LIÈGE université
Sciences Appliquées

Module 1.2:
Distribution network expansion planning

ELEC0448 - Planning and operation of electric power and energy systems

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Overview

1. Introduction
2. Distribution network expansion planning
3. Turning the DNEP problem into a MISOCP

Introduction

Module objectives

1.1 Formalize the problem of optimal power flow (OPF) in an electrical network.

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1.1 Understand different mathematical formulations of the problem and their interest.

1.2 **From the OPF problem, derive a way to plan the construction / extension of an electrical disctribution network.**

1.3 Read, understand, explain, and criticise scientific articles on the subject (graded).

1.4 Apply the last concept to your case study of ENERG001 (graded).

Distribution network expansion planning

Distribution network expansion planning: definition

Constraints: We want to serve the demand while satisfying the physical and technical constraints of the expanded network.

Decisions: We do an expansion plan to determine

- The conductor types and line construction routes
- The substations installation and reinforcement

Objective: Distribution network expansion planning (DNEP) aims at minimizing the capital and operational cost of the expansion plan

We will now see how to solve the DNEP problem

The formulation of the problem will follow the scientific article

Jabr, Rabih A. "Polyhedral formulations and loop elimination constraints for distribution network expansion planning." IEEE Transactions on Power Systems 28.2 (2012): 1888-1897.

We will understand the formulation, but we will only detail some of the numerical tricks proposed to facilitate the resolution of the problem.

DNEP constraints

Satisfy

- the Kirchhoff's voltage and current laws
- the technical constraints:
  - load bus voltage limits
  - line current-carrying capacity
  - the network acyclic structure.

Nodes and loads

- Consider a distribution network having n nodes,
- each node has either:
  - a real P_{D_i} and a reactive load Q_{D_i}
  - or a substation (P_{S_i}, Q_{S_i}) connected to it, meaning that it is connected to a higher voltage, non-represented network that can inject or absorb power. In our case as we assume the network contains only loads, then globally it consumes active power.

Routes

Every distribution route between nodes *i* and *j* can have one line installed with a conductor of type *k*.

- It has a length l_{ij}.
- Variable $\alpha_{ij}^{(k)} = 1$ if conductor of type *k* is placed, 0 otherwise

Substations

- There are a total of n_s substations that can be reinforced or installed
- they are numbered to correspond to the first nodes in the network
- variable $\beta_i = 1$ if a substation is built or reinforced, $\beta_i = 0$ otherwise

DNEP Jabr I

Objective function to minimize:

$$K_\ell \sum_{ij\in\Omega_s} \sum_{k=1}^{K} \alpha_{ij}^{(k)} C_{ij}^{(k)} \ell_{ij} + K_S \sum_{i=1}^{n_S} \beta_i C_i^f \tag{1}$$
$$+ 8760\left(1+\tau_\ell\right)\phi_\ell c_\ell \sum_{i=1}^{n} \left(P_{Si} - P_{Di}\right)$$
$$+ 8760\left(1+\tau_S\right)\phi_S \sum_{i=1}^{n_S} C_i^v \left(P_{Si}^2 + Q_{Si}^2\right)$$

Cost of lines, cost of substations, cost of losses, and production costs

DNEP Jabr II

Is there a line between *i* and *j* ?

$$x_{ij} = \sum_{k=1}^{K} \alpha_{ij}^{(k)} \leq 1, \quad ij \in \Omega_s \tag{2}$$

Substation capacity limit

$$\sqrt{P_{Si}^2 + Q_{Si}^2} \leq S_i^0 + \beta_i S_i^{\max}, \beta_i \in \{0, 1\}, i = 1, \ldots, n_S \tag{3}$$

DNEP Jabr III

Power balances

$$P_{Si} - P_{Di} = \sum_{j \in N(i)} P_{ij}$$

$$= \sum_{j \in N(i)} \sum_{k=1}^{K} \alpha_{ij}^{(k)} p_{ij}^{(k)}, \quad i = 1, \ldots, n \tag{4}$$

$$Q_{Si} - Q_{Di} = \sum_{j \in N(i)} Q_{ij}$$

$$= \sum_{j \in N(i)} \sum_{k=1}^{K} \alpha_{ij}^{(k)} q_{ij}^{(k)}, \quad i = 1, \ldots, n \tag{5}$$

DNEP Jabr IV

These equations are non-linear.

