Fourier representation of signals

MATLAB tutorial series (Part 1.1)

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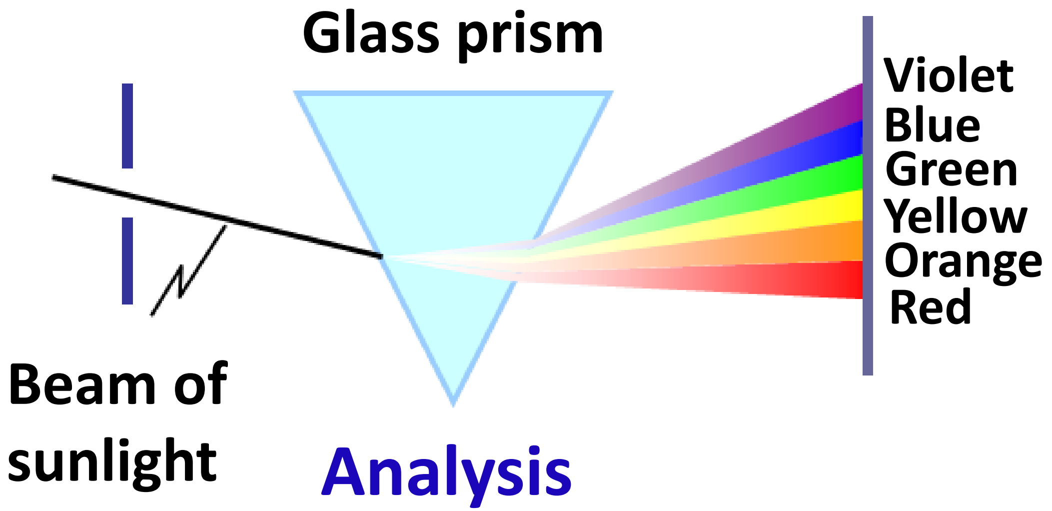
Fourier analysis is like a glass prism



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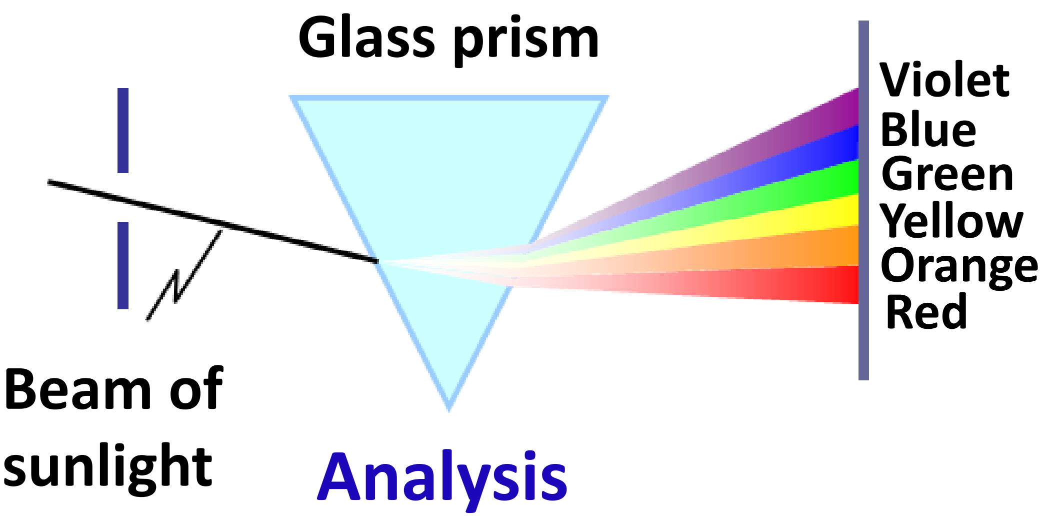

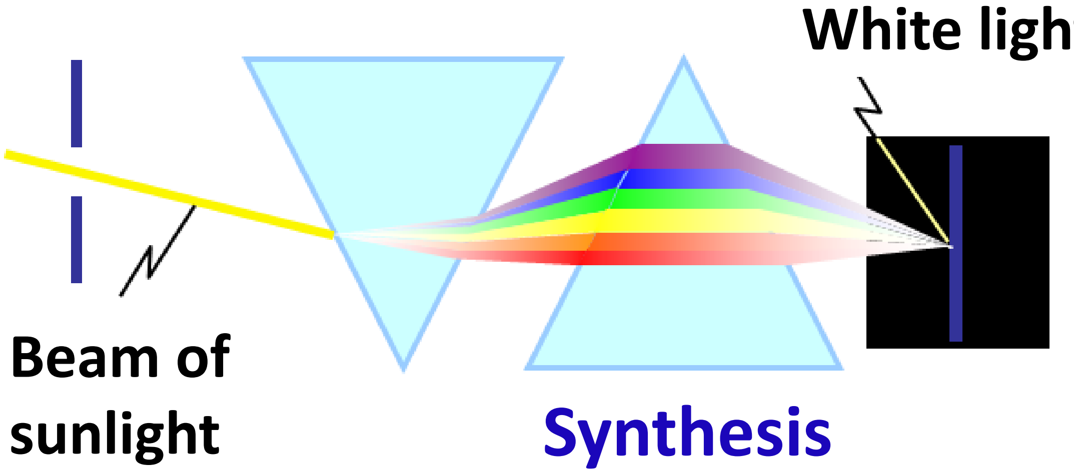

Fourier analysis in signal processing

- Fourier analysis is the decomposition of a signal into frequency components, that is, complex exponentials or sinusoidal signals.

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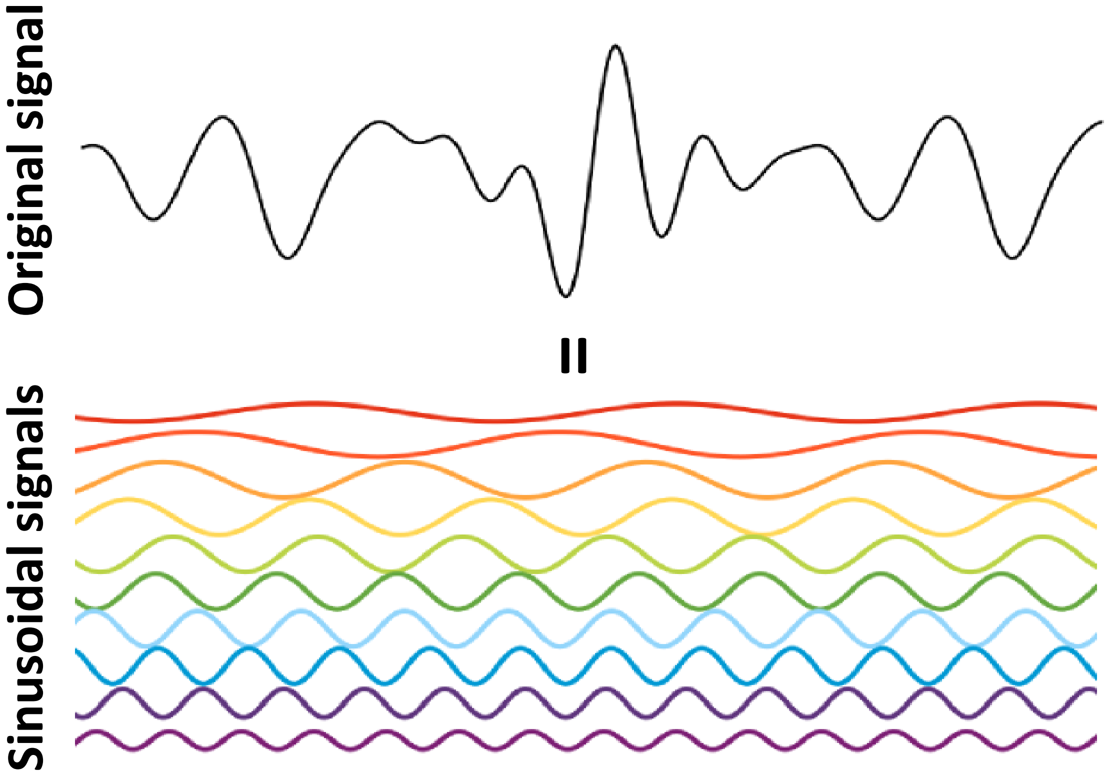

Joseph Fourier
1768-1830

Motivation

Question: what is our motivation to describe each signal as a sum or integral of sinusoidal signals?

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Answer: the major justification is that LTI systems have a simple behavior with sinusoidal inputs.

Notice: the response of a LTI system to a sinusoidal is sinusoid with the same frequency but different amplitude and phase.

Motivation

Question: what is our motivation to describe each signal as a sum or integral of sinusoidal signals?

Answer: the major justification is that LTI systems have a simple behavior with sinusoidal inputs.

Interesting application: we can remove selectively a desired frequency Ω_i from the original signal using an LTI system (i.e. "Filter") by setting $H\left(e^{j\Omega_i}\right) = 0$.

Notations and abbreviations

Mathematical tools for frequency analysis depends on,

- **Nature of time: continuous or discrete**
- **Existence of harmonic: periodic or aperiodic**

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The signal could be,

Continuous-time and **periodic**
Continuous-time and **aperiodic**

Discrete-time and **periodic**
Discrete-time and **aperiodic**

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The signal could be,

Continuous-time and periodic (freq. dom. CTFS)
Continuous-time and aperiodic (freq. dom. CTFT)

Discrete-time and periodic (freq. dom. DTFS)
Discrete-time and aperiodic (freq. dom. DTFT)

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Notice: when the signal is periodic, we talk about Fourier series (FS).

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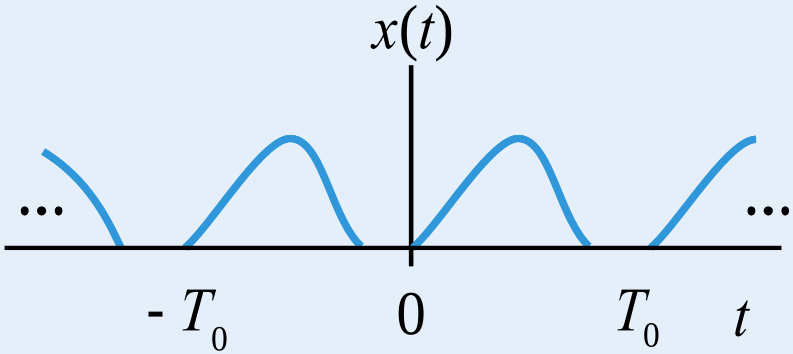
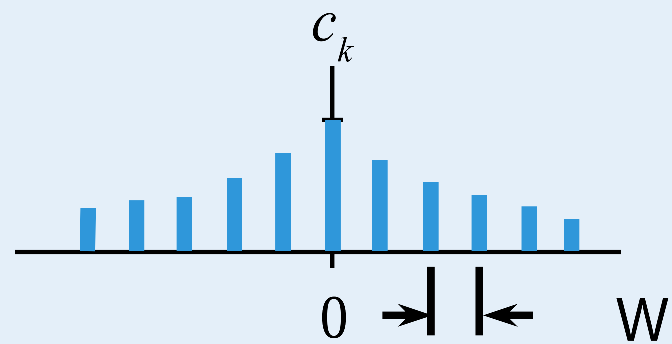
Notice: when the signal is aperiodic, we talk about Fourier transform (FT).

