LIÈGE université
Sciences Appliquées

Module 1.1: Optimal Power Flow

ELEC0448 - Planning and operation of electric power and energy systems

Bertrand Cornélusse

Overview

1. Introduction
2. A network flow model
3. Power flow equations – reminder
4. AC optimal power flow model
5. OPF in rectangular coordinates

Introduction

Module objectives

1.1 Formalize the problem of optimal power flow (OPF) in an electrical network.

1.1 Understand different mathematical formulations of the problem and their interest.

Module objectives

1.1 Formalize the problem of optimal power flow (OPF) in an electrical network.

1.1 Understand different mathematical formulations of the problem and their interest.

1.2 From the OPF problem, derive a way to plan the construction / extension of an electrical disctribution network.

Module objectives

1.1 Formalize the problem of optimal power flow (OPF) in an electrical network.

1.1 Understand different mathematical formulations of the problem and their interest.

1.2 From the OPF problem, derive a way to plan the construction / extension of an electrical disctribution network.

1.3 Read, understand, explain, and criticise scientific articles on the subject (graded).

1.4 Apply the last concept to your case study of ENERG001 (graded).

Optimal power flow: problem statement

Let us consider a power system with

- some generators either from large power plants (control of P and V) or distributed interfaced with *grid-following*[1] inverters (control of P and Q, if any)
- loads that are more of less flexible
- transmission lines, transformers, and other grid devices that can be controlled to some extent.

[1] Any idea what this means?

Optimal power flow: problem statement (...)

We take the role of the network operator[2], who wants to

- ensure voltage levels stay within the limits, e.g. $[0.95, 1.05]$ *pu*
- ensure currents in devices stay within the limits: $i_l \leq I_l^{max}$
- keep some margin for other operational constraints, or in case of contingency
- optimize some objective function: minimize losses, generation costs, renewable energy curtailment.

---

[2] At the transmission and distribution levels the problems are different, but the mathematical formulation is similar. What are the key differences, according to you?

Optimal power flow: problem statement (...)

The operator has a set of control actions[3], often incurring some costs:

- change the setpoints of generators
- change the setpoints of reactive power compensators / shunt capacitors
- change the topology of the network (reconfiguration)
- change some transformer taps
- shed some loads.

[3] who owns the devices listed here?

A network flow model

A network flow model

We first consider a very naive representation of a microgrid and its network.

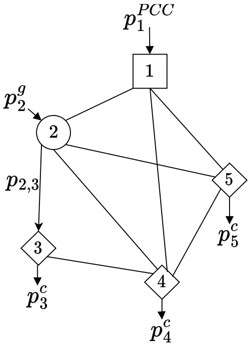
We derive a linear program to optimize the power flows in this network, and solve it.

The main goal is to familiarize you with the concept of graphs, network flows and linear programming, before moving to the more complex AC power flow equations.

A network flow model

We assume the network is a graph containing

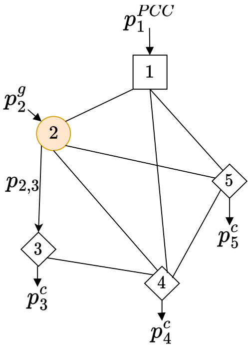
- source nodes that can inject power
  - one source node models the connection to the public grid; we call it the PCC (point of common coupling)
  - other source nodes are generators
- sink nodes that always consume power
- edges that can transmit power from one node to another.



A first basic generator model

A generator is attached to a node u and can output a power $p_u^g \geq 0$ $[MW]$ limited by a maximum power $\bar{P}_u^g$ $[MW]$. The associated production cost $[EUR/h]$ is

$$c_{u,0}^g + c_{u,1}^g p_u^g.$$

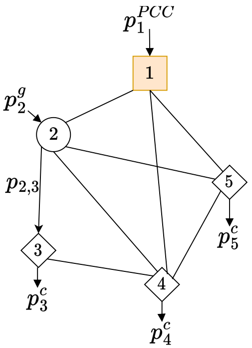


Point of common coupling I

The point of common coupling can be seen as a special generator that can either inject or consume power. Let

$$p^{PCC} = p^{PCC,+} - p^{PCC,-} \quad [MW]$$

be the power injected by the PCC in the microgrid, with $p^{PCC,+} \geq 0$ and $p^{PCC,-} >= 0$. When it consumes power ($p^{PCC} \leq 0$), it means that the power generated in the microgrid exceeds the consumption and is thus pushed into the public grid.