Ohm's law (power version)

$$p_{ij}^{(k)} + \mathrm{i} q_{ij}^{(k)} = \left(g_{ij}^{(k)} - \mathrm{i} b_{ij}^{(k)} \right) \left(V_i^2 - \bar{V}_i \bar{V}_j^* \right) \tag{6}$$

$$ij \in \Omega_s \cup \Omega_r \quad k = 1, \ldots, K$$

Variables:

- $\bar{V}_i$ the voltage phasor at node i
- $p_{ij}^{(k)}$ the active power entering line from i to j

DNEP Jabr V

- $q_{ij}^{(k)}$ the reactive power entering line from i to j

$g_{ij}^{(k)} + \mathrm{i} b_{ij}^{(k)}$ is the line admittance for conductor k.

$$\left(I_{ij}^{(k)}\right)^2 = \left(\left(g_{ij}^{(k)}\right)^2 + \left(b_{ij}^{(k)}\right)^2\right)\left(V_i^2 + V_j^2 - 2\Re\left\{\bar{V}_i \bar{V}_j^*\right\}\right), \quad ij \in \Omega_s \cup \Omega_r, k = 1, \ldots, K \tag{7}$$

$$\left(I_{ij}^{(k)}\right)^2 \leq \alpha_{ij}^{(k)} \left(I_{ij\,\max}^{(k)}\right)^2 \quad ij \in \Omega_s \quad k = 1, \ldots, K \tag{8}$$

DNEP Jabr VI

$$V_i^{\min} \leq V_i \leq V_i^{\max}, i = 1, \dots, n \tag{9}$$

Radiality constraints

The modeling of radiality constraints is of particular importance for DNEP problems. Another related constraint is that two sub-stations cannot feed a load node.

For an n-node network having n_s installed substations and load nodes with non-zero demand, the following constraints guarantee that the network is acyclic:

- Kirchhoff's current law is enforced at all nodes
- the sum of all binary variables representing line connection equals $n - n_s$.

$$\sum_{ij \in \Omega_s} x_{ij} = n - n_{S0} - \sum_{i=n_{S0}+1}^{n_S} \beta_i. \tag{10}$$

(In case a node practically has zero loads connected to it, then it is assumed to carry a fictitious load of small value, for instance 0.001 pu)

Comments

- As we have formulated the problem, it is a mixed-integer non-linear program.
- It can be solved but without an optimality guarantee.
- In the next section, we will see a possible way to reformulate the problem to solve it more efficiently and with an optimality guarantee.

Turning the DNEP problem into a MISOCP

- the conic relaxation of the power flow equations in an acyclic network
- the disjunctive programming technique (for choosing conductors)
- the linear representation of conic constraints by tight polyhedral approximations (then MISOCP turns into MILP)

Conic reformulation I

Let X denote an $n \times n$ Hermitian matrix ($X = X^*$) with X_{ij} being an element in the i^{th} row and j^{th} column. If X is given by

$$X = \begin{bmatrix} V_1^2 & \cdots & \bar{V}_1\bar{V}_i^* & \dots & \bar{V}_1\bar{V}_n^* \\ \vdots & \ddots & \vdots & & \vdots \\ \bar{V}_i\bar{V}_1^* & \cdots & V_i^2 & \dots & \bar{V}_i\bar{V}_n^* \\ \vdots & & \vdots & \ddots & \vdots \\ \bar{V}_n\bar{V}_1^* & \cdots & \bar{V}_n\bar{V}_i^* & \cdots & V_n^2 \end{bmatrix} \tag{11}$$

Then we can rewrite constraints (6), (7) and (8) using elements of X.

Conic reformulation II

$$p_{ij}^{(k)} = g_{ij}^{(k)} \left(X_{ii} - \Re\left\{X_{ij}\right\}\right) - b_{ij}^{(k)} \Im\left\{X_{ij}\right\} \tag{12}$$

$$q_{ij}^{(k)} = - b_{ij}^{(k)} \left(X_{ii} - \Re\left\{X_{ij}\right\}\right) - g_{ij}^{(k)} \Im\left\{X_{ij}\right\} \tag{13}$$

$$\left(l_{ij}^{(k)}\right)^2 = \left(\left(g_{ij}^{(k)}\right)^2 + \left(b_{ij}^{(k)}\right)^2\right) \left(X_{ii} + X_{jj} - 2\Re\left\{X_{ij}\right\}\right) \tag{14}$$

$$\left(V_i^{\min}\right)^2 \leq X_{ii} \leq (V_i^{\max})^2 \tag{15}$$

- We now have only linear expressions!

Conic reformulation III

- For acyclic networks, it has been shown that the rank-1 constraint over X will always be satisfied provided load over-satisfaction is allowed.
- If we have also $X \succeq 0$ we can replace (6) – (9) by (12)–(15).
- This semi-definiteness constraint in the modified problem can be further replaced with a set of rotated conic quadratic constraints

$$X_{ii}X_{jj} \geq \Re\left\{X_{ij}\right\}^2 + \Im\left\{X_{ij}\right\}^2, \quad ij \in \Omega_s \tag{16}$$

- Remark: load over-satisfaction is only a sufficient condition. In practice if works even if it is not present.

Choosing conductors I

Now, the binary variables are removed from the product terms in (4) and (5).

There are **two possible formulations for conductor selection**. the performance of contemporary mixed-integer programming solvers may be affected by even minor changes in the problem formulation Hence, here, the author proposed two equivalent representations for modeling the choice between conductor types.

1. **the disjunctive model chooses one of several circuits, each representing one conductor type**
2. each of the line conductor types is modeled as the parallel equivalent of circuits that are switched sequentially.

