Fast Fourier transform (FFT)

MATLAB tutorial series (Part 2.1)

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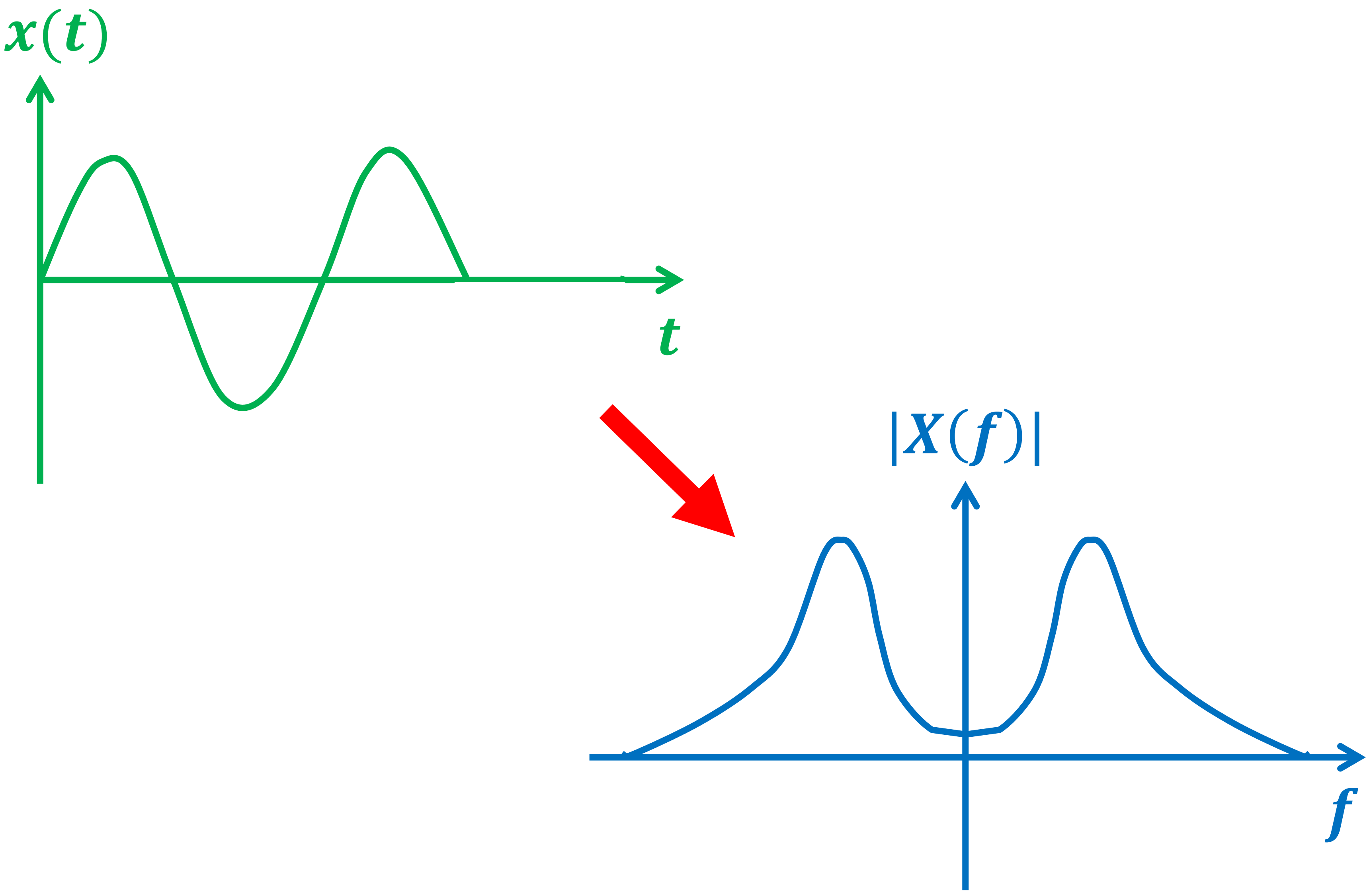
Introduction



