Information theory and coding

Source coding: Some guessing games and notion of typical set [MacKay, 2003]

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The weighing problem (see [MacKay, 2003])

Greedy strategy: not necessarily the best one

The game of "Sixty-three" (see [MacKay, 2003])

What's the **smallest** number of **yes/no** questions needed to identify an integer x between 0 and 63?

$$x \in \{0, 1, 2, 3, 4, 5, 6, \ldots, 63\}$$

The game of "Sixty-three"

Let's play

$$x = ?$$

The game of "Sixty-three"

Solution

$$x = 42$$

The game of "Sixty-three"

Strategy

Q		q_i
1:	is $x \geqslant 32$?	yes
2:	is $x \mod 32 \geqslant 16$?	no
3:	is $x \mod 16 \geqslant 8$?	yes
4:	is $x \mod 8 \geqslant 4$?	no
5:	is $x \mod 4 \geqslant 2$?	yes
6:	is $x \mod 2 = 1$?	no

The game of "Sixty-three"

Binary point of view

Q		q_i	
1:	is $x \geqslant 32$?	**yes**	1
2:	is $x \mod 32 \geqslant 16$?	**no**	0
3:	is $x \mod 16 \geqslant 8$?	**yes**	1
4:	is $x \mod 8 \geqslant 4$?	**no**	0
5:	is $x \mod 4 \geqslant 2$?	**yes**	1
6:	is $x \mod 2 = 1$?	**no**	0

101010 is 42 in binary

What's the information content gained for each question?
And for all questions?

The game of "Sixty-three"

Information theory point of view

For each question:

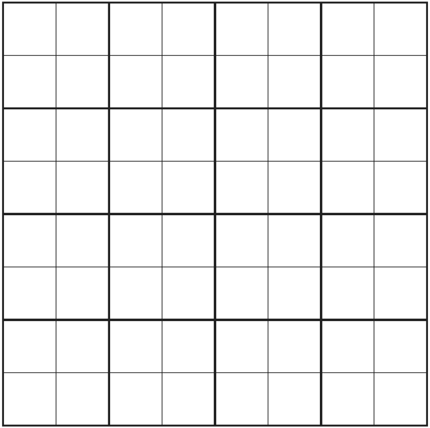
$$h(q_i) = -\log_2 \frac{1}{p(q_i)} = -\log_2 \frac{1}{0.5} = 1 \; \mathcal{S}hannon$$

For all six questions:

$$\begin{aligned}
\sum_{i=1}^{6} h(q_i) = 1 + \cdots + 1 &= 6 \; \mathcal{S}hannon \\
&= \log_2(64) \\
&= H\left(\frac{1}{64}, \frac{1}{64}, \ldots, \frac{1}{64}\right) \\
&= -64 \frac{1}{64} \log_2 \left(\frac{1}{64}\right)
\end{aligned}$$

The game of submarine (see [MacKay, 2003] and TP1)

Find the submarine (one square) in this eight-by-eight grid.



The game of submarine

1. First question: $p(\text{found}) = \frac{1}{64}$ and $p(\text{miss}) = \frac{63}{64}$

   If found: $h(\text{found}) = -\log_2\left(\frac{1}{64}\right) = 6 \text{ Shannon}$
   if miss: $h(\text{miss}) = -\log_2\left(\frac{63}{64}\right) = 0.0227 \text{ Shannon}$
2. 2^{nd}, 3^{rd}, 4^{th}, ...

   $h(\text{miss}) = -\log_2\left(\frac{62}{63}\right) = 0.0230 \text{ Shannon}$
   ... (very bad luck)
   $h(\text{miss}) = -\log_2\left(\frac{1}{2}\right) = 1$
3. $0.0227 + 0.0230 + \cdots + 1 = 6 \text{ Shannon}$

The bent coin lottery
Principle

A coin with $p_1 = 0.1$ will be tossed $N = 1000$ times.
For example, $x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 0, x_5 = 0, \ldots$

The outcome is $\mathbf{x} = x_1 x_2 x_3 \ldots x_N$.
In our example, the ticket is $10100\ldots$

You can buy any of the 2^N possible tickets for 1 euro each, before the coin-tossing.

If you own ticket $\mathbf{x}$, you win 1 billion euros.
Note that $2^{1000} \gg 10^9$.