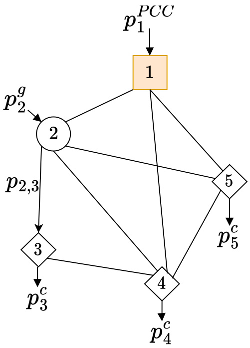
Point of common coupling II

The power exchanged with the public grid is limited, either physically or contractually, to $\bar{P}^{PCC}$:

$$-\bar{P}^{PCC} \leq p^{PCC} \leq \bar{P}^{PCC} \quad [MW].$$

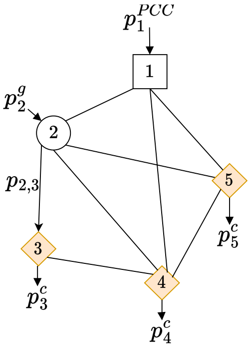
There is a cost $c^{PCC,+}$ $[EUR/MWh]$ associated with energy bought from the public grid and a revenue $c^{PCC,-}$ $[EUR/MWh]$ associated with energy injected into the public grid. We have

$$c^{PCC,+} > c^{PCC,-}.$$



Consumption nodes

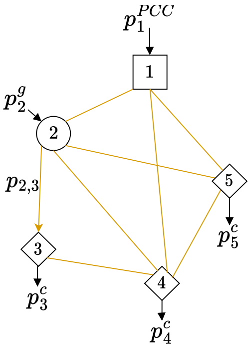
A load is attached to a consumption node u and consumes a power $p_u^c \geq 0$ that cannot be modified.



Edges

An edge (u, v) allows sending power $p_{u,v}$ from node u to node v. However, it has a maximum capacity $\bar{P}_{u,v}$ $[MW]$:

$$-\bar{P}_{u,v} \leq p_{u,v} \leq \bar{P}_{u,v}.$$



Objective

We aim to minimize the total cost of satisfying the demand while satisfying the constraints of the generators, PCC, edges, and the power balance at each node.

Formulate this problem as a linear program and solve it!

- A **template Google Colab is available here**
- It uses
  - Python as a general programming language
  - Pyomo as mathematical programming modeling library for Python
  - Ipopt as a solver, which receives the problem from Pyomo and returns a solution, if any.
- More instruction in the Colab template.

Power flow equations – reminder

Power flow equations – reminder

Nowadays, the vast majority of powergrids are AC, with some DC point-to-point connections.

The goal of this section is to remind you how an AC network can be modeled in sinusoidal steady-state analysis using phasors and impedances. Then we derive the power flow equations of a network as a function of bus voltages and power injections.

Power and energy I

- Power measures the rate of use of energy
- It is expressed in Watt [W]: 1 W = 1 Joule/second
- In an electric system,

$$p(t) = u(t) \times i(t)$$

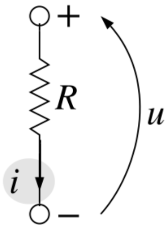
- $u(t)$ is the voltage measured in volt [V], the line integral of the electric field between two points.
- $i(t)$ is the current measured in amps [A]
- t is the time

Power and energy II

- To measure energy in power systems, we use units ranging from a kWh (a microgrid) to a TWh (a country)
- Devices have power ratings ranging from W to GW (although we generally speak in VA for ratings)

Motor convention (or standard reference)

When using the motor convention to direct u w.r.t. i, $p(t)$ represents the power **consumed** by a device (here a resistor):



- The power consumed can be < 0, = 0, or > 0 depending on the device
- E.g. for a resistor we always have $p(t) \geq 0$
- The **opposite** convention is the **generator convention**
- We will sometimes use a mix of both conventions based on intuition, so that, in general, we have few negative numbers: pay attention to the orientations!