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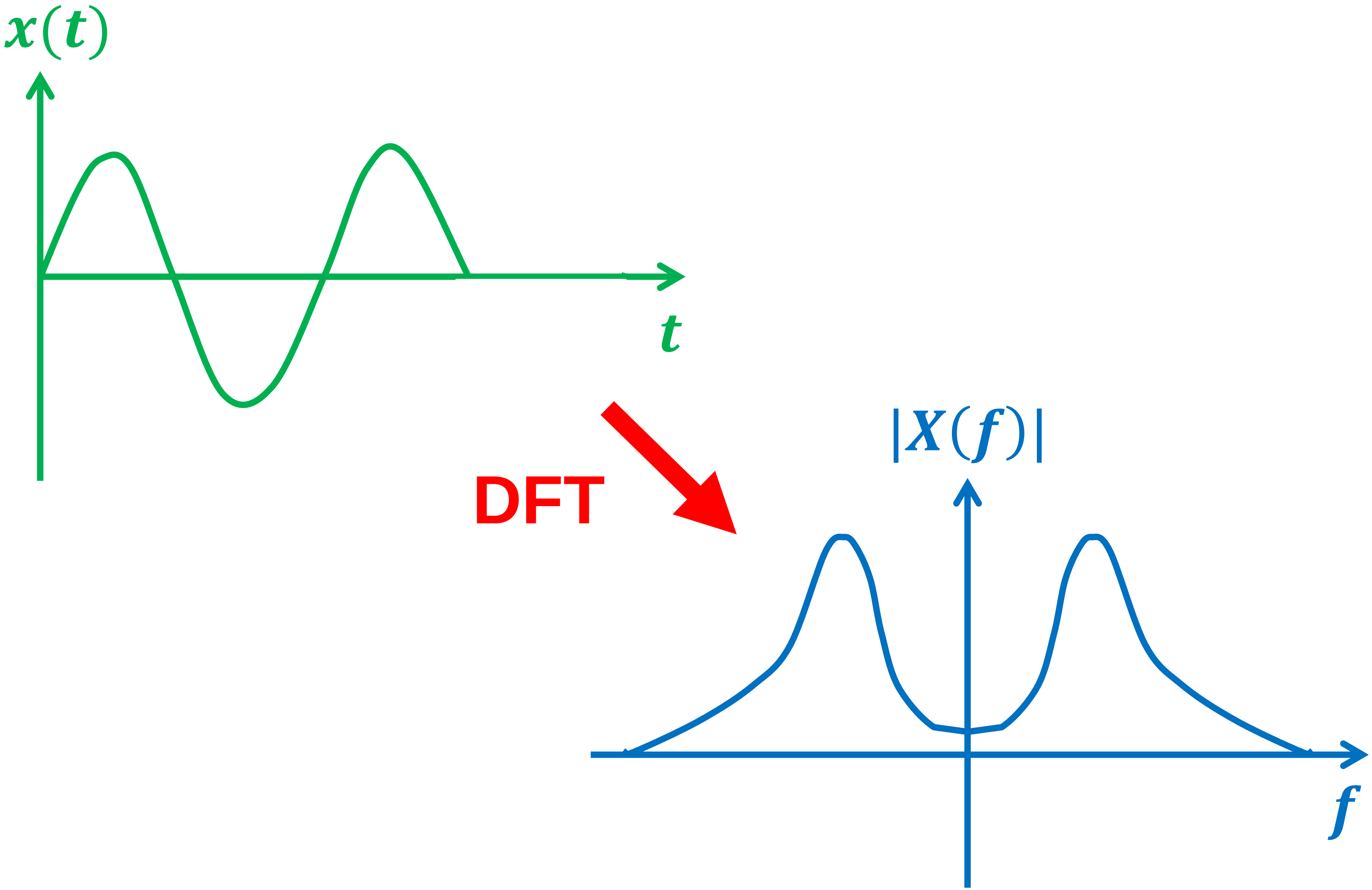

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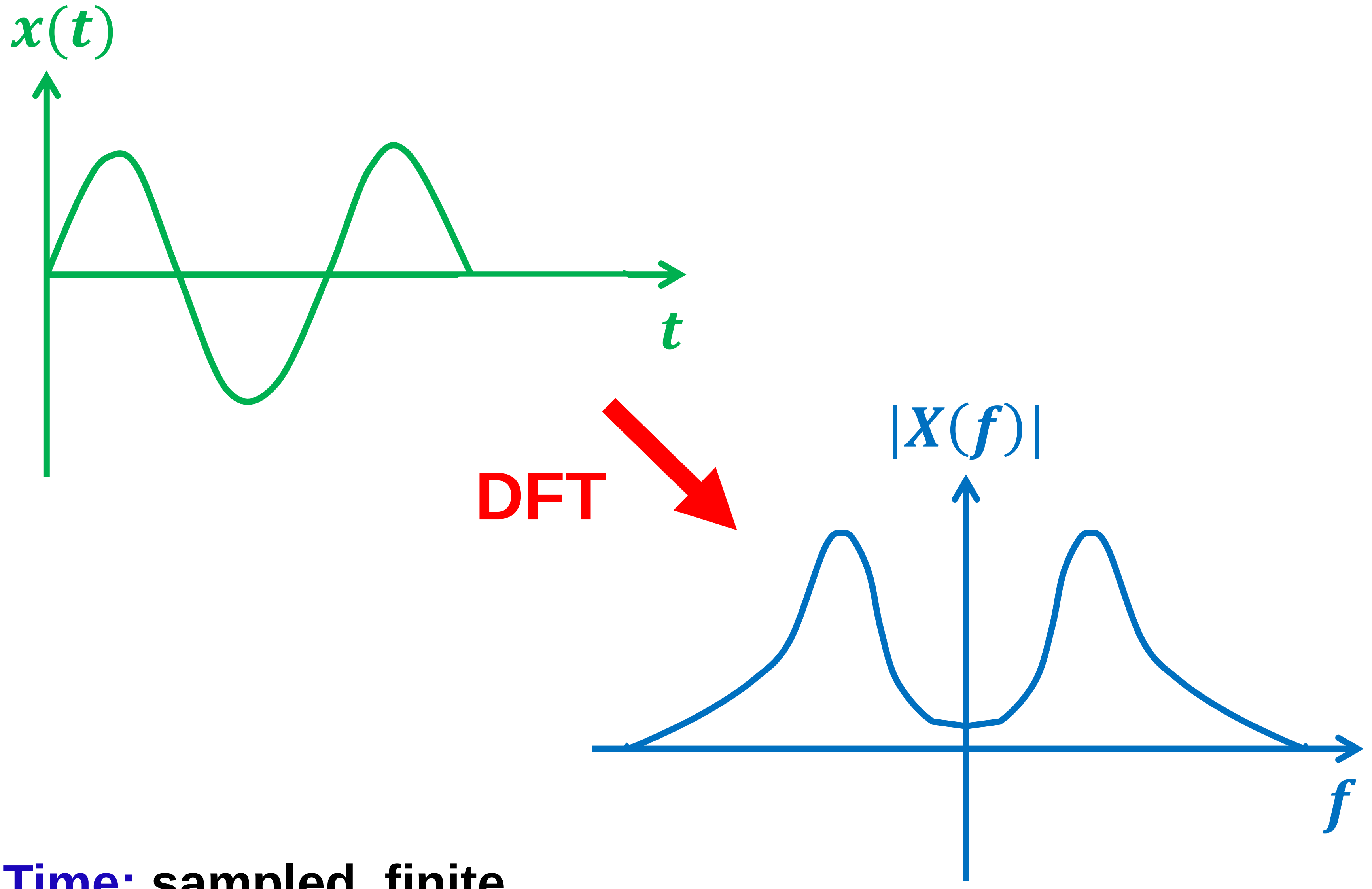