Continuous-time periodic signal: CTFS

Continuous - time signals

Time-domain	**Frequency-domain**





$$c_k = \frac{1}{T_0}\int_{T_0} x(t)\mathrm{e}^{-\mathrm{j}k\Omega_0 t}\mathrm{d}t$$

CTFS →

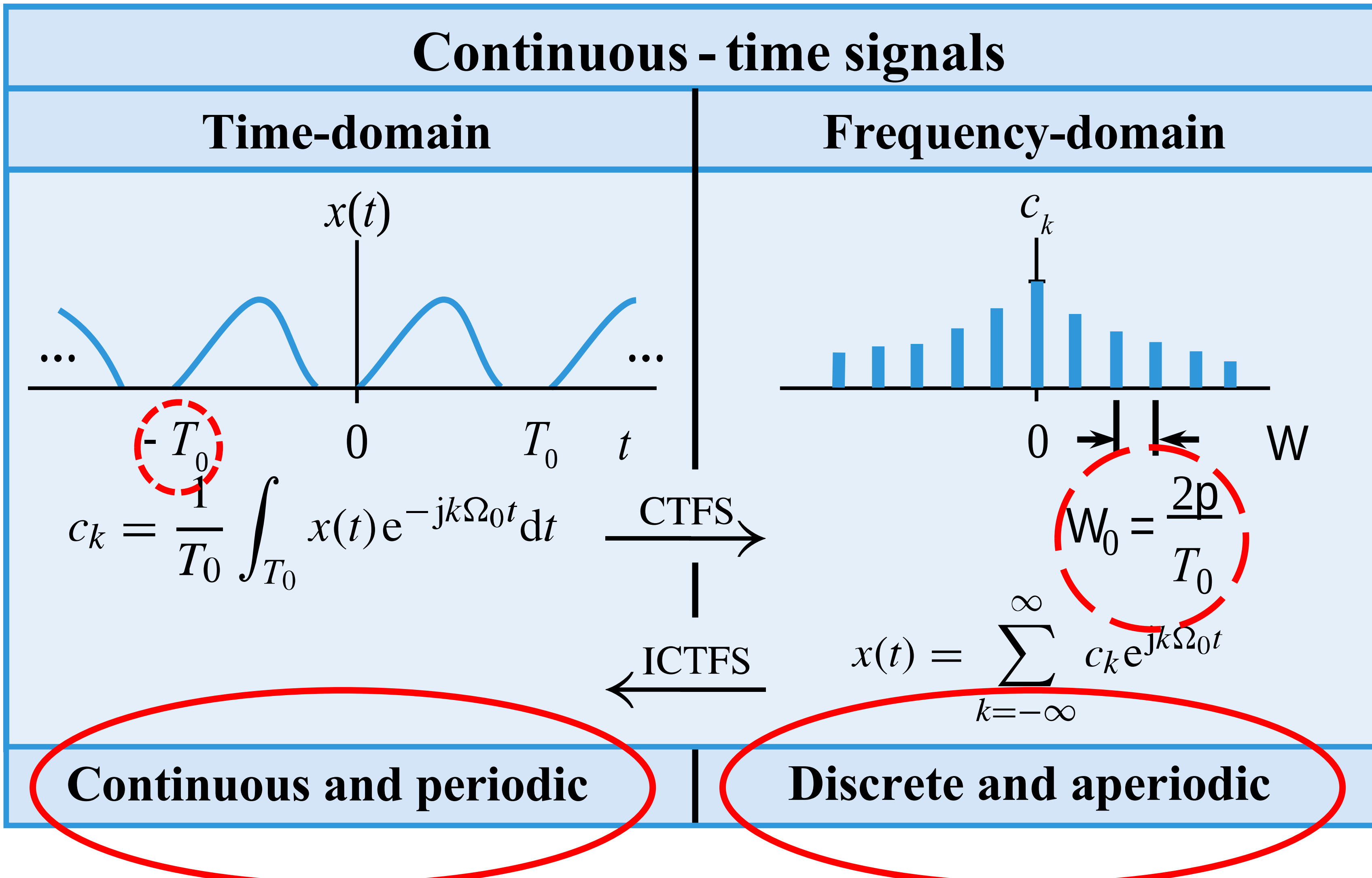
$$\Omega_0 = \frac{2\pi}{T_0}$$

← ICTFS

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\mathrm{j}k\Omega_0 t}$$

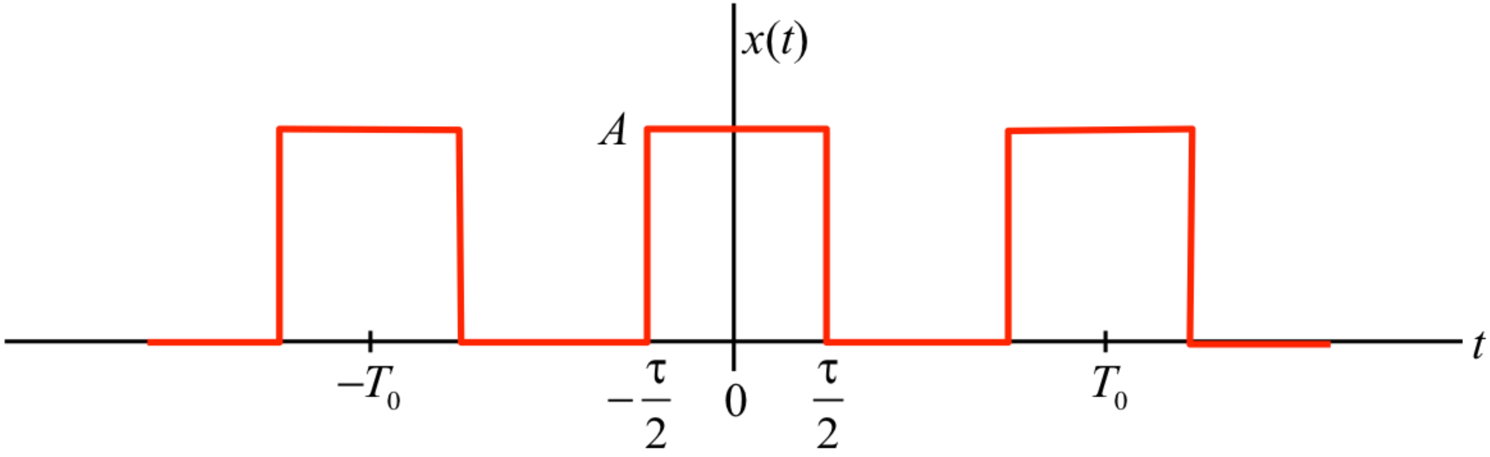
Continuous and periodic	**Discrete and aperiodic**

Continuous-time periodic signal: CTFS



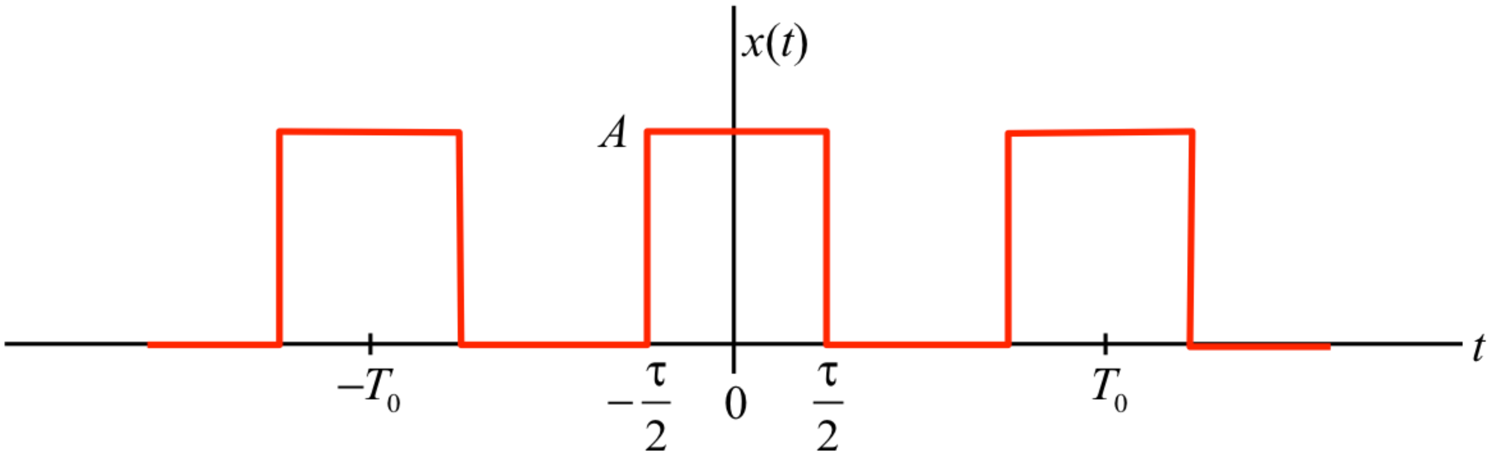
From CTFS to CTFT

Example: consider the following signal,



From CTFS to CTFT

Example: consider the following signal,

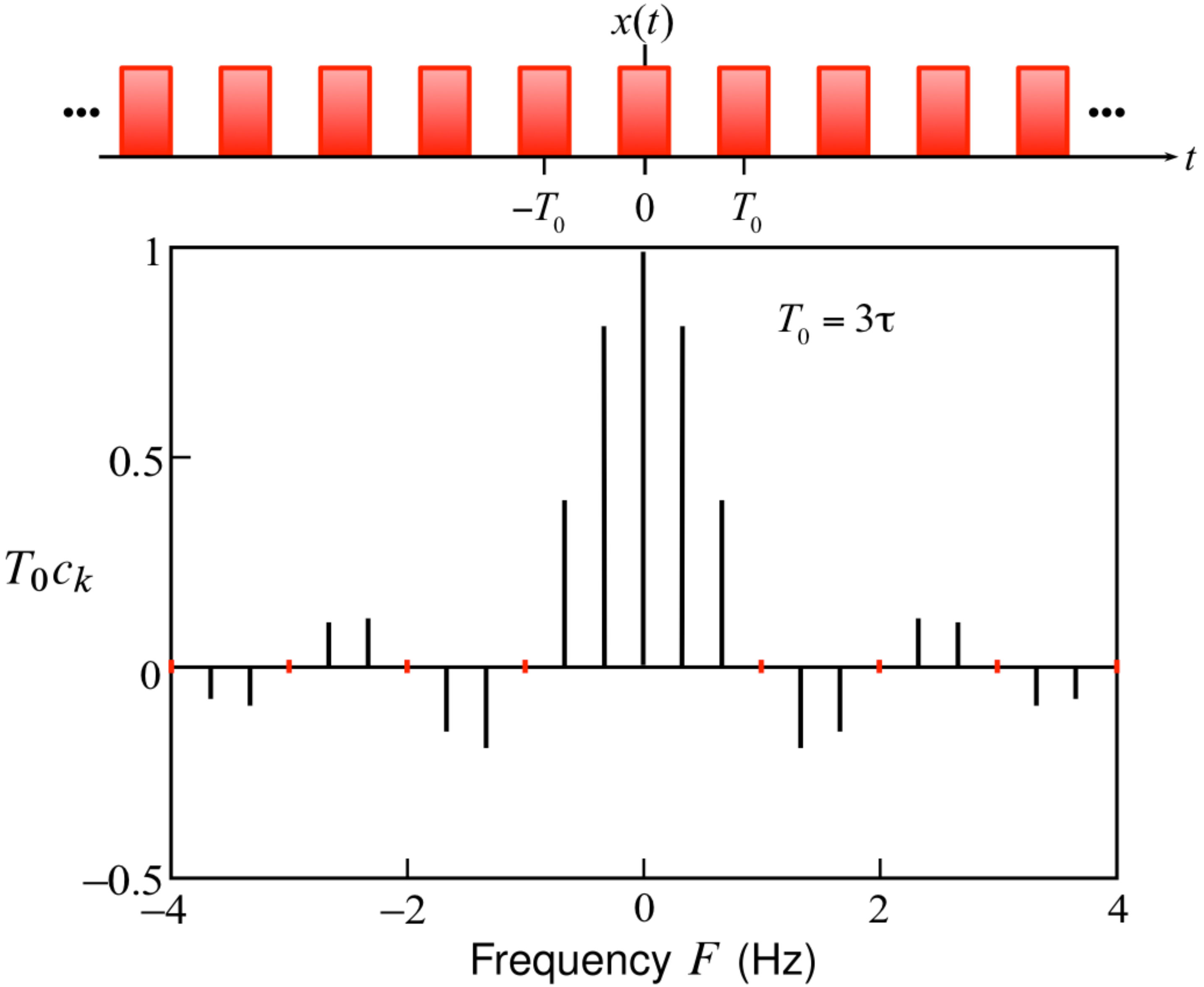


$$c_k = \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} A e^{-j2\pi k F_0 t}\, dt = \frac{A}{T_0}\left[\frac{e^{-j2\pi k F_0 t}}{-j2\pi k F_0}\right]_{\tau/2}^{\tau/2}$$

$$= \frac{A}{\pi k F_0 T_0}\, \frac{e^{j\pi k F_0 t} - e^{-j\pi k F_0 t}}{2j}$$

$$= \frac{A\tau}{T_0}\, \frac{\sin(\pi k F_0 \tau)}{\pi k F_0 \tau}, \quad k = 0, \pm 1, \pm 2, \ldots$$

