1 Exercises

Exercise 1. An American submarine sank in January 1970 in Atlantic Ocean. The probability density of the submarine position is uniform in the research zone. This zone has been delimited by a grid of size 3×4 such that the area covered by each case is the same (see Figure 1). Variables $\mathcal{X}$ and $\mathcal{Y}$ correspond to abscissa and ordinate positions in the grid.

a. What is the probability to find the submarine in $(X, Y) = (i, j)$?

b. What is the entropy of variable $\mathcal{X}$?

c. Can this problem be modelled as a probabilistic graphical model with these two variables? Justify your answer using information theory measures and if such a model exists, draw it.

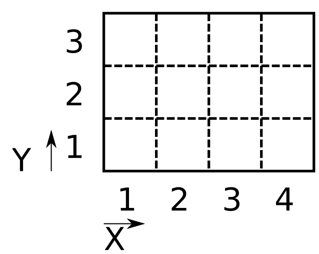


Figure 1: Research grid.

(a) $P(X_i, Y_j) = \frac{1}{m} = \frac{1}{12}$

(b) $H(X) = -\sum_{i=1}^{m} P(X_i) \log_2 P(X_i)$

$\to P(X_i) = \frac{1}{4} \quad \forall X_i$

$= -\underbrace{4 \cdot \frac{1}{4}} \log_2 \left(\frac{1}{4}\right) = 2$ Shannon bits

(c) $I(X; Y) = + \sum_{i=1}^{m} \sum_{j=1}^{m} P(X_i \cap Y_j) \log_2 \frac{P(X_i \cap Y_j)}{\underbrace{P(X_i) P(Y_j)}}$

$\left[P(X_i \cap Y_j) = P(X_i) P(Y_j) \right.$

$X \perp Y$

X Y

$= 0$

Exercise 2. [IT 2005-2, note 11]

Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be three binary random variables. One gives the following information:

- $P(\mathcal{X} = 0) = P(\mathcal{Y} = 0) = 0.5$,
- $P(\mathcal{X}, \mathcal{Y}) = P(\mathcal{X})P(\mathcal{Y})$,
- $\mathcal{Z} = (\mathcal{X} + \mathcal{Y}) mod 2$ (i.e., $\mathcal{Z} = 1 \Leftrightarrow \mathcal{X} \neq \mathcal{Y}$)

(a) What is the value $P(\mathcal{Z} = 0)$?

(b) What is the value of $H(\mathcal{X})$, $H(\mathcal{Y})$, $H(\mathcal{Z})$?

(c) What is the value of $H(\mathcal{X}, \mathcal{Y}), H(\mathcal{X}, \mathcal{Z}), H(\mathcal{Y}, \mathcal{Z}), H(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$?

(d) What is the value of $I(\mathcal{X}; \mathcal{Y}), I(\mathcal{X}; \mathcal{Z}), I(\mathcal{Y}; \mathcal{Z})$?

(e) What is the value of $I(\mathcal{X}; \mathcal{Y}, \mathcal{Z}), I(\mathcal{Y}; \mathcal{X}, \mathcal{Z}), I(\mathcal{Z}; \mathcal{X}, \mathcal{Y})$?

(f) What is the value of $I(\mathcal{X}; \mathcal{Y}|\mathcal{Z}), I(\mathcal{Y}; \mathcal{X}|\mathcal{Z}), I(\mathcal{Z}; \mathcal{X}|\mathcal{Y})$?

(g) Can you draw a Venn diagram which summarizes the situation?

Exercise 3. You are given 12 balls, all of which are equal in weight except for one which is either lighter or heavier. You are also given a two-pan balance to use. In each use of the balance you may put any number of the 12 balls on the left pan, and the same number (of the remaining) balls on the right pan. The result may be one of three outcomes : equal weights on both pans; left pan heavier, right pan heavier. Your task is to design a strategy to determine which is the odd ball *and* whether it is lighter or heavier *in as few (expected) uses of the balance as possible*.

Exercise 4. Consider the following contingency table

	Y_1	Y_2
X_1	1/3	1/3
X_2	0	1/3

Compute:

(a) $H(\mathcal{X}), H(\mathcal{Y})$

(b) $H(\mathcal{X}|\mathcal{Y})$, $H(\mathcal{Y}|\mathcal{X})$

(c) $H(\mathcal{X}, \mathcal{Y})$

(d) $H(\mathcal{Y}) - H(\mathcal{Y}|\mathcal{X})$

(e) $I(\mathcal{X}; \mathcal{Y})$

(f) Draw a Venn diagram

Exercise 5. Consider $\mathcal{Y} = g(\mathcal{X})$ a deterministic function of the discrete random variable $\mathcal{X}$. Are the following assertions true or not ? If true, give an example and if false, prove it. In both cases, justify your answers.

(a) There exist a random variable $\mathcal{X}$ and a function $\mathcal{Y} = g(\mathcal{X})$ such that $H(\mathcal{Y}) < H(\mathcal{X})$.

(b) There exist a random variable $\mathcal{X}$ and a function $\mathcal{Y} = g(\mathcal{X})$ such that $H(\mathcal{Y}) = H(\mathcal{X})$.

(c) There exist a random variable $\mathcal{X}$ and a function $\mathcal{Y} = g(\mathcal{X})$ such that $H(\mathcal{Y}) > H(\mathcal{X})$.