Time: **sampled, finite**

Frequency: **sampled, finite**

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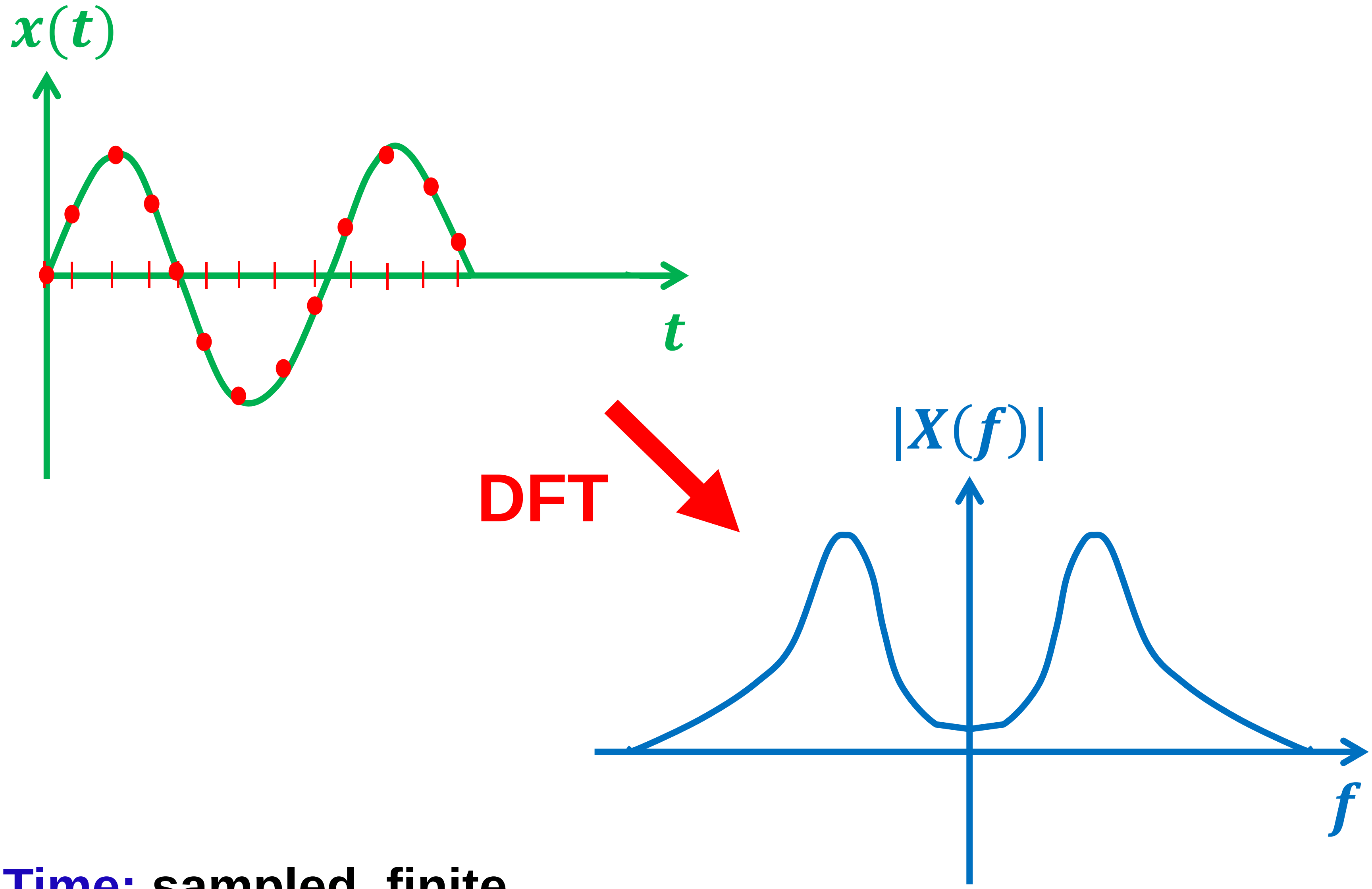