From CTFS to CTFT



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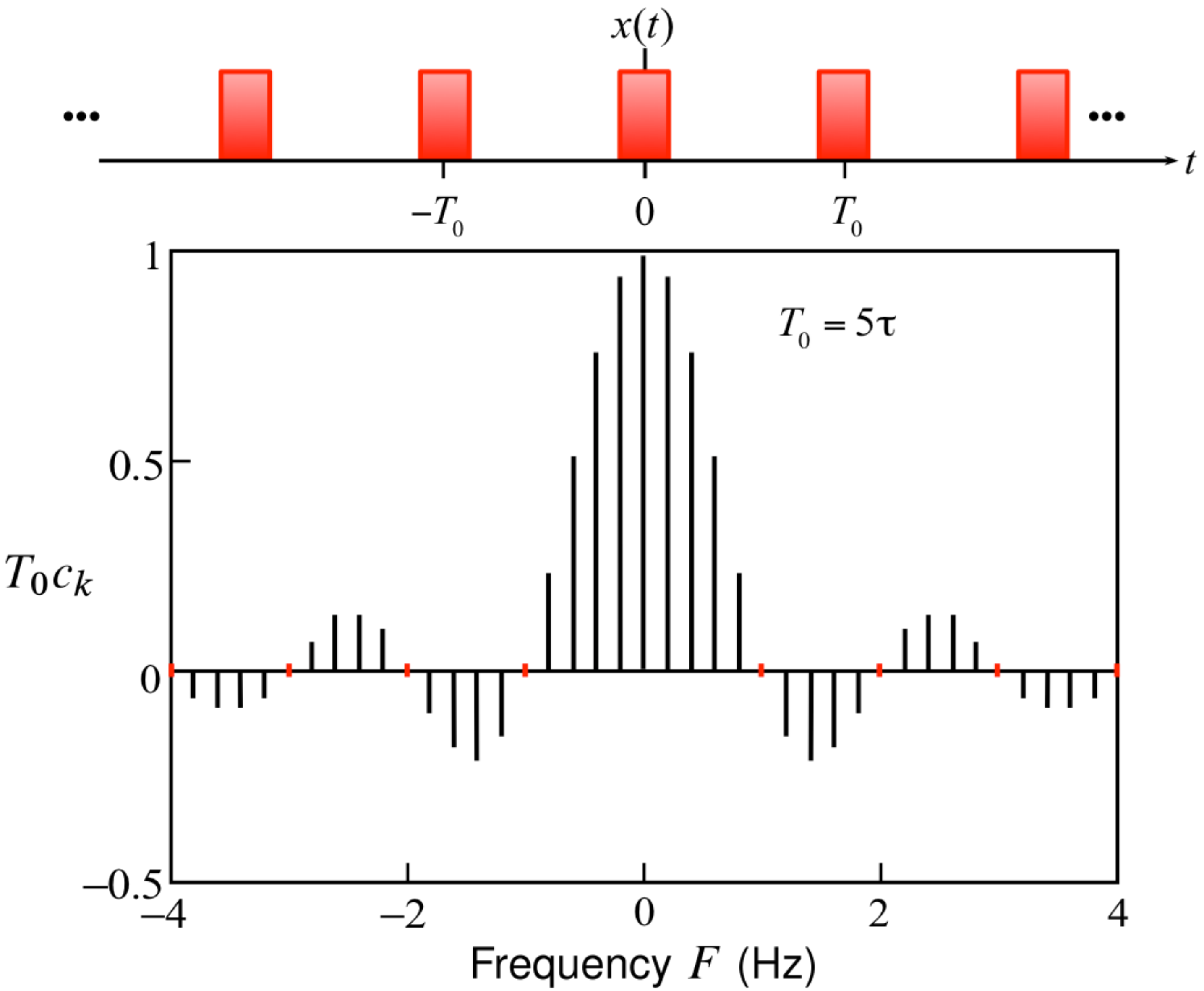

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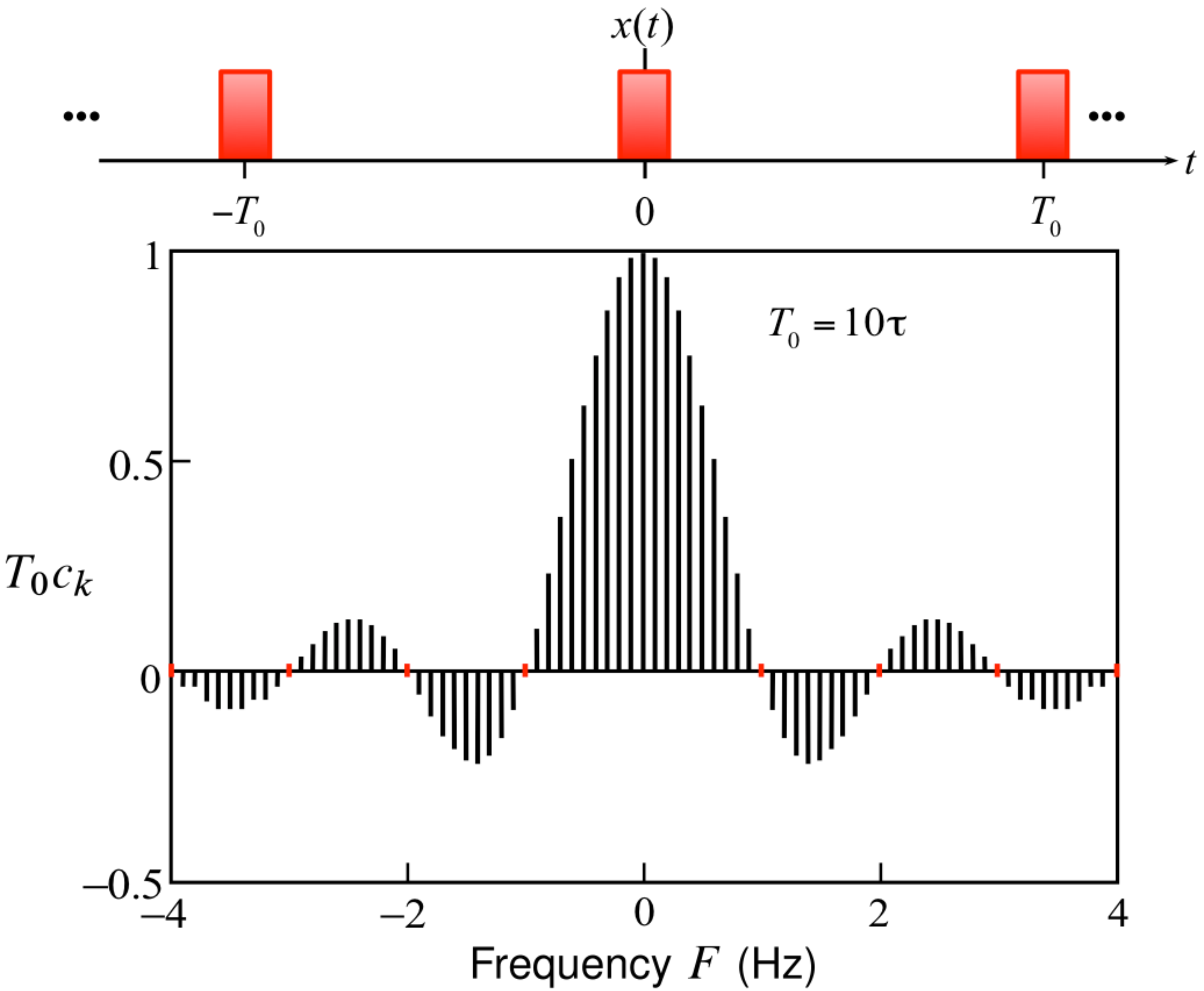

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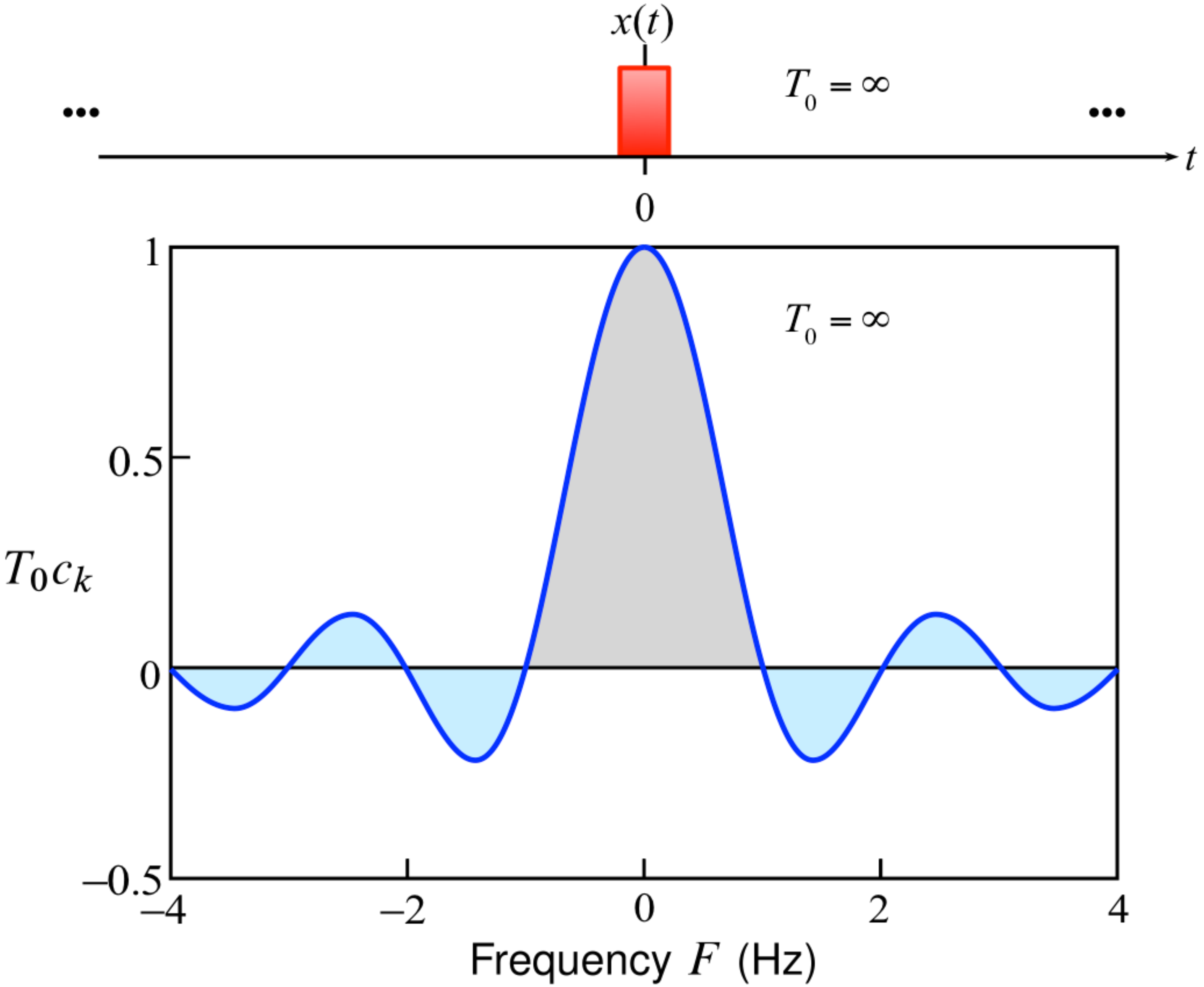
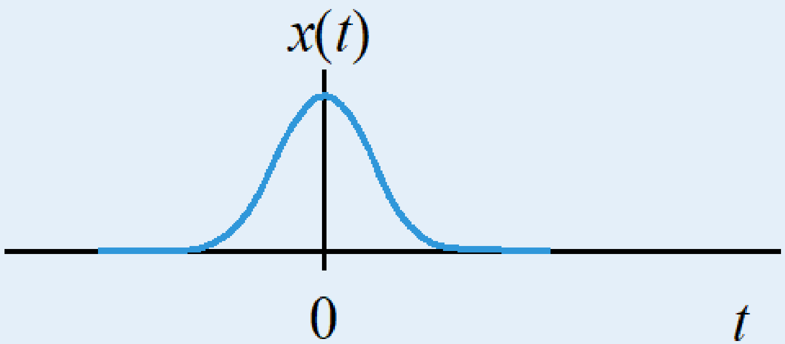
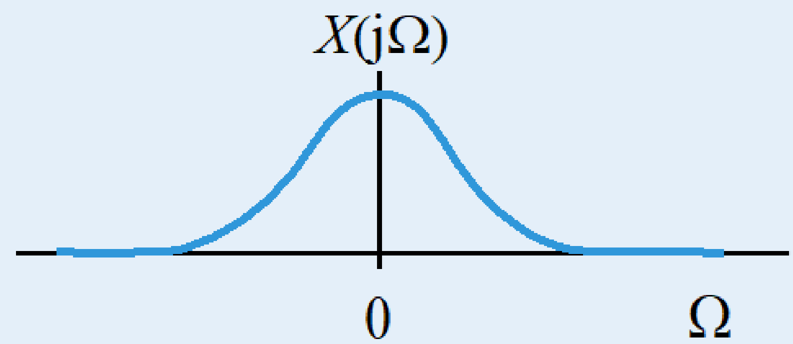