Sinusoidal signals and phasor representation

Unless otherwise specified, we will always work with sinusoidal signals and in steady state:

$$y(t) = \sqrt{2}Y\cos(\omega t + \phi_y).$$

Y is the *rms* value of the signal, ϕ_y its phase and ω its angular frequency.

At a specific frequency $f = \frac{\omega}{2\pi}$, the signal can be represented as a phasor

$$\bar{Y} = Y\angle\phi_y = Ye^{j\phi_y}$$

Phasors allow working in the frequency domain, which is much nicer for computations.

How do you get the time expression from the phasor?

See https://en.wikipedia.org/wiki/Phasor

Impedance I

Let $u(t)$ and $i(t)$ be the voltage and current across a one-port, respectively, in sinusoidal steady state and with the motor convention.

- For a resistor, $u(t) = Ri(t)$ hence

$$\bar{U} = R\bar{I}$$

- For an inductor, $u(t) = L\frac{di(t)}{dt}$ hence

$$\bar{U} = j\omega L\bar{I}$$

- For a capacitor, $i(t) = C\frac{du(t)}{dt}$ hence

$$\bar{I} = j\omega C\bar{U}$$

Impedance II

The *impedance*, a complex number, generalizes this notion

$$Z = R + jX \ \ [\Omega]$$

such that $\bar{U} = Z\bar{I}$ with

- for a resistor, $Z = R$
- for a self, $Z = jX = j\omega L$
- for a capacitor, $Z = jX = -j\frac{1}{\omega C}$

Impedance, admittance, etc.

The imaginary part of the impedance, X, is called reactance

The admittance Y is the inverse of the impedance, expressed in Siemens [S]:

$$Y = G + jB$$

- G is the conductance
- B is the susceptance

Complex calculus

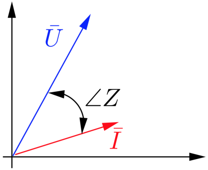
$$|Z| = \sqrt{R^2 + X^2}$$

$$\angle Z = \arctan \frac{X}{R}$$

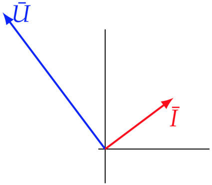
$$Z = \frac{\bar{U}}{\bar{I}} = \frac{U}{I} \angle (\phi_u - \phi_i)$$

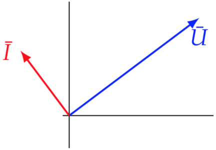
Phasor diagrams

Plot the phasors in the complex plane!



Inductive or capacitive? Which is which?





The notions of power I

The complex power is defined as

$$S = \bar{U}\bar{I}^*$$

Let

$$\phi = \phi_u - \phi_i$$

then

$$S = UIe^{j\phi} = P + jQ$$

- $P = UI\cos\phi$ is the active power, measured in watt
- $Q = UI\sin\phi$ is the reactive power, measured in var
- $\cos\phi$ is the power factor

Reactive power is, in general, undesirable.

The apparent power is $|S| = UI$, measured in VA

Useful formulas

$$P = RI^2 = \frac{U^2}{R}$$

$$Q = XI^2 = \frac{U^2}{X}$$

$$\tan\phi = \frac{Q}{P}$$

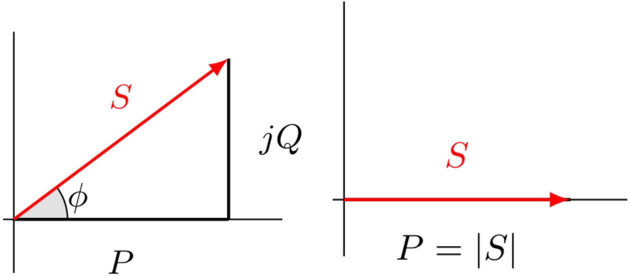
$$\cos\phi = \frac{P}{|S|}$$

The power factor does not tell you whether the system is leading or lagging

- in an inductive system, $u(t)$ precedes $i(t)$, $i(t)$ is lagging, thus $Q > 0$ (motor convention)
- in a capacitive system, this is the opposite (leading).