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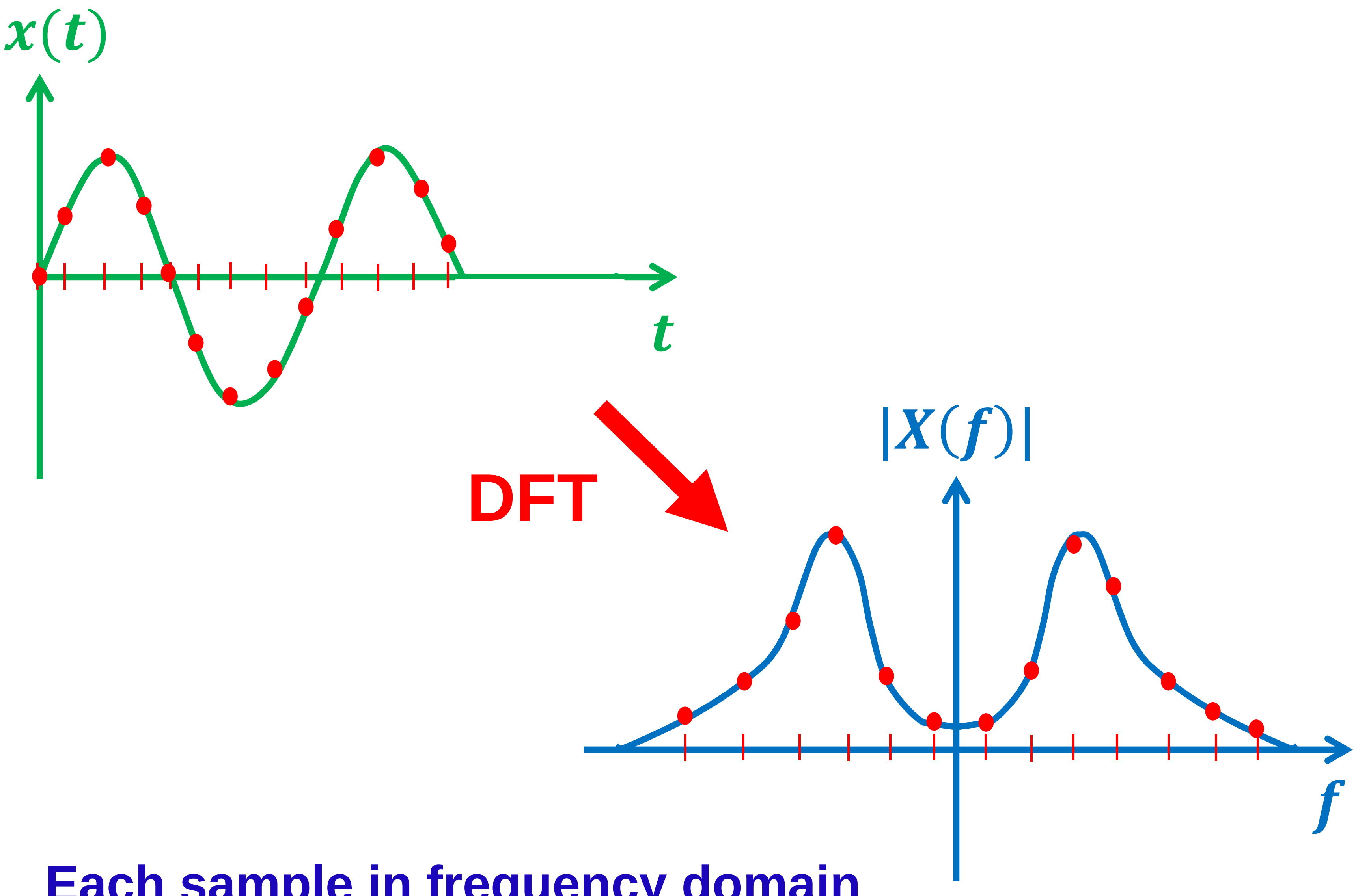

Each sample in frequency domain corresponds to a sample in time domain !

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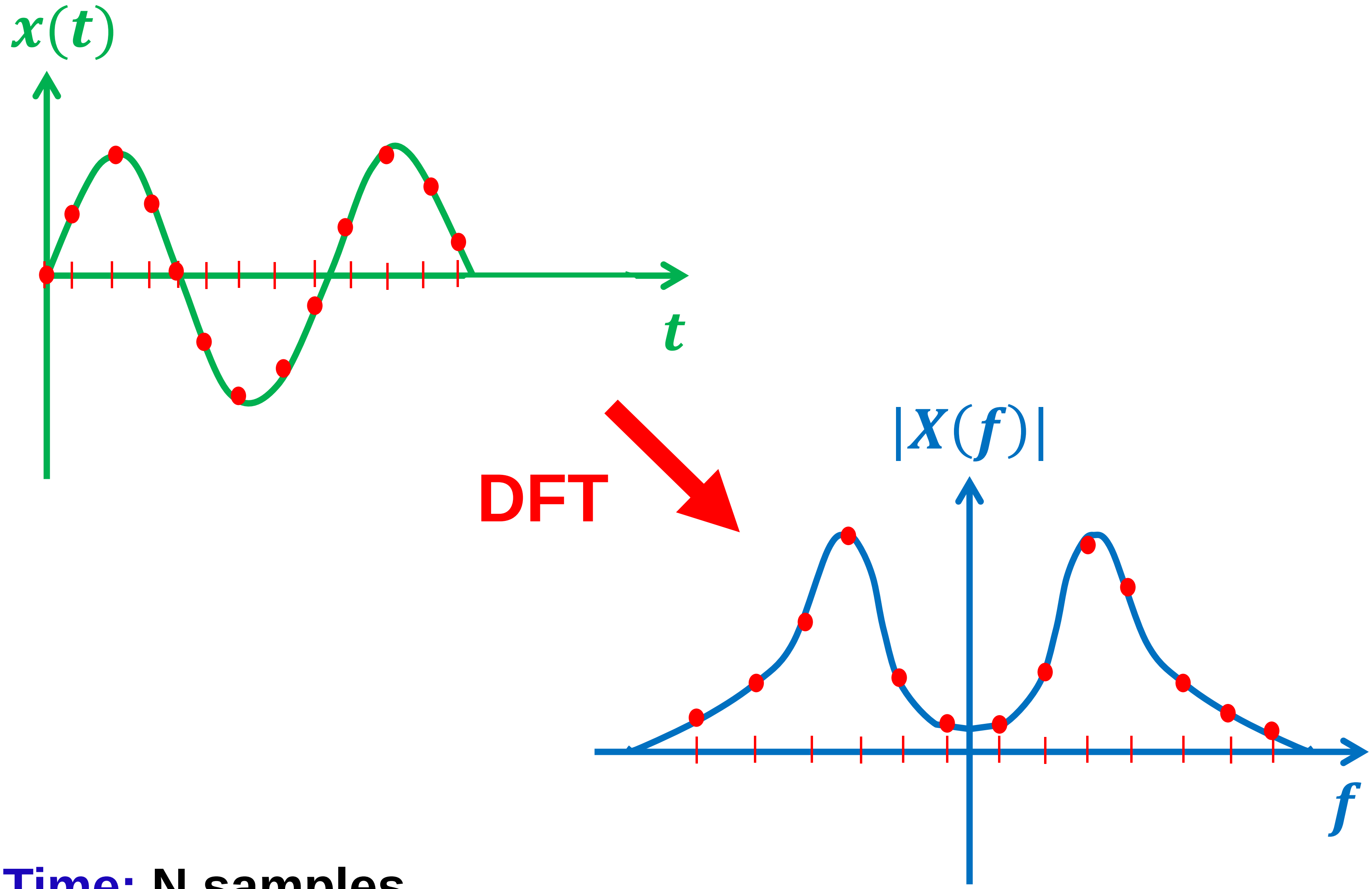