Continuous-time aperiodic signal: CTFT

Continuous - time signals

Time-domain	Frequency-domain
	
Continuous and aperiodic	Continous and aperiodic

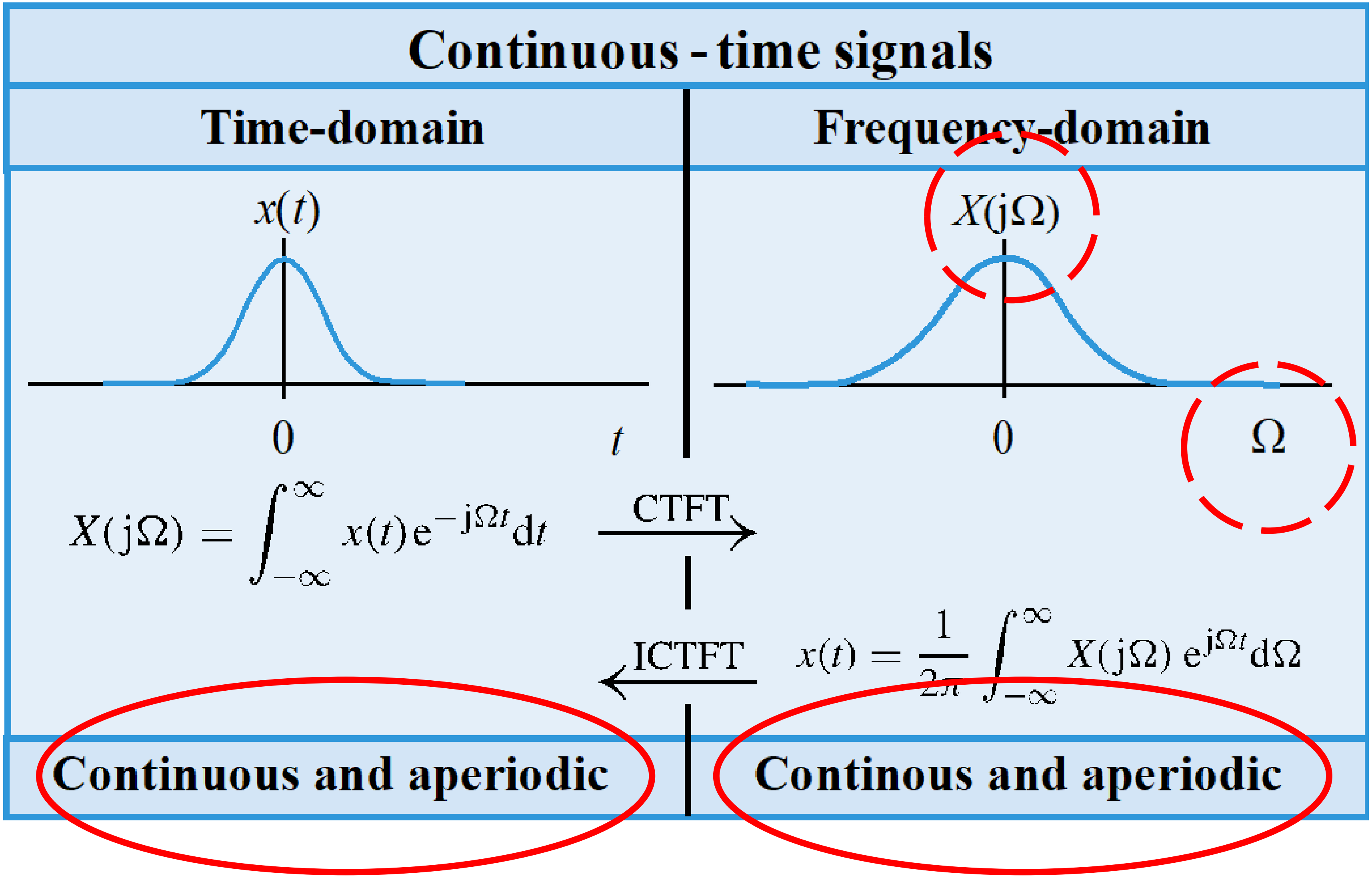
$$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

CTFT →

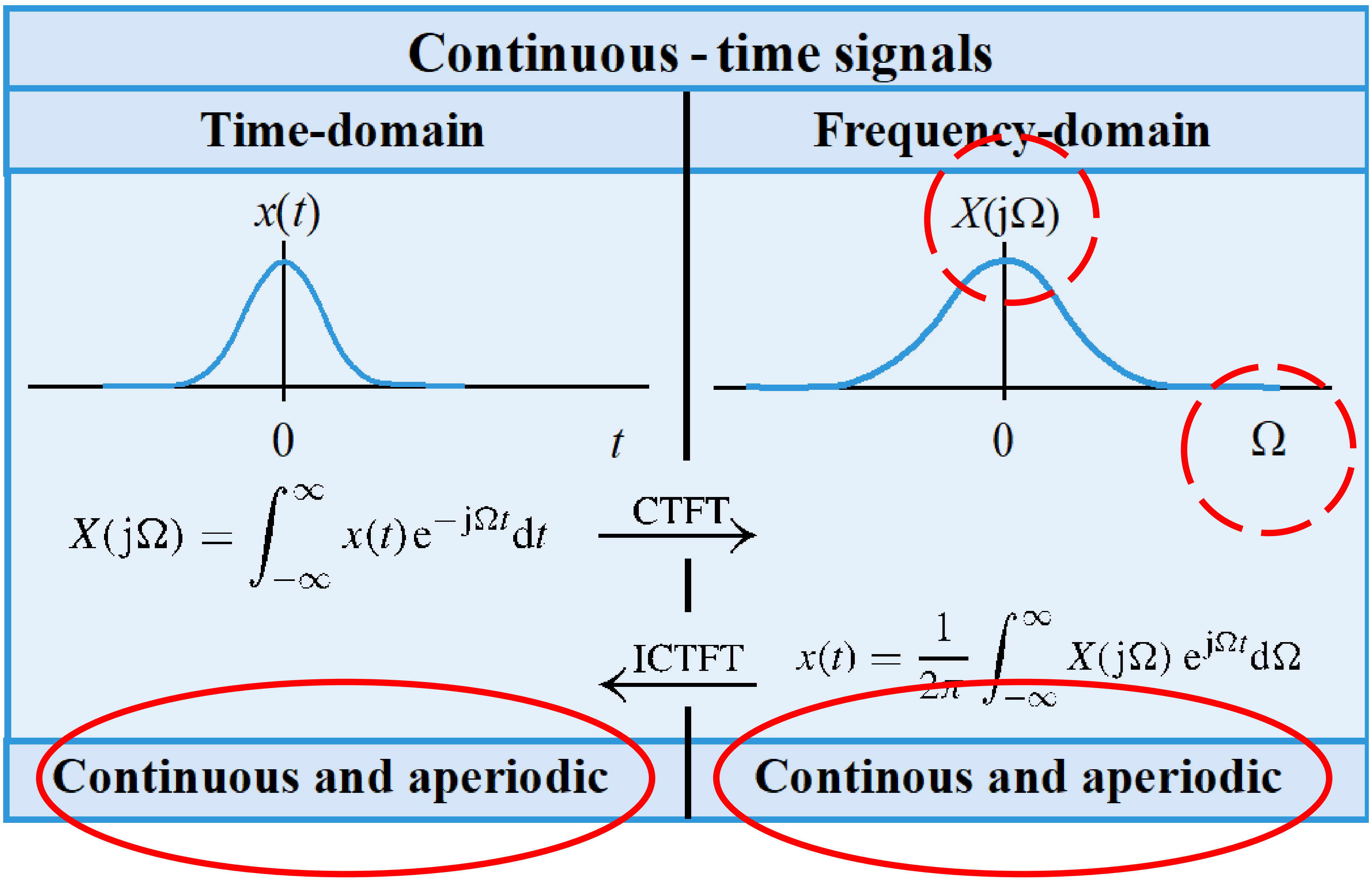
← ICTFT

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega t} d\Omega$$

Continuous-time aperiodic signal: CTFT

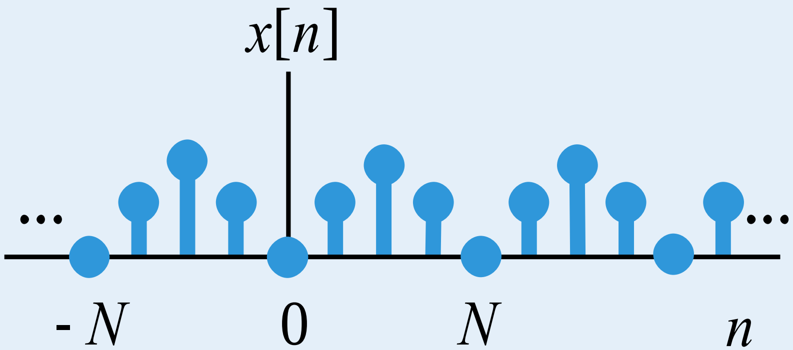
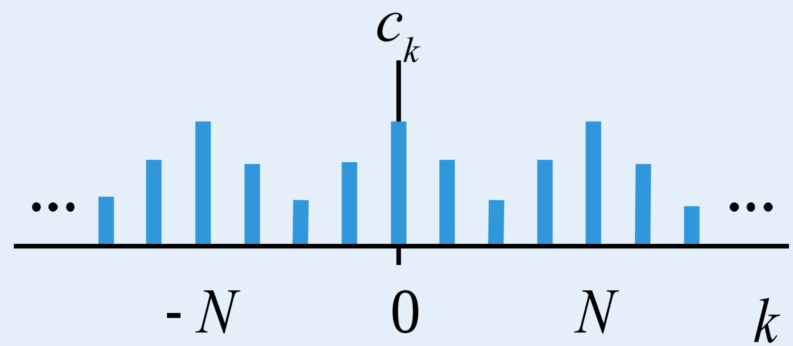


Continuous-time aperiodic signal: CTFT



Discrete-time periodic signal: DTFS

Discrete - time signals

Time-domain	Frequency-domain
	
Discrete and periodic	Discrete and periodic

DTFS →

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

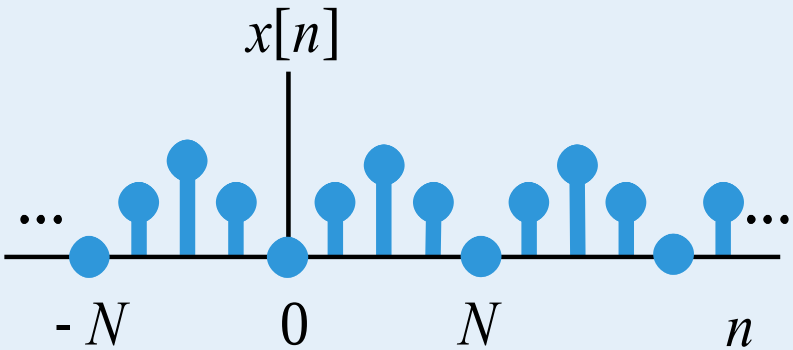
← IDTFS

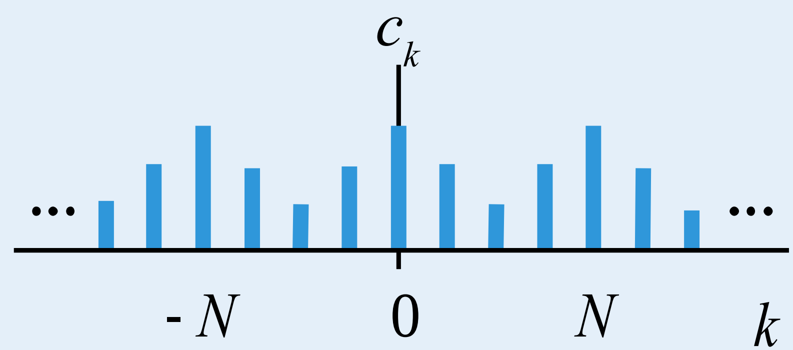
$$x[n] = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi}{N}kn}$$

