basis

In an iteration of the simplex algorithm, in the basis, we replace the variable \(i\) such that

\[
i = \arg \min \left\{ -\frac{x_{B(i)}}{d_{B(i)}} \mid i \in B \text{ with } d_{B(i)} < 0 \right\}
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by the variable \(j\) entering the basis (that was chosen with \(\bar{c}_j < 0\)).

**Theorem**

(i) The columns \(A_{B(k)}\), \(k \neq i\) and \(A_j\) are linearly independent

(ii) The vector \(y = x + \theta^* d\) is a basic feasible solution associated with the new basis matrix.
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Finding an initial basis

2 issues: finding a feasible solution and finding the corresponding basis.

One easy case

Consider a problem of the type \( Ax \leq b \) where \( b \geq 0 \).
The standard form, if we add the slack variables

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\begin{align*}
    a_{11}x_1 + \cdots + a_{1n}x_n + s_1 &= b_1 \\
    & \vdots \\
    a_{m1}x_1 + \cdots + a_{mn}x_n + s_m &= b_m
\end{align*}
\]

The point \((x_1 = 0, \cdots, x_n = 0)\) is a feasible solution and \((s_1 \cdots s_m)\) is the corresponding basis.

In general

It is not easy to find a feasible basis.
One can try any basis (choice of \(m\) variables) but there is no guarantee that it is feasible.
In some cases, the problem is infeasible \(\rightarrow\) it might be impossible to find a basic feasible solution.
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Using a linear program and the simplex algorithm to find an initial basis

**Phase I**

Consider a problem of the type $Ax = b, x \in \mathbb{R}^n$ where we assume (after suitable multiplication of rows by $-1$) that $b \geq 0$.

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\begin{align*}
\text{min} & \quad \xi_1 + \cdots + \xi_m \\
\text{subject to} & \quad a_{11}x_1 + \cdots + a_{1n}x_n + \xi_1 = b_1 \\
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- If the optimal solution of the phase 1 problem is $> 0 \Rightarrow$ the initial problem is infeasible.
- If the initial problem is feasible $\Rightarrow$ the phase 1 problem has 0 as optimal solution.
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