Time: N samples

Frequency: N samples

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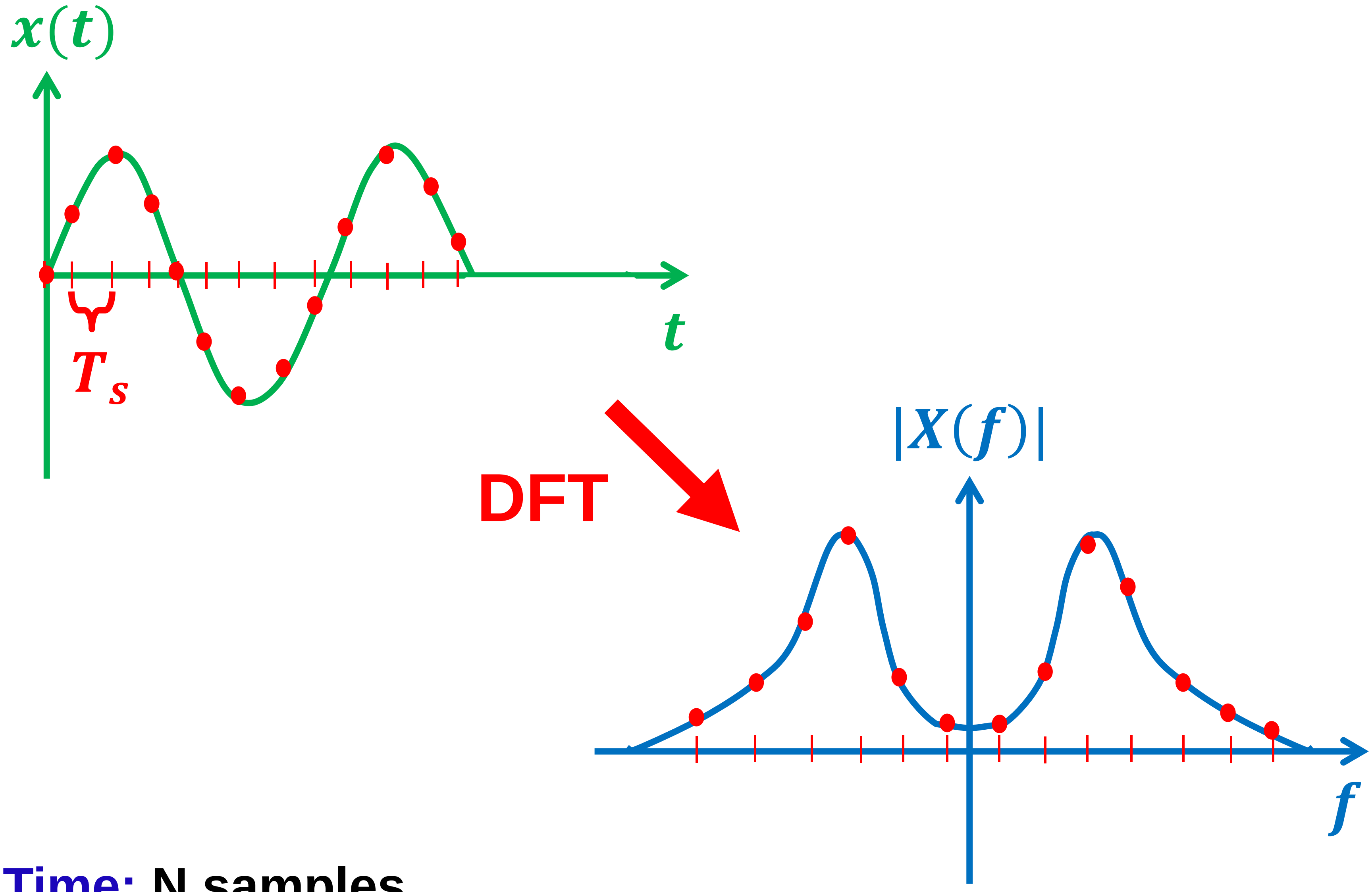