Discrete-time periodic signal: DTFS

Discrete - time signals

Time-domain	Frequency-domain





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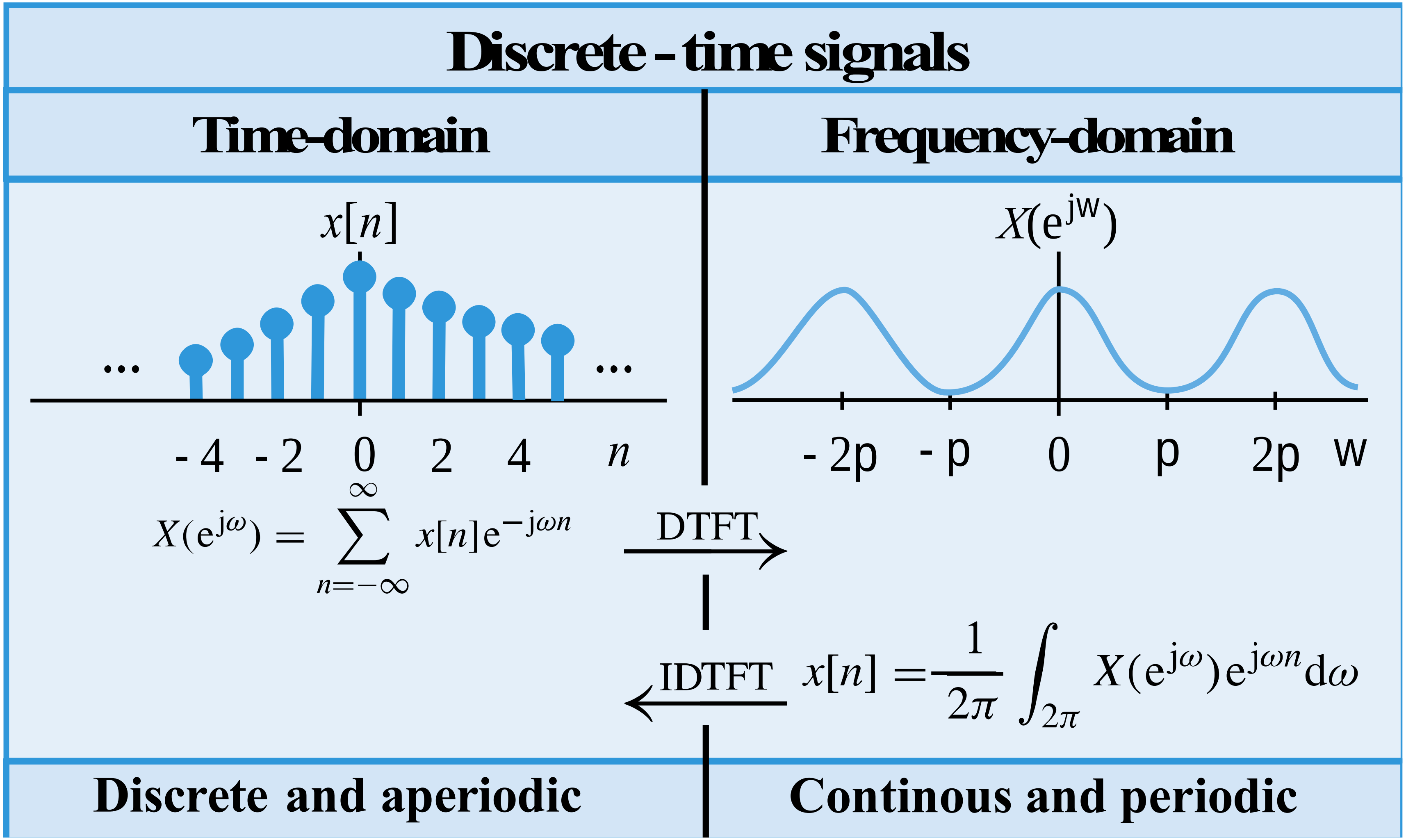
DTFS →

← IDTFS

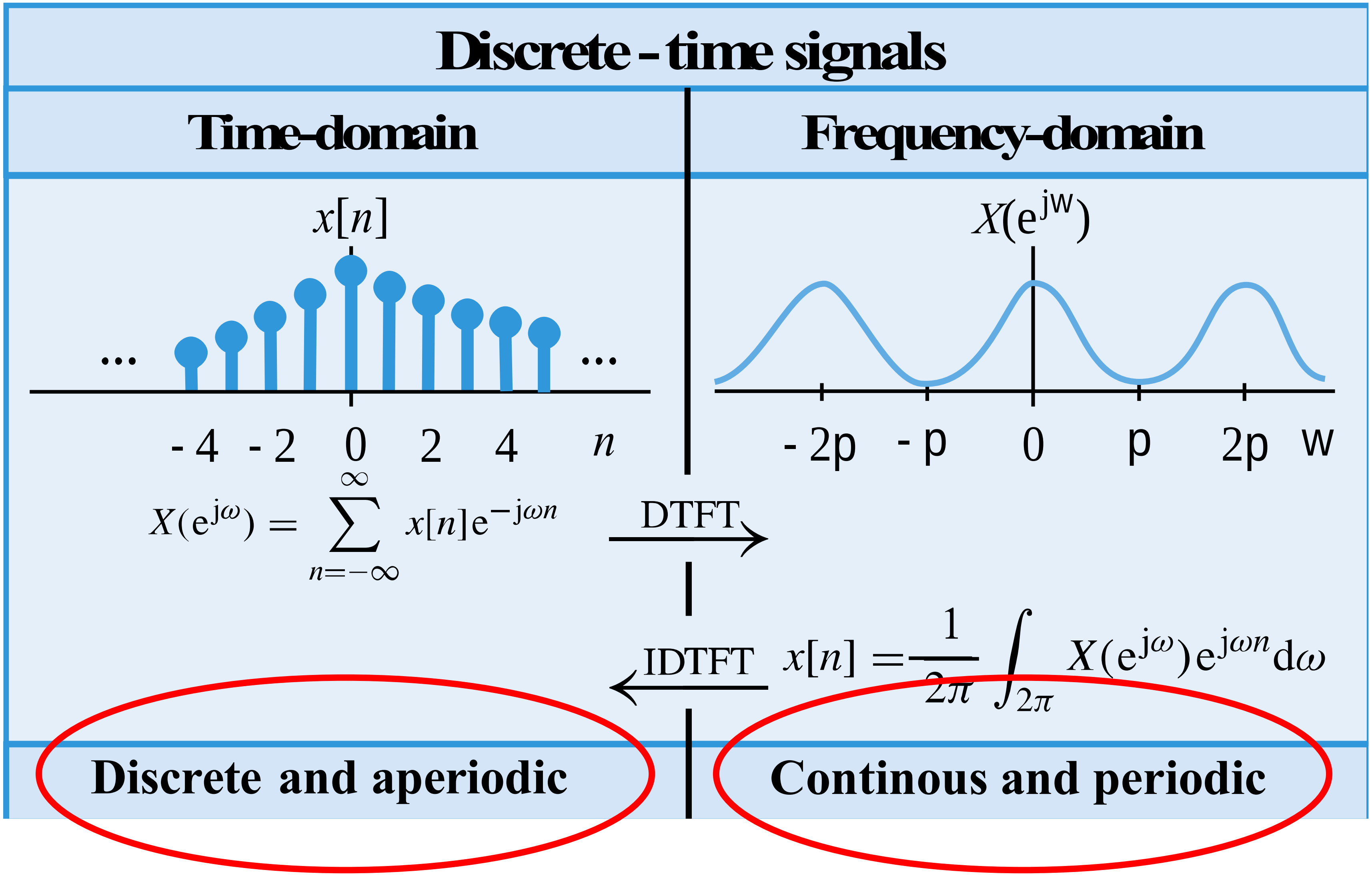
$$x[n] = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi}{N}kn}$$

Discrete and periodic	Discrete and periodic

Discrete-time aperiodic signal: DTFT

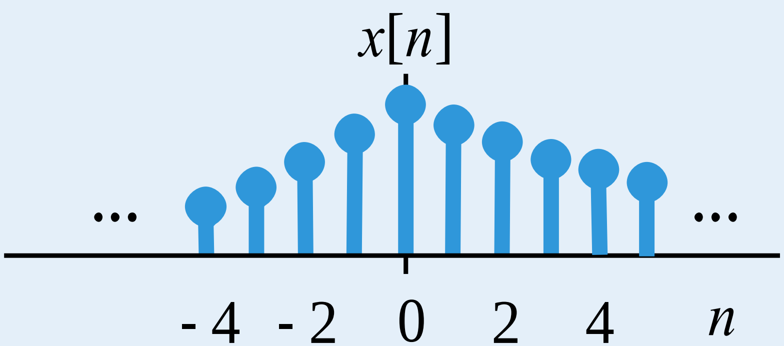
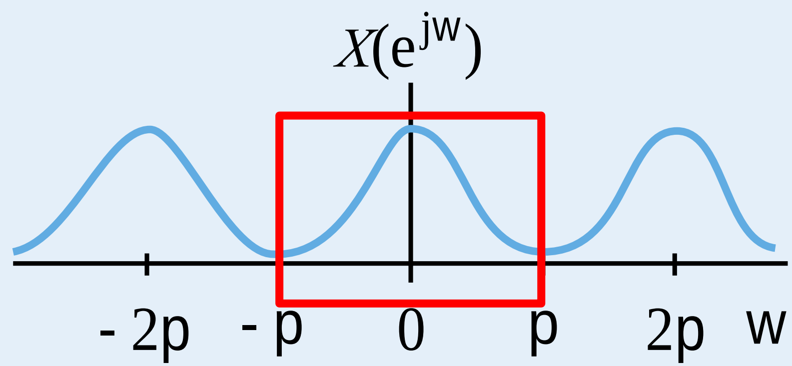