Power factor compensation

Produce some Q to cancel out ϕ:

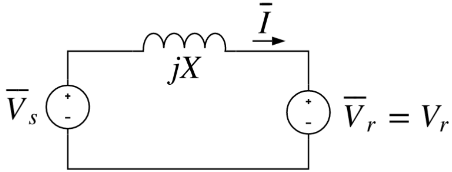


Example:

A 120V voltage source at 60 Hz that feeds an R-L load $1858.4 + j1031.4$ *VA*

Power transfer between AC systems I

Consider the following simple system



We have

$$\bar{I} = \frac{\bar{V}_s - \bar{V}_r}{jX}$$

Let δ be the angle between $\bar{V}_r$ and $\bar{V}_s$, then

$$S_r = \bar{V}_r\bar{I}^* = V_r\left(\frac{V_s\angle - \delta - V_r}{-jX}\right) = \frac{V_sV_r\sin\delta}{X} + j\frac{V_sV_r\cos\delta - V_r^2}{X}$$

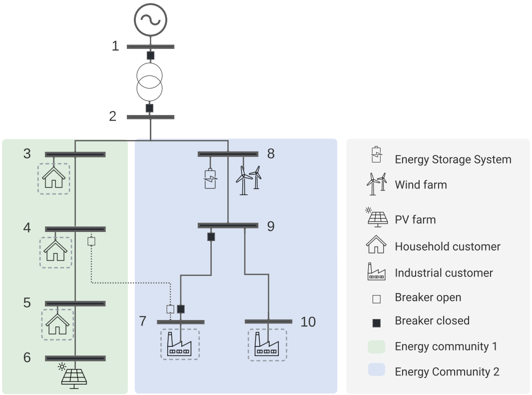
Power transfer between AC systems II

Let's remember two things:

- The **active** power is highly sensitive to δ
- The **reactive** power acts on the **voltage magnitude** (look at what happens for $\delta = 0$)

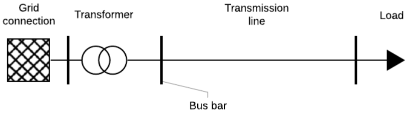
One-line diagram I

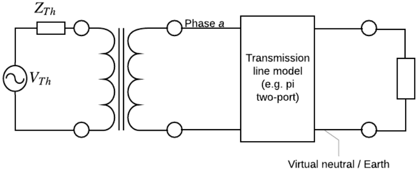
We usually represent a power system with a *one-line diagram*.



One-line diagram II

Looking, e.g., at the top portion of this schematic,
it actually means



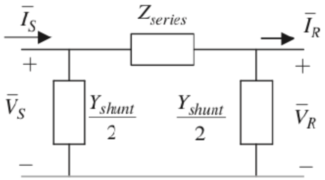


This is a simplification since most AC networks are three-phase. Here we assume the three phases are perfectly balanced to be able to represent it as a single-line equivalent.

Lumped transmission line model in steady state:

The π model

If the length l [km] of a transmission line is relatively small ($< 300km$), we can **approximate** the line with lumped parameters.



with,

- $Z_{series} = Rl + j\omega Ll$
- $\frac{Y_{shunt}}{2} = j\frac{\omega Cl}{2}$

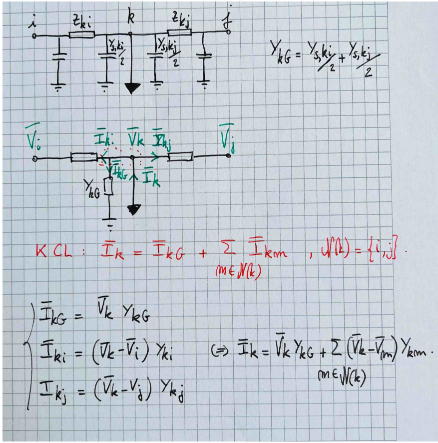
R, L and C are per km values.