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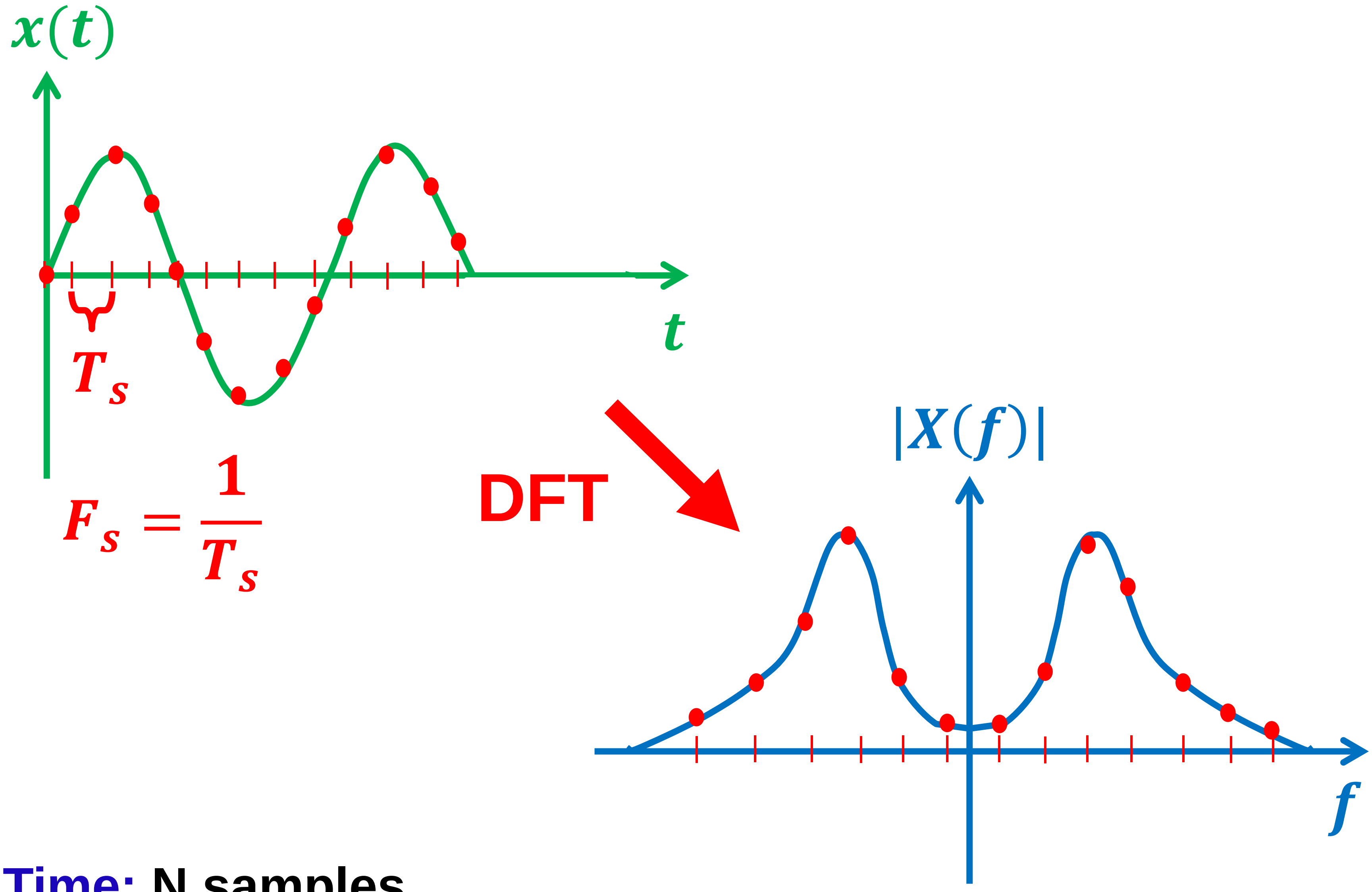

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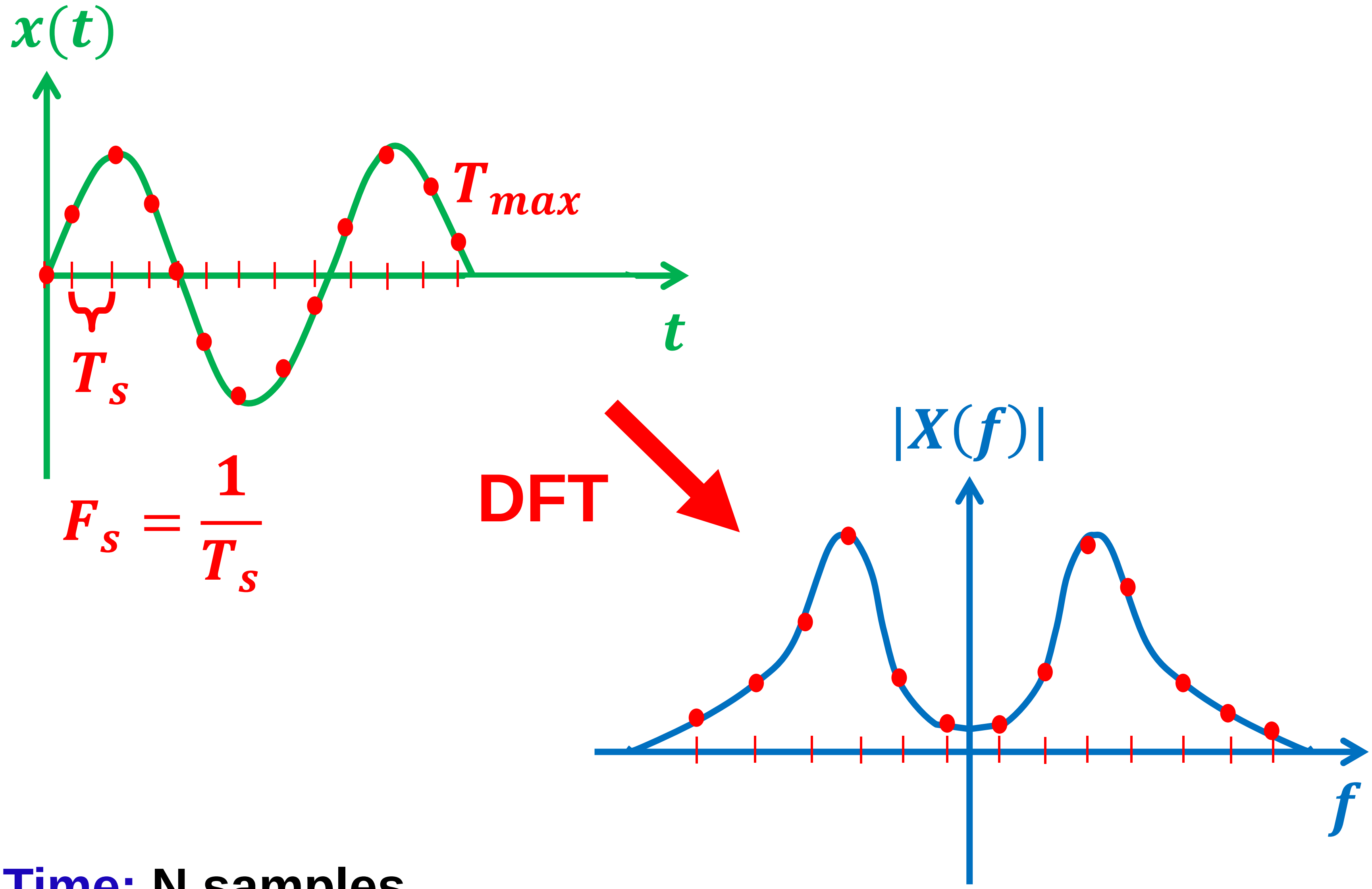