Discrete-time aperiodic signal: DTFT



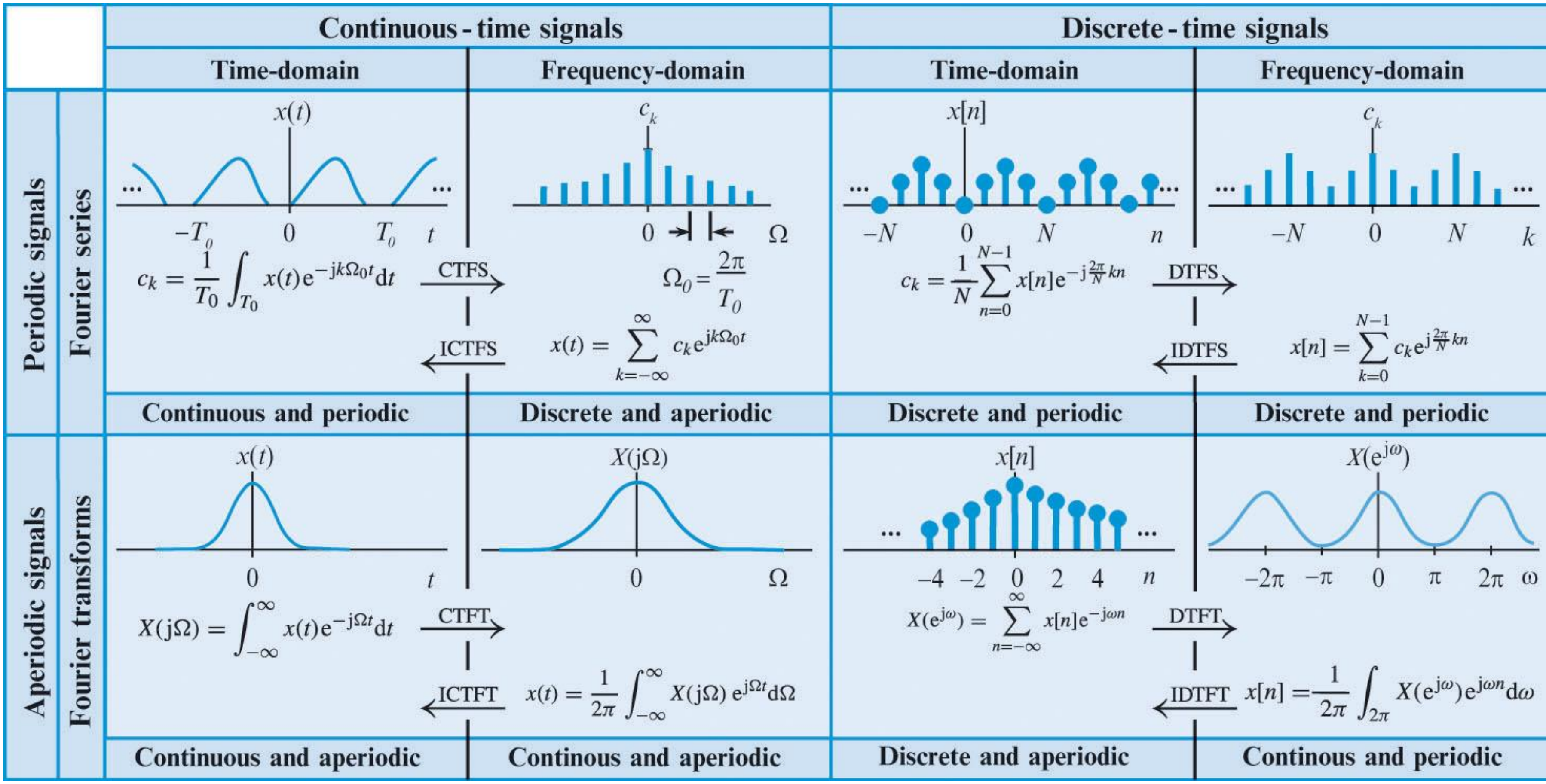
Discrete-time aperiodic signal: DTFT

Discrete-time signals

Time-domain	Frequency-domain
	
$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$	
DTFT →	
← IDTFT	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$
Discrete and aperiodic	**Continous and periodic**

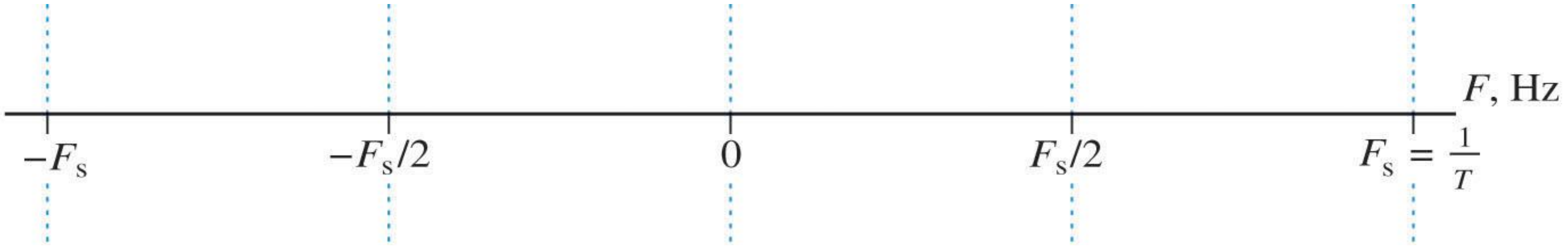
Everything you need to know !

Summary of Fourier series and transforms

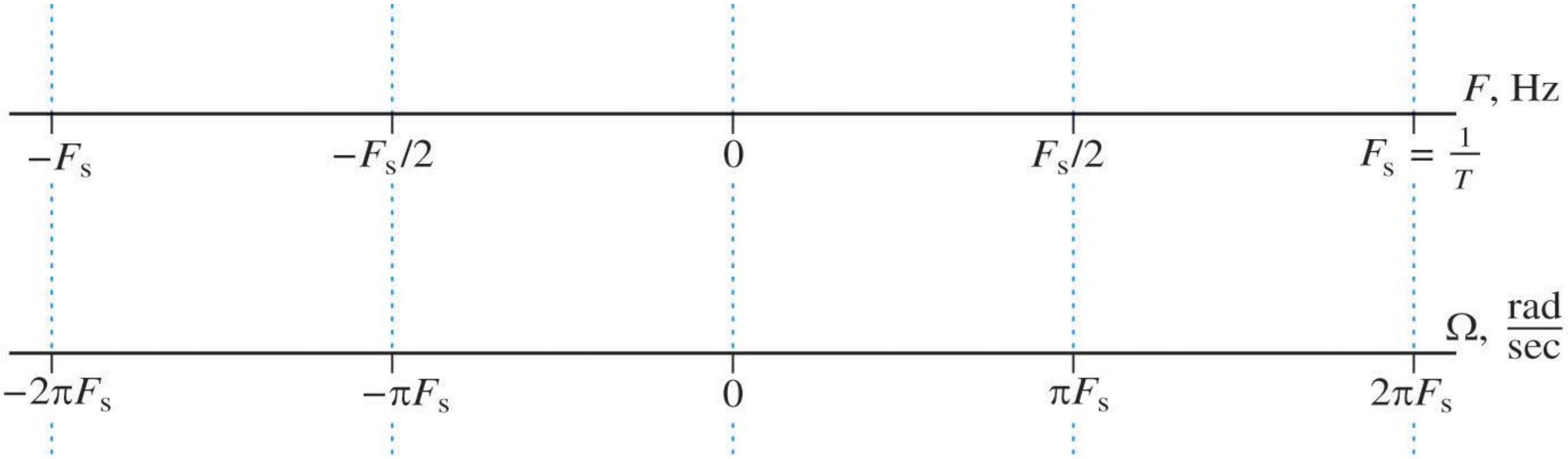


Periodicity with “period” α in one domain implies discretization with “spacing” $1/\alpha$ in the other domain, and vice versa.

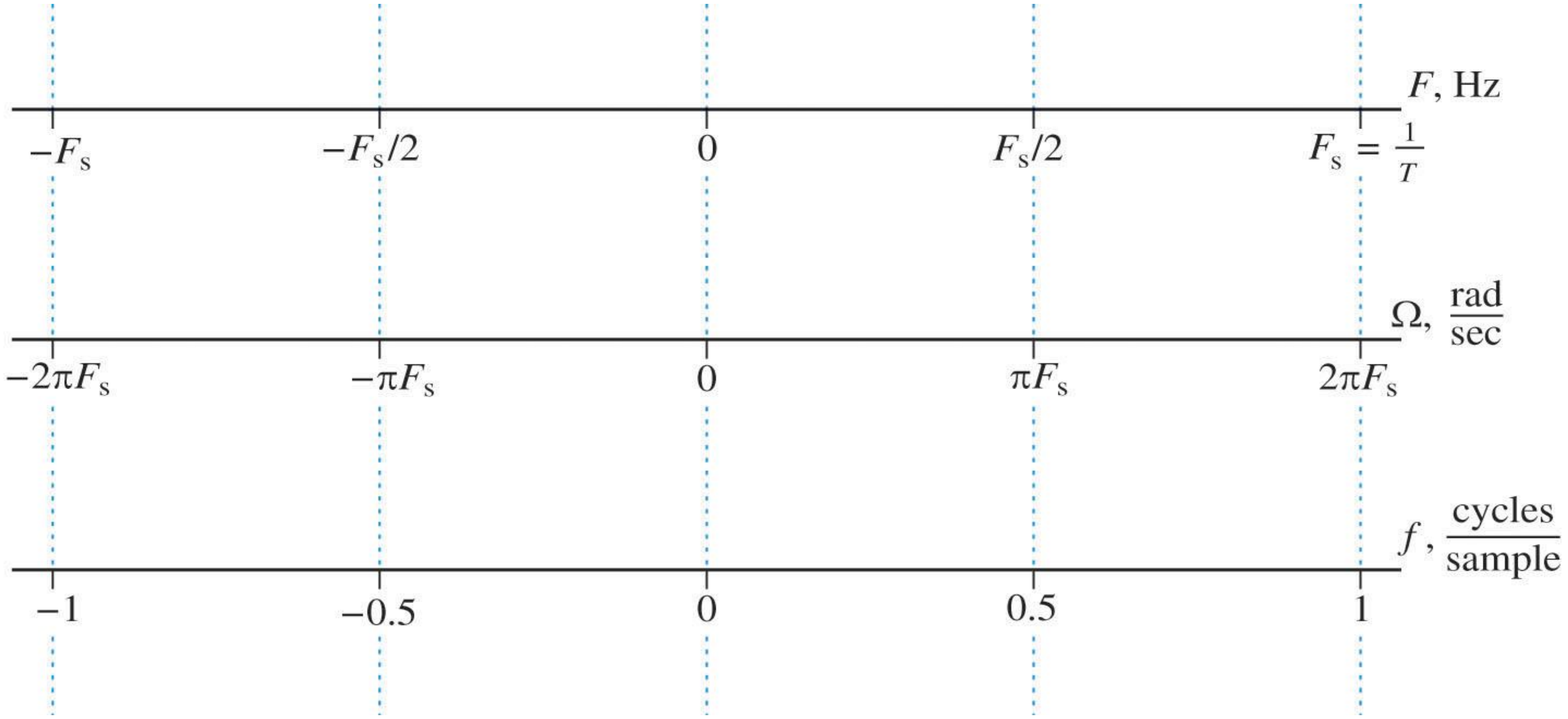
Frequency : *F* (Hz)



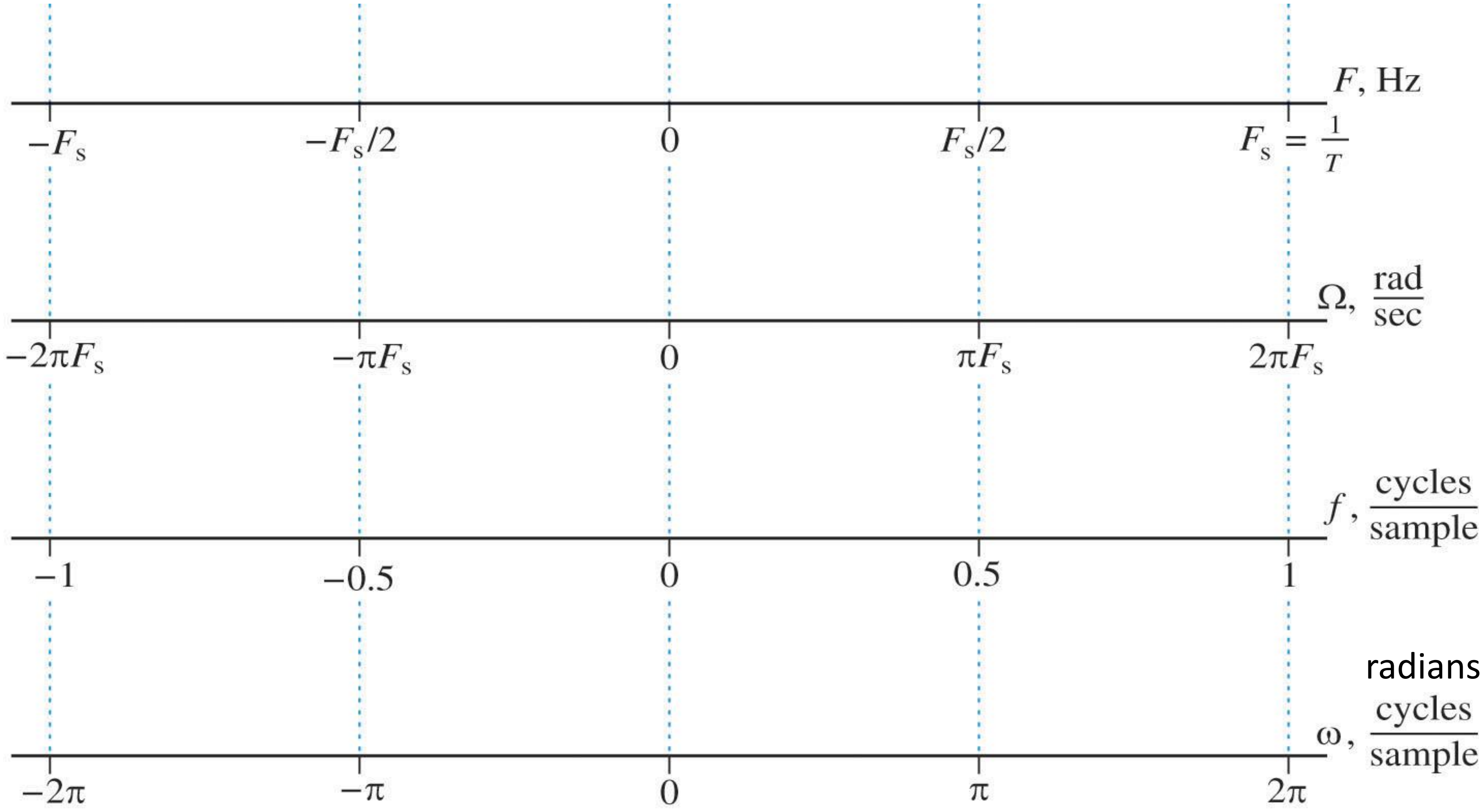
Angular frequency: $\Omega = 2\pi F$ (rad/sec)