Choosing conductors II

$$\begin{aligned} P_{Si} - P_{Di} &= \sum_{j \in N(i)} P_{ij} \\ &= \sum_{j \in N(i)} \sum_{k=1}^{K} p_{ij}^{(k)}, \quad i = 1, \ldots, n \end{aligned} \tag{17}$$

$$\begin{aligned} Q_{Si} - Q_{Di} &= \sum_{j \in N(i)} Q_{ij} \\ &= \sum_{j \in N(i)} \sum_{k=1}^{K} q_{ij}^{(k)}, \quad i = 1, \ldots, n \end{aligned} \tag{18}$$

Choosing conductors III

$$p_{ij}^{(k)} = g_{ij}^{(k)} \left(X_{i(ij)}^{(k)} - \Re\left\{ X_{ij}^{(k)} \right\} \right) - b_{ij}^{(k)} \Im\left\{ X_{ij}^{(k)} \right\} \tag{19}$$

$$q_{ij}^{(k)} = -\, b_{ij}^{(k)} \left(X_{i(ij)}^{(k)} - \Re\left\{ X_{ij}^{(k)} \right\} \right) - g_{ij}^{(k)} \Im\left\{ X_{ij}^{(k)} \right\} \tag{20}$$

$$\left(I_{ij}^{(k)} \right)^2 = \left(\left(g_{ij}^{(k)} \right)^2 + \left(b_{ij}^{(k)} \right)^2 \right) \left(X_{i(ij)}^{(k)} + X_{j(ji)}^{(k)} - 2\Re\left\{ X_{ij}^{(k)} \right\} \right) \tag{21}$$

What are all these $X_{\ldots}$ variables?

Choosing conductors IV

Bounds on X components as a function of lines constructed

$$V_i^{\min} V_j^{\max} \alpha_{ij}^{(k)} \leq X_{i(ij)}^{(k)} \leq V_i^{\max} V_j^{\max} \alpha_{ij}^{(k)}, \quad ij \in \Omega_s \cup \Omega_r \tag{22}$$

$$0 \leq \Re\left\{X_{ij}^{(k)}\right\} \leq V_i^{\max} V_j^{\max} \alpha_{ij}^{(k)}, \quad ij \in \Omega_s \cup \Omega_r \tag{23}$$

$$-V_i^{\max} V_j^{\max} \alpha_{ij}^{(k)} \leq \Im\left\{X_{ij}^{(k)}\right\} \leq V_i^{\max} V_j^{\max} \alpha_{ij}^{(k)}, \quad ij \in \Omega_s \cup \Omega_r \tag{24}$$

If circuit Wk is installed, $X_{i(ij)}^{(k)}$ is set equal to the squared voltage magnitude at node i.

$$\left(V_i^{\min}\right)^2 \left(1 - \alpha_{ij}^{(k)}\right) \leq |V_i|^2 - X_{i(ij)}^{(k)} \leq (V_i^{\max})^2 \left(1 - \alpha_{ij}^{(k)}\right), \quad ij \in \Omega_s \cup \Omega_r, \quad k = 1, \ldots, K. \tag{25}$$

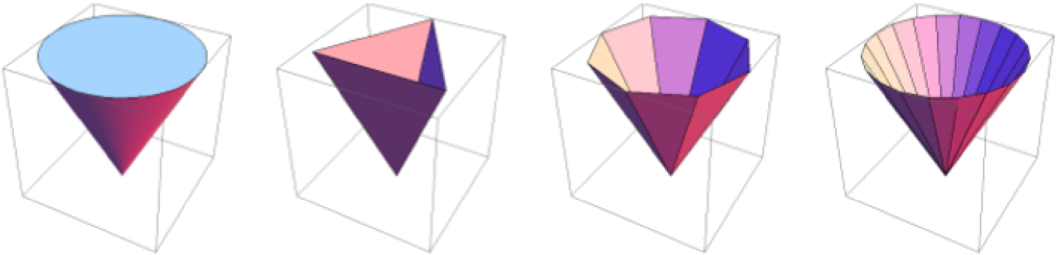
Choosing conductors V

To complete the formulation, the rotated conic quadratic constraint (16) is rewritten in terms of the newly indexed variables of X:

$$X_{i(ij)}^{(k)} X_{j(ji)}^{(k)} \geq \Re\left\{X_{ij}^{(k)}\right\}^2 + \Im\left\{X_{ij}^{(k)}\right\}^2, \quad ij \in \Omega_s, \quad k = 1, \ldots, K. \tag{26}$$

This is thus a MISOCP formulation.

In the paper, cones are approximated by polyhedra



(a) $F(0,0,1)$ (b) $F_3(0,0,1)$ (c) $F_8(0,0,1)$ (d) $F_{20}(0,0,1)$

- This turns the mixed-integer conic program into a mixed-integer linear program
- Modern solvers can deal with cones natively

Comments

- The number of possible network topologies grows exponentially with the number of buses.
- It is known that introducing auxiliary variables can strengthen the linear relaxations of MILP formulations.
- In fact, the solution time using MILP depends on the strength of the formulation's relaxation, i.e., tighter formulations produce better (larger) lower bounds making it more likely that the MILP solver closes the optimality gap within a shorter time.