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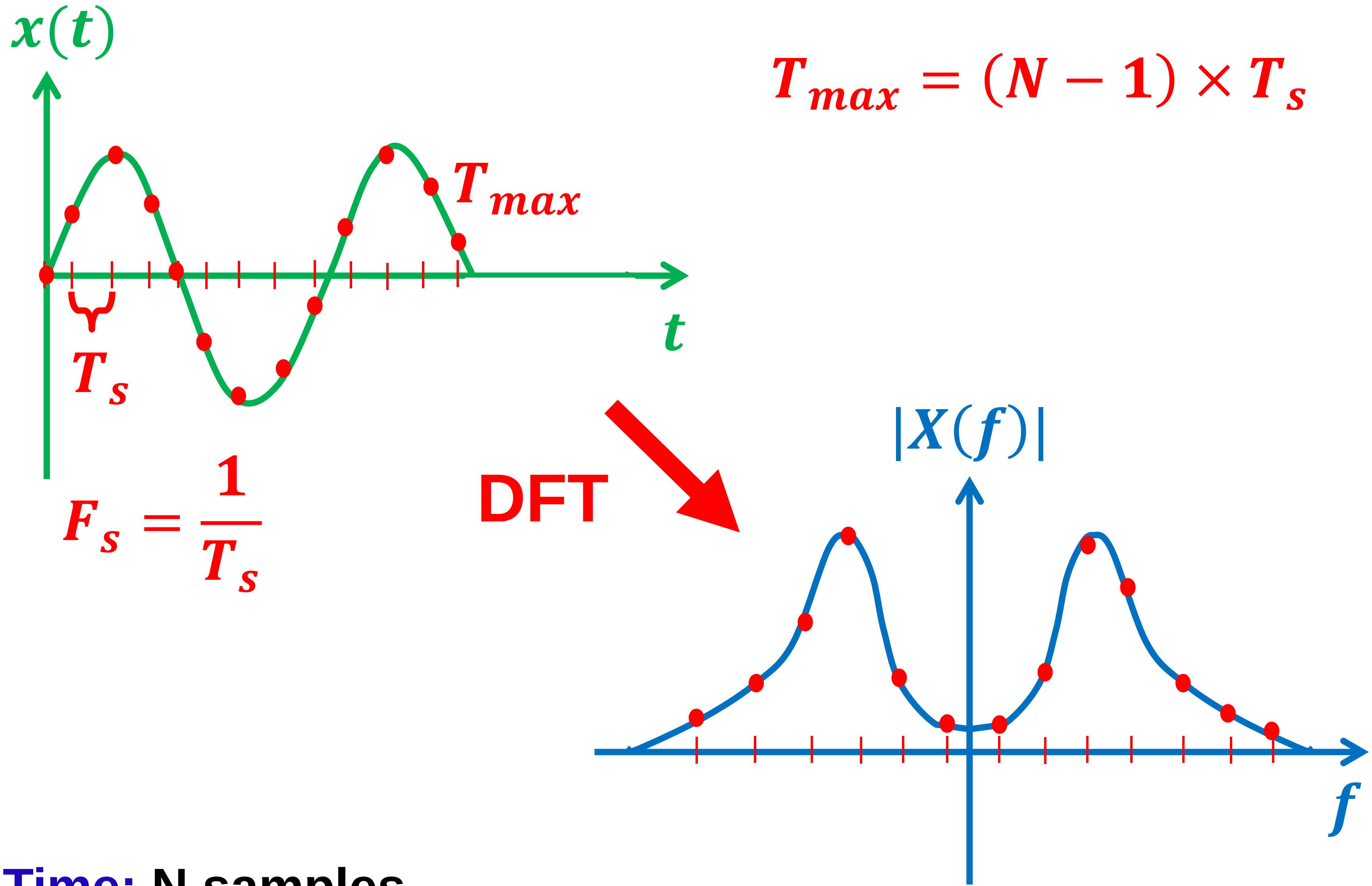