Normalized frequency: $f = F/F_s$ (cycles/samples)



Normalized angular frequency:
$\omega = 2\pi \times F/F_s$ (radians x cycles/samples)



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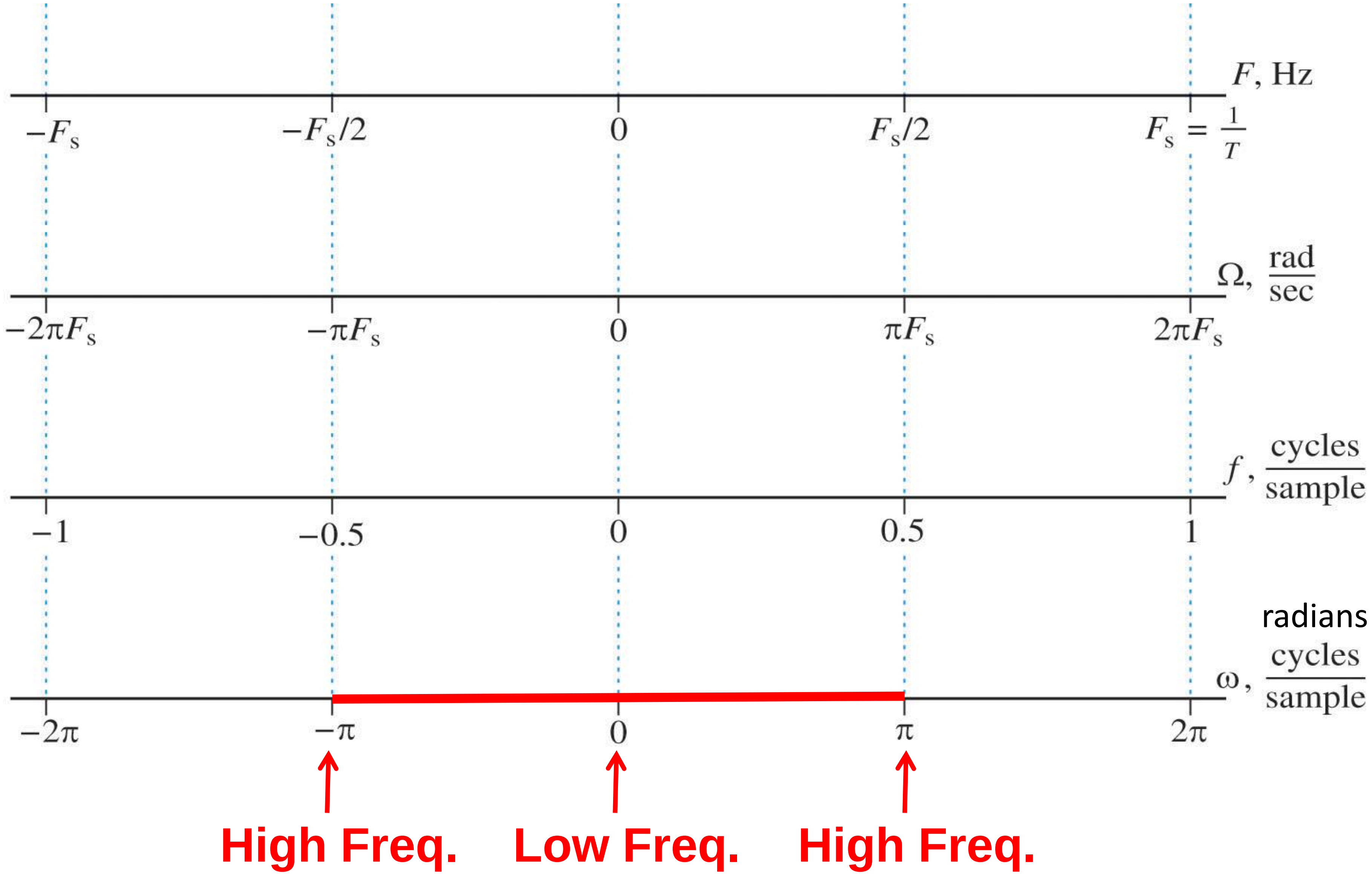


Numerical computation of DTFS

Let $\boldsymbol{x}[\boldsymbol{n}]$ be periodic and $\boldsymbol{x} = \left[\boldsymbol{x}[\mathbf{0}]\ \boldsymbol{x}[\mathbf{1}], \cdots, \boldsymbol{x}[\boldsymbol{N}-\mathbf{1}]\right]$ includes first $\boldsymbol{N}$ sampls.

Formula	MATLAB function
$c_k = \dfrac{1}{N} \sum_{n=0}^{N-1} x[n]\, \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}kn}$	`ck = (1/N)*fft(xn) % dtfs`
$x[n] = \sum_{k=0}^{N-1} c_k\, \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}kn}$	`xn = N*ifft(ck) % idtfs`

Example 1.1: use of fft and ifft

Example 1: Compute the DFTS of pulse train with L=2 and $N = 10$.

```
% signal
x=[1 1 1 0 0 0 0 0 1 1]
% N
N=length(x);
% ck
c=fft(x)/N
x1=ifft(c)*N
% plot x1
stem(x1)
title('ifft(c)*N')
```

Numerical computation of DTFT

The computation of a finite length sequence $x[n]$ that is nonzero between 0 and $N-1$ at frequency ω_k is given by,

Formula

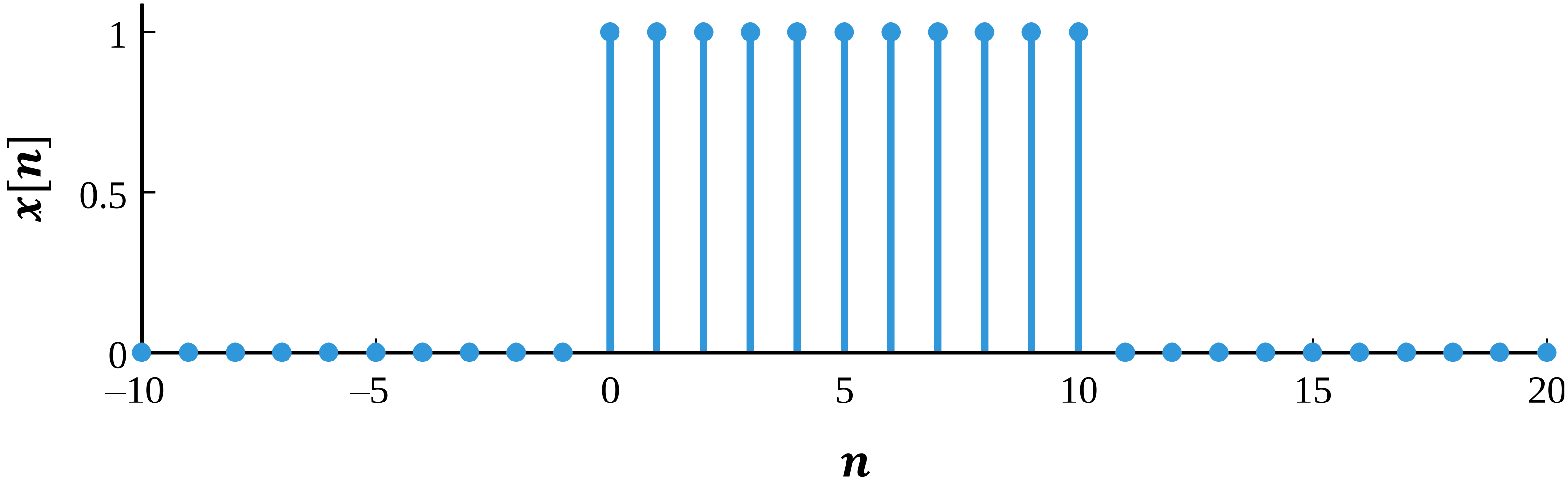
$$X(e^{j\omega_k}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega_k n}, \ k = 1, 2, \ldots, K$$

MATLAB function

```
X=freqz(x,1,om) % DTFT
```

Example 1.2: use of freqz

Example 1.2: plot magnitude and phase spectrum of the following signal



Example 1.2: use of freqz

```
% signal
x=[1 1 1 1 1 1 1 1 1 1 1];
```

Example 1.2: use of freqz

```
% signal
x=[1 1 1 1 1 1 1 1 1 1 1];
% define omega
om=linspace(-pi,pi,500);
```

Example 1.2: use of freqz

```
% signal
x=[1 1 1 1 1 1 1 1 1 1 1];
% define omega
om=linspace(-pi,pi,500);
% Compute DTFT
X=freqz(x,1,om);
```

Example 1.2: use of freqz

```
% signal
x=[1 1 1 1 1 1 1 1 1 1 1];
% define omega
om=linspace(-pi,pi,500);
% Compute DTFT
X=freqz(x,1,om);
% |X|
X1=abs(X);
```

Example 1.2: use of freqz

```
% signal
x=[1 1 1 1 1 1 1 1 1 1 1];
% define omega
om=linspace(-pi,pi,500);
% Compute DTFT
X=freqz(x,1,om);
% |X|
X1=abs(X);
% plot  magnitude spectrum
figure(1)
plot(om,X1,'LineWidth',2.5)
xlabel('Normalized angular frequency')
ylabel('Magnitude |X|')
```

Example 1.2: use of freqz

```
% signal
x=[1 1 1 1 1 1 1 1 1 1 1];
% define omega
om=linspace(-pi,pi,500);
% Compute DTFT
X=freqz(x,1,om);
% phase
p=angle(X);
% plot phase spectrum
figure(2)
plot(om,p,'LineWidth',2.5)
xlabel('Normalized angular frequency')
ylabel(‘Phase')
```

Example 1.3: use of freqz

Example 1.2: plot magnitude and phase spectrum of $\boldsymbol{x[n] = 0.6 \times \mathrm{sinc}\,(0.6n)}$ for $\boldsymbol{n = -200{:}\,1{:}\,200}$.

Example 1.3: use of freqz

```
% time t or n
t=-200:1:200;
```

Example 1.3: use of freqz

```
% time t or n
t=-200:1:200;
% signal
x=0.6*sinc(0.6.*t);
% plots signal
figure(1)
plot(t,x,'LineWidth',2.5)
title('x')
```

Example 1.3: use of freqz

```
% time t or n
t=-200:1:200;
% signal
x=0.6*sinc(0.6.*t);
% plots signal
figure(1)
plot(t,x,'LineWidth',2.5)
title('x')
% define omega
om=linspace(-pi,pi,500);
% compute DTFT
X=freqz(x,1,om);
```

Example 1.3: use of freqz

```
% time t or n
t=-200:1:200;
% signal
x=0.6*sinc(0.6.*t);
% plots signal
figure(1)
plot(t,x,'LineWidth',2.5)
title('x')
% define omega
om=linspace(-pi,pi,500);
% compute DTFT
X=freqz(x,1,om);
% plot magnitude spectrum
figure(2)
plot(om,abs(X),'LineWidth',2.5)
```

Example 1.3: use of freqz

```
% time t or n
t=-200:1:200;
% signal
x=0.6*sinc(0.6.*t);
% plots signal
figure(1)
plot(t,x,'LineWidth',2.5)
title('x')
% define omega
om=linspace(-pi,pi,500);
% compute DTFT
X=freqz(x,1,om);
% scale it by factor pi
figure(2)
plot(om/pi,abs(X),'LineWidth',2.5)
```

Example 1.4: use of freqz

Example 1.2: plot magnitude and phase spectrum of $x[n] = 0.6 \times \mathrm{sinc}\,(0.6n)$ for $n = -200{:}\,0.1{:}\,200$.