The bent coin lottery

Questions

1. If you are forced to buy **one** ticket, which would you buy?
2. To have a 99% chance of winning, at lowest possible cost,
   - **which** tickets would you buy?
   - And how **many** tickets is that?

The bent coin lottery
Only one ticket

Any idea?

The bent coin lottery
Binomial distribution

Given $p = 0.1$ and $1 - p = 0.9$, the probability to have exactly k ones among N coin-tossing is

$$C_N^k p^k (1-p)^{N-k}$$

But for a given order

$$p^k (1-p)^{N-k}$$

The bent coin lottery

Only one ticket

The all-zeros ticket.

Examples:

1. $k = 0 \Rightarrow p(k = 0) = 1.74e - 46$
2. $k = 1 \Rightarrow p(k = 1) = 1.94e - 44$
3. But... $10000\ldots000 \Rightarrow p(100\ldots000) = 1.94e - 47$

The bent coin lottery

99% chance of winning

Any idea?

The bent coin lottery

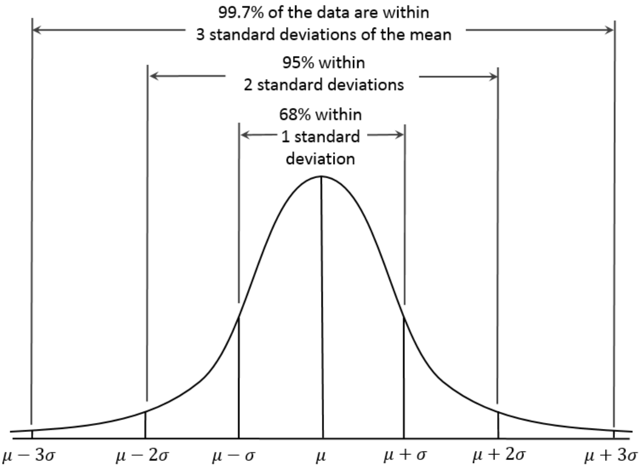
Binomial distribution

For a binomial distribution of probability p of success:

$$\begin{aligned} mean &= Np \\ variance &= Np(1-p) \end{aligned}$$

The bent coin lottery

Gaussian distribution



The bent coin lottery

Number of tickets

$$\#\ \text{tickets} = 1 + C^{1}_{1000} + C^{2}_{1000} + \cdots + C^{100}_{1000} + \cdots + C^{123}_{1000}$$

The bent coin lottery

Approximation of C_N^k (Stirling's approximation)

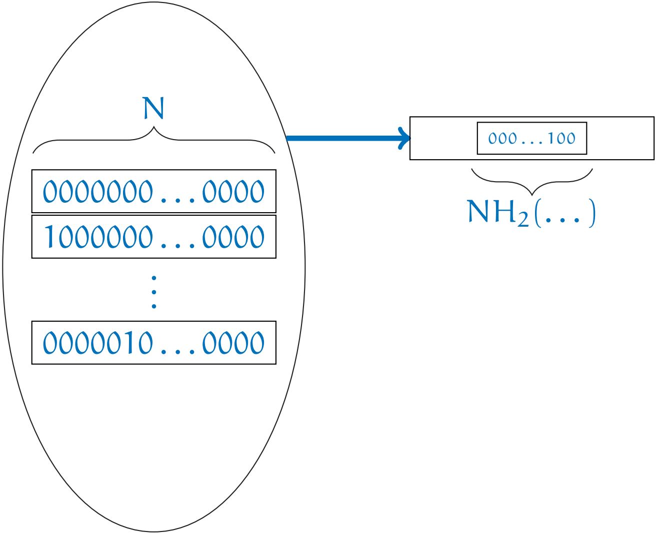
$$\log_2 C_N^k \approx N H_2\left(\frac{k}{N}\right)$$

Example:

$$\begin{aligned} & \log_2 C_{1000}^{123} \approx 1000 H_2\left(\frac{123}{1000}\right) = 537 \\ \Leftrightarrow \quad & C_{1000}^{123} \approx 2^{537} \end{aligned}$$

The bent coin lottery

A lot of tickets



The bent coin lottery
Typical set

99 % chance of owning the winning ticket!

The bent coin lottery
Typical set with uniform distribution

What can you notice?

References

MacKay, D. J. (2003). Information theory, inference, and learning algorithms.