This π model is symmetrical by design.

The power flow equations I

- Let $\mathcal{N}$ be the set of buses of the network
- Some buses are interconnected by transmission lines, given by their π models
- Let Y_{kG} be the sum of admittances connected between node k and the ground:
  - the shunt admittances of the lines incident to k,
  - the admittances of the devices connected at node k if any
  - but not the admittance related to the injection or withdrawal at node k, which is usually unknown.
- For two nodes k and m, let Z_{km} be the series impedance of the line connecting them and $Y_{km} = Z_{km}^{-1}$ ($Y_{km} = 0$ if there is no line)

The power flow equations II



The power flow equations III

The current injection at node k is

$$\bar{I}_k = Y_{kG}\bar{V}_k + \sum_{m \in \mathcal{N} \setminus k} (\bar{V}_k - \bar{V}_m) Y_{km}$$

This last equation can be rewritten as

$$\bar{I}_k = \left(Y_{kG} + \sum_{m \in \mathcal{N} \setminus k} Y_{km} \right) \bar{V}_k - \sum_{m \in \mathcal{N} \setminus k} Y_{km} \bar{V}_m$$

The power flow equations IV

The complex power injected at bus k is

$$S_k = \bar{V}_k \bar{I}_k^\star,$$

we develop this relation and separate the real and imaginary parts:

$$P_k = G_{kk} V_k^2 \quad + V_k \sum_{m \in \mathcal{N} \setminus k} V_m (G_{km} \cos \theta_{km} + B_{km} \sin \theta_{km})$$

$$Q_k = -B_{kk} V_k^2 \quad + V_k \sum_{m \in \mathcal{N} \setminus k} V_m (G_{km} \sin \theta_{km} - B_{km} \cos \theta_{km})$$

with

The power flow equations V

- $Y_{km} = G_{km} + jB_{km}$
- $Y_{kk} = G_{kk} + jB_{kk}$ the sum of all the admittances connected to bus k
- $\theta_{km} = \theta_k - \theta_m$ the phase difference between voltages at nodes k and m

The equations are non-linear.

AC optimal power flow model

AC optimal power flow model I

Starting from the network flow model, consider now that

- the network lines are specified by their π model (we will neglect the shunt susceptances).
- power flows in the network according to the equations we have established in Section "Power flow equations" (You may need to reformulate the equations to avoid having trigonometric functions. How can you do this?)
- the absolute value of the current in each line is limited to avoid damaging the lines (line rating).

AC optimal power flow model II

- voltages must stay within a range around a nominal voltage value to guarantee that grid devices can function properly (the voltage at the PCC sets the reference)
- loads absorb active and reactive power you cannot control
- generators can, within some limits, generate active power and generate or consume reactive power.

You are asked to

- Extend the network flow model to include all these aspects.
- Minimize the total generation costs.

AC optimal power flow model III

- Compare the solution to the solution of the problem modeled as a network flow problem.
- Compute the losses in the system.
- Is the solution that minimizes the costs also minimizing the losses?

In practice:

- Use **the template Google Colab available here.**
- It uses the same Python libraries and solvers as the first problem.