(a) X uniform r.v. $H(X) > 0$ TRUE

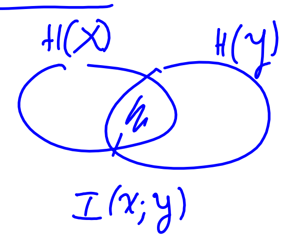
$g(X) = 0 = Y \rightarrow H(Y) = 0$

(b) $g(X) = X = Y$ TRUE

$H(X) = H(Y)$

(c) FALSE

Given $Y = g(X)$, it comes that $\underline{H(Y|X) = 0}$

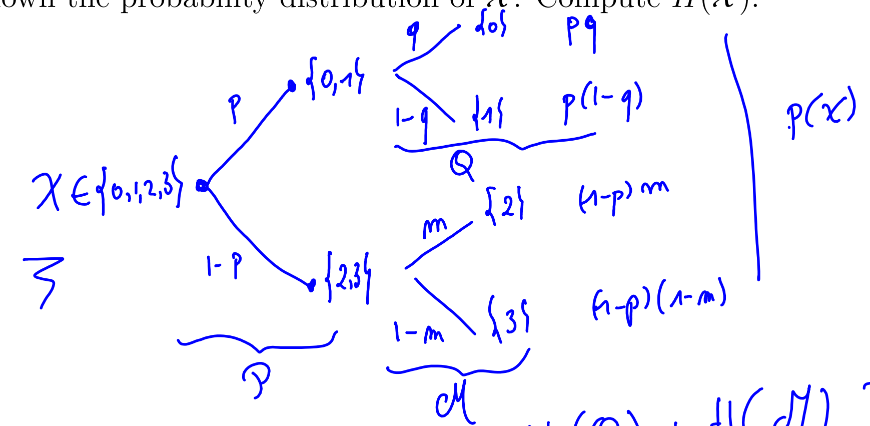


$$H(X,Y) = H(Y) + H(X|Y)$$
$$= H(X) + \underbrace{H(Y|X)}_{=0}$$

$$\Rightarrow H(Y) + \underbrace{H(X|Y)}_{\geq 0} = H(X)$$

$$H(Y) \leq H(X)$$

Exercise 6. A random variable $\mathcal{X} \in \{0, 1, 2, 3\}$ is selected by flipping a bent coin with probability p of coming up heads to determine whether the outcome is in $\{0, 1\}$ or $\{2, 3\}$ corresponding to heads and tails respectively; then either flipping a second bent coin (with probability q of coming up heads) to determine whether the outcome is 0 or 1 or a third bent coin (with probability m of coming up heads) to determine whether the outcome is 2 or 3. Write down the probability distribution of $\mathcal{X}$. Compute $H(\mathcal{X})$.



$$H(\mathcal{X}) = \underbrace{H(\mathcal{P}) + H(\mathcal{Q}) + H(\mathcal{M})}\ ?$$

FALSE

$$\begin{aligned} H(\mathcal{X}) = & -pq \log_2(pq) \\ & - p(1-q)\log_2(p(1-q)) \\ & - (1-p)\, m \log_2((1-p)m) \\ & - (1-p)(1-m)\log_2((1-p)(1-m)) \\ = & \ \ldots \\ = & \ H(\mathcal{P}) + p\, H(\mathcal{Q}) + (1-p)\, H(\mathcal{M}). \end{aligned}$$

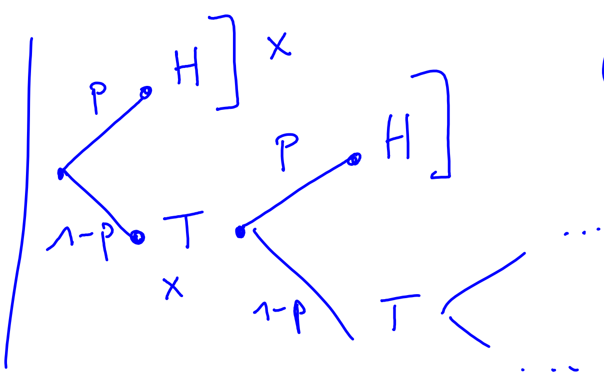
$$\log(ab) = \log(a) + \log(b)$$

Exercise 7. An unbiased coin is flipped until one head is thrown. What is the entropy of the random variable $\mathcal{X} \in \{0, 1, 2, 3, \dots\}$, the number of flips? Repeat the calculation for the case of a biased coin with probability p of coming up heads.

H, TH, TTH, TTTH, ...

Prob. that there are exactly $(x-1)$ tails and then one head

is $p(x) = (1-p)^{x-1} p^1$



$$H(X) = H_2(p) + (1-p)H(X) + p \cdot 0$$

$$\boxed{H(X) = \frac{H(P)}{p}}$$

$$H(X) = H(P) + H(X) - p(H(X))$$

$$\Leftrightarrow pH(X) = H(P)$$

$$\Leftrightarrow H(X) = \frac{H(P)}{p}$$

$$H_2(p) = -p\log_2(p) - (1-p)\log_2(1-p)$$
$$= H_2(1-p)$$
$$= H(P)$$
$$P(P) = [p \quad 1-p]$$

Exercise 8. Are these assertions necessarily true? Justify.

(a) If $\mathcal{Y} = f(\mathcal{X})$ and $\mathcal{Z} = g(\mathcal{X})$, then $H(\mathcal{X}) \leq I(\mathcal{X}; \mathcal{Y}, \mathcal{Z})$.

(b) If $H(\mathcal{X}|\mathcal{Y}) \leq H(\mathcal{Y}|\mathcal{Z})$ then $I(\mathcal{X}; \mathcal{Y}) \geq I(\mathcal{Y}; \mathcal{Z})$.