You should scale freqz(x,1,om) by Ts (i.e. X=Ts* freqz(x,1,om)).

More details in the future sessions…

Main application of freqz(b,a,om)

$$H(z) = \frac{B(z)}{A(z)} = \frac{b(1) + b(2)z^{-1} + \cdots + b(n)z^{-(n-1)}}{a(1) + a(2)z^{-1} + \cdots + a(m)z^{-(m-1)}}$$

Main application of freqz(b,a,om)

$$H(z) = \frac{B(z)}{A(z)} = \frac{b(1) + b(2)z^{-1} + \cdots + b(n)z^{-(n-1)}}{a(1) + a(2)z^{-1} + \cdots + a(m)z^{-(m-1)}}$$

For $z = e^{j\omega}$ one can write,

$$H\left(e^{j\omega}\right) = \frac{B(e^{j\omega})}{A(e^{j\omega})} = \frac{b(1) + b(2)e^{-j\omega} + \cdots + b(n)e^{-j(n-1)\omega}}{a(1) + a(2)e^{-j\omega} + \cdots + a(m)e^{-j(m-1)\omega}}$$

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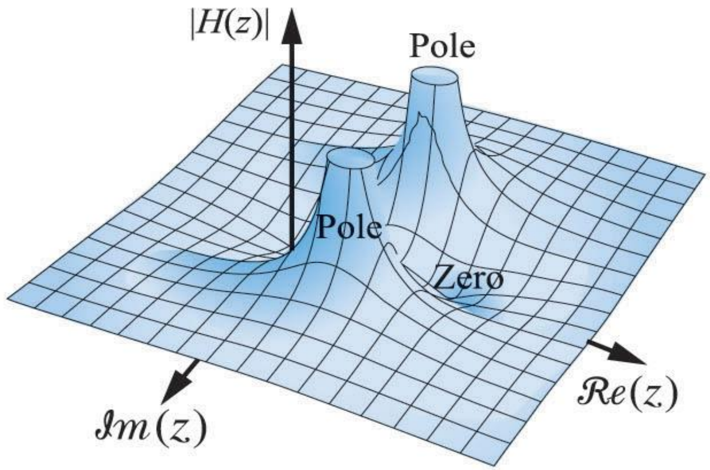
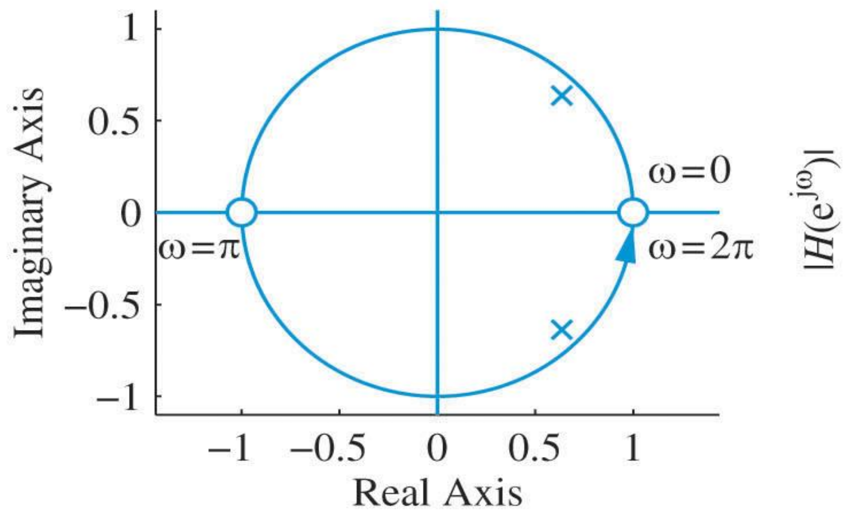
$$H(e^{j\omega}) = \frac{B(e^{j\omega})}{A(e^{j\omega})} = \frac{b(1) + b(2)e^{-j\omega} + \cdots + b(n)e^{-j(n-1)\omega}}{a(1) + a(2)e^{-j\omega} + \cdots + a(m)e^{-j(m-1)\omega}}$$

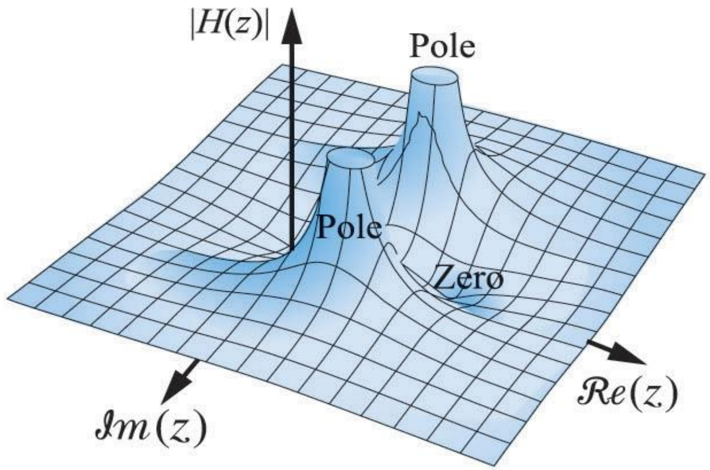
```
b= [b(1),…,b(n)]; % vector b numerator
a= [a(1),…,a(n)]; % vector a denominator
om=linspace(-pi,pi,k); % desired frequency range
H=freqz(b,a,om); % system frequency response
```

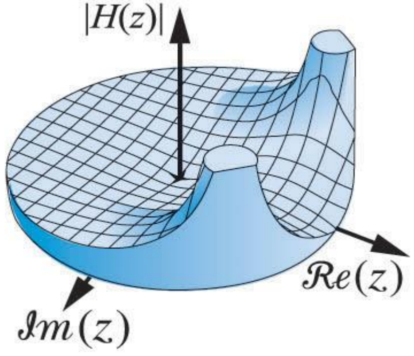
From ZT to DTFT

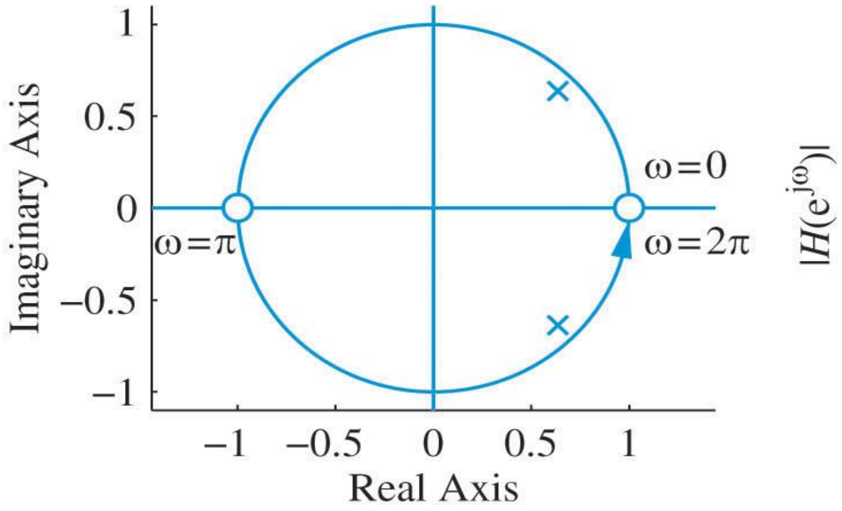




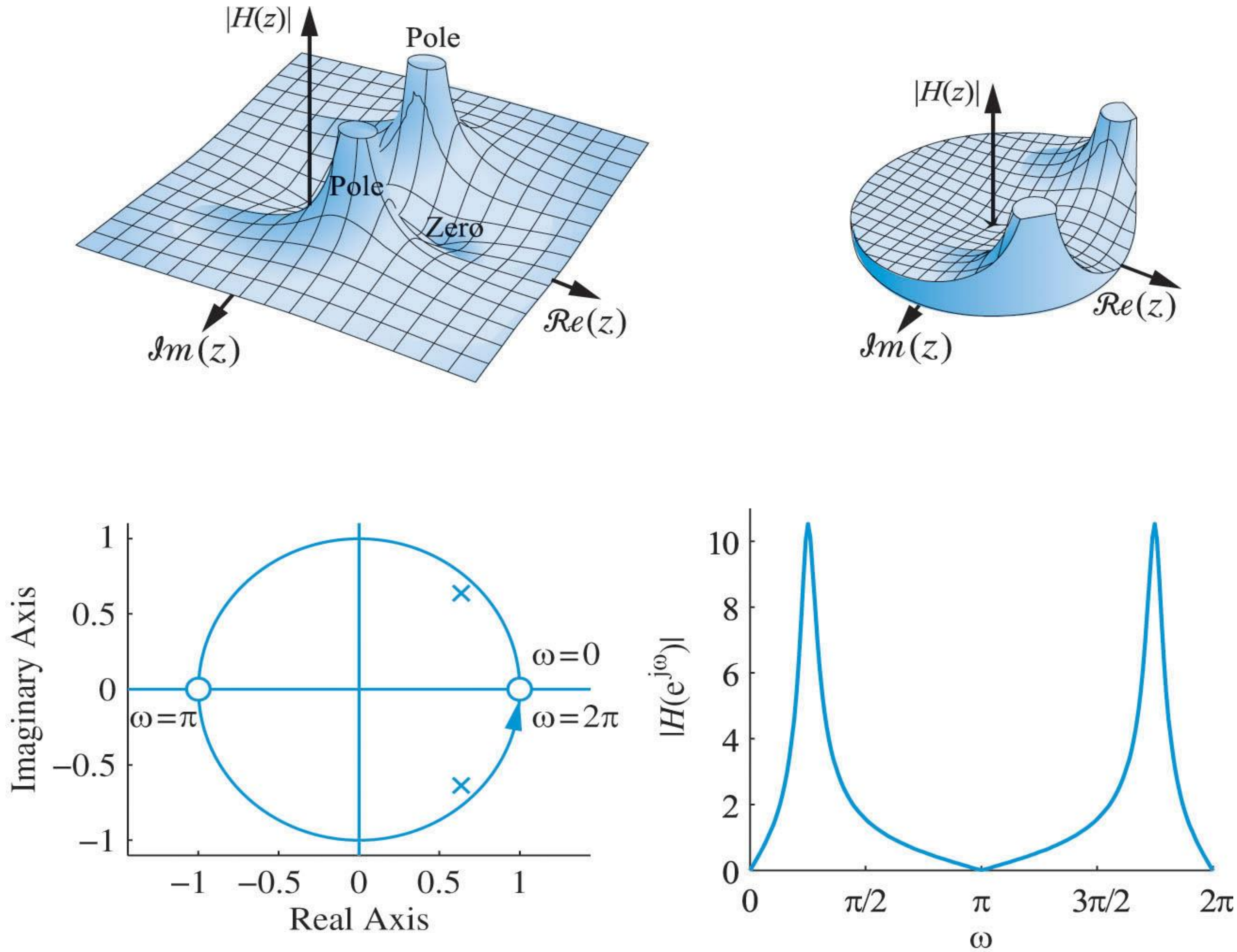
From ZT to DTFT







From ZT to DTFT



Example 1.5: frequency response

Example 1.2: plot magnitude and phase spectrum of a system with zeros $z_{1,2} = \pm 1$ and $p_{1,2} = 0.9e^{\pm j\pi/4}$.

Example 1.5: frequency response

```
% zeros
zer = [-1 1];
% ploes
pol=0.9*exp(1i*pi*1/4*[-1 +1]);
```

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% zeros
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% Turn it to rational transfer function
[b,a]=zp2tf(zer',pol',1);
```

Example 1.5: frequency response

```
% zeros
zer = [-1 1];
% ploes
pol=0.9*exp(1i*pi*1/4*[-1 +1]);
% Turn it to rational transfer function
[b,a]=zp2tf(zer',pol',1);
% omega
om=linspace(-pi,pi,500);
% freq. response
X=freqz(b,a,om);
% magnitude response scaled by pi
figure(1)
plot(om/pi,abs(X),'LineWidth',2.5)
xlabel('Normalized frequency (\pi x rad/sample) ')
```

Useful links

- https://nl.mathworks.com/help/signal/ref/freqz.html
- https://nl.mathworks.com/help/signal/ref/angle.html
- https://nl.mathworks.com/help/matlab/ref/fft.html
- https://www.12000.org/my_notes/on_scaling_factor_for_ftt_in_matlab/index.htm