OPF in rectangular coordinates

Bus injection model in rectangular coordinates I

The goal is to minimize the total generation cost:

$$\min \sum_{k \in \mathcal{G}} \left(c^g_{k,2} P^2_{G_k} + c^g_{k,1} P_{G_k} + c^g_{k,0} \right) \tag{1}$$

Power Flow Constraints. The active and reactive power injections at each bus k are:

$$P_{G_k} - P_{D_k} = \sum_{m \in \mathcal{N} \setminus k} \underbrace{\left[G_{km}(v_k^2 + w_k^2) - G_{km}(v_k v_m + w_k w_m) + B_{km}(v_k w_m - w_k v_m) \right]}_{\text{Active power flow from } k \text{ to } m} \tag{2}$$

$$Q_{G_k} - Q_{D_k} = \sum_{m \in \mathcal{N} \setminus k} \underbrace{\left[-B_{km}(v_k^2 + w_k^2) + B_{km}(v_k v_m + w_k w_m) + G_{km}(v_k w_m - w_k v_m) \right]}_{\text{Reactive power flow from } k \text{ to } m} \tag{3}$$

Bus injection model in rectangular coordinates II

where

- G_{km} and B_{km} are the real and imaginary parts of the admittance matrix coefficient Y_{km}
- we neglect the shunt elements
- P_{D_k} and Q_{D_k} are the active and reactive power demands at bus k, respectively,
- v_k and w_k are the real and imaginary parts of voltage $\bar{V}_k$
- we assume there is at most a generator per bus.

Bus injection model in rectangular coordinates III

This is the so called **Bus Injection Model**, because we have only voltage variables and power injections at each bus.

Generator Constraints. Each generator k must operate within its limits:

$$P_{G_k}^{\min} \leq P_{G_k} \leq P_{G_k}^{\max} \tag{4}$$

$$Q_{G_k}^{\min} \leq Q_{G_k} \leq Q_{G_k}^{\max} \tag{5}$$

Voltage Magnitude Constraints. Voltage magnitudes must remain within limits:

$$(V_k^{\min})^2 \leq v_k^2 + w_k^2 \leq (V_k^{\max})^2 \tag{6}$$

Bus injection model in rectangular coordinates IV

Slack Bus Constraint. The slack bus is assigned a fixed reference voltage (e.g. $1pu$):

$$v_{\text{slack}} = 1, \quad w_{\text{slack}} = 0 \tag{7}$$

Branch Flow Model in Rectangular Coordinates I

Alternatively, we can state the problem by defining explicitely the branch flows. This is the **Branch Flow Model** (here in rectangular coordinates). For a line connecting bus *k* to bus *m*, the active and reactive power flows from bus *k* to bus *m* are:

$$P_{km} = G_{km}(v_k^2 + w_k^2) - G_{km}(v_k v_m + w_k w_m) + B_{km}(v_k w_m - w_k v_m) \tag{8}$$

$$Q_{km} = -B_{km}(v_k^2 + w_k^2) + B_{km}(v_k v_m + w_k w_m) + G_{km}(v_k w_m - w_k v_m) \tag{9}$$

(Similarly, the power flows from bus *m* to bus *k* are:

$$P_{mk} = G_{km}(v_m^2 + w_m^2) - G_{km}(v_m v_k + w_m w_k) + B_{km}(v_m w_k - w_m v_k) \tag{10}$$

$$Q_{mk} = -B_{km}(v_m^2 + w_m^2) + B_{km}(v_m v_k + w_m w_k) + G_{km}(v_m w_k - w_m v_k)) \tag{11}$$

Branch Flow Model in Rectangular Coordinates II

At each bus k, the power balance equations then become

$$P_{G_k} - P_{D_k} = \sum_{m \in \mathcal{N}_k} P_{km} \tag{12}$$

$$Q_{G_k} - Q_{D_k} = \sum_{m \in \mathcal{N}_k} Q_{km} \tag{13}$$

where

- P_{G_k} and Q_{G_k} are the active and reactive power generated at bus k, respectively,
- P_{D_k} and Q_{D_k} are the active and reactive power demands at bus k, respectively,
- $\mathcal{N}_k$ denotes the set of buses connected to bus k.

Branch Flow Model in Rectangular Coordinates III

We can then express line flow constraints. Each transmission line (k, m) must satisfy its thermal limit:

$$P_{km}^2 + Q_{km}^2 \leq (S_{km}^{\max})^2 \tag{14}$$