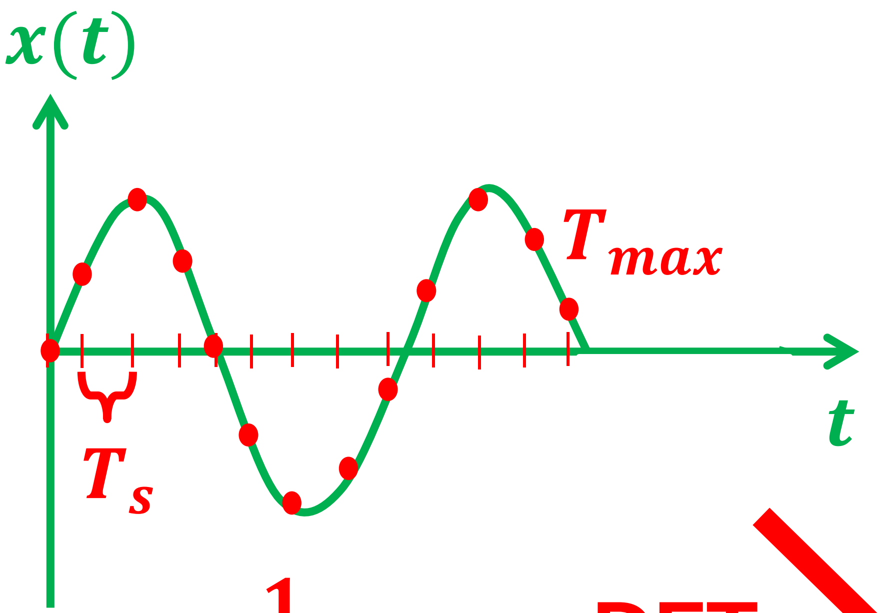
$$T_{max} = (N-1) \times T_s$$

$$F_s = \frac{1}{T_s}$$

Time: **N samples**

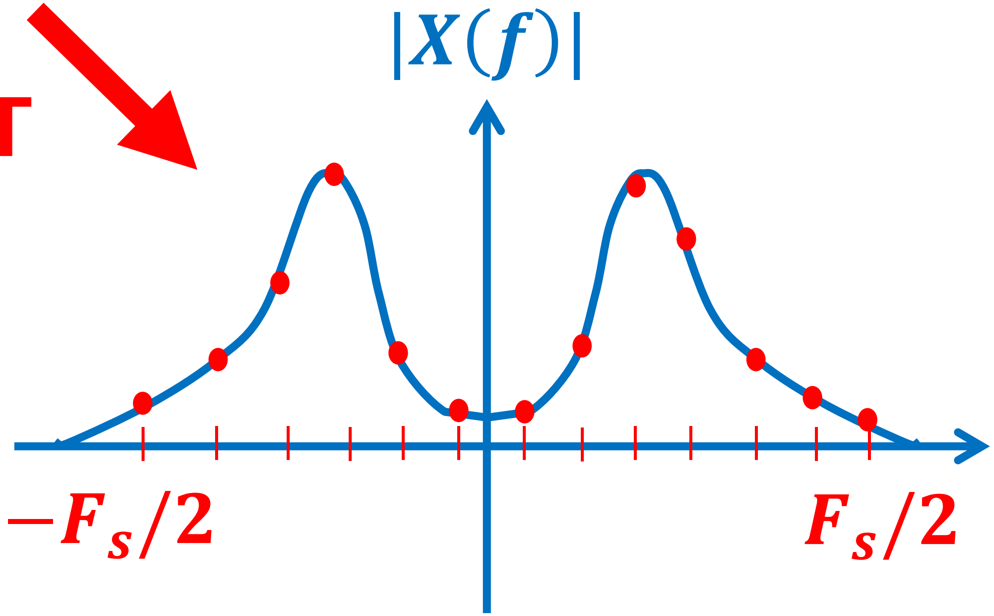
Frequency: **N samples**

Introduction



$$T_{max} = (N - 1) \times T_s$$

DFT

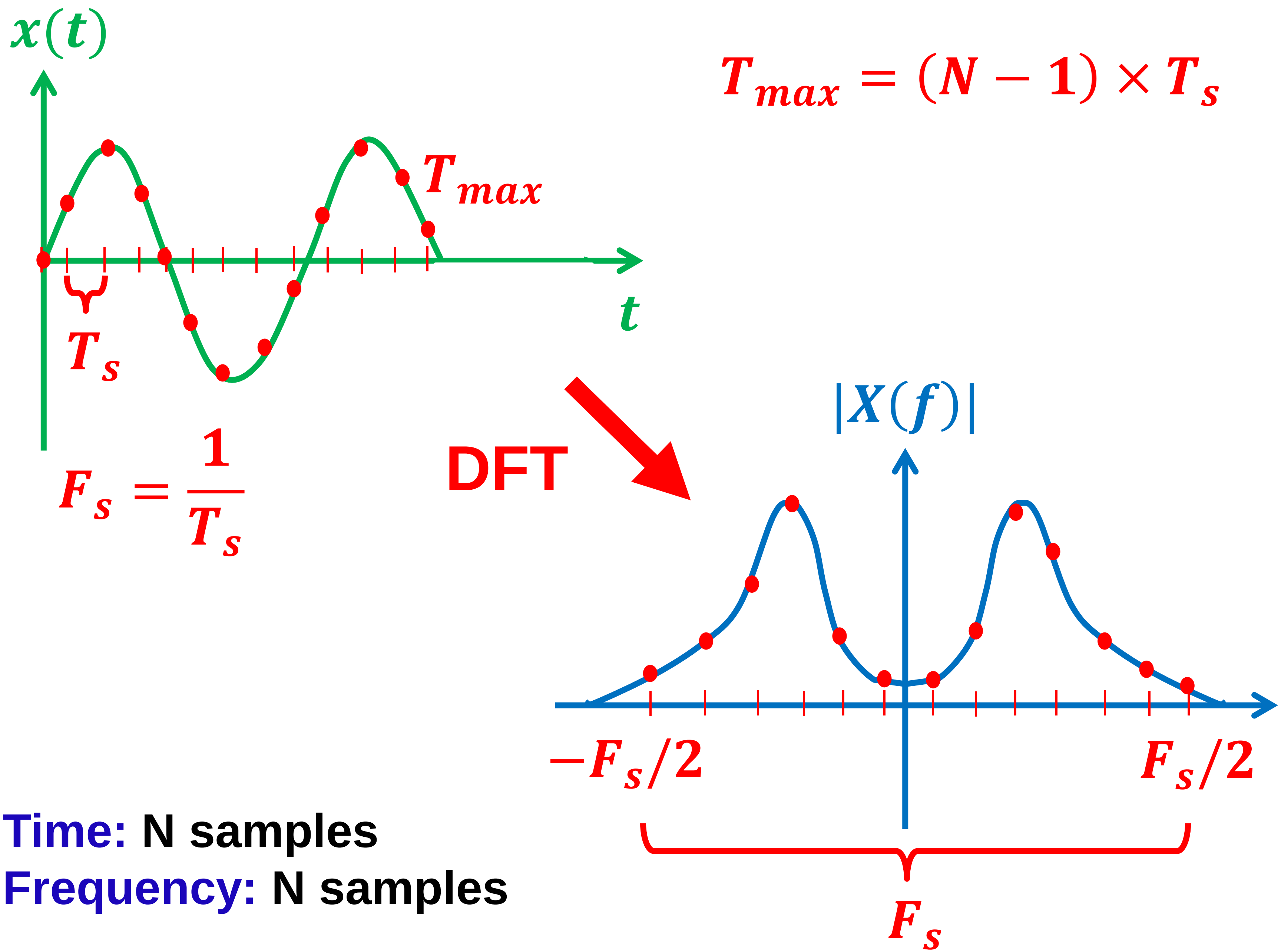


Time: **N samples**

Frequency: **N samples**

From sampling theorem

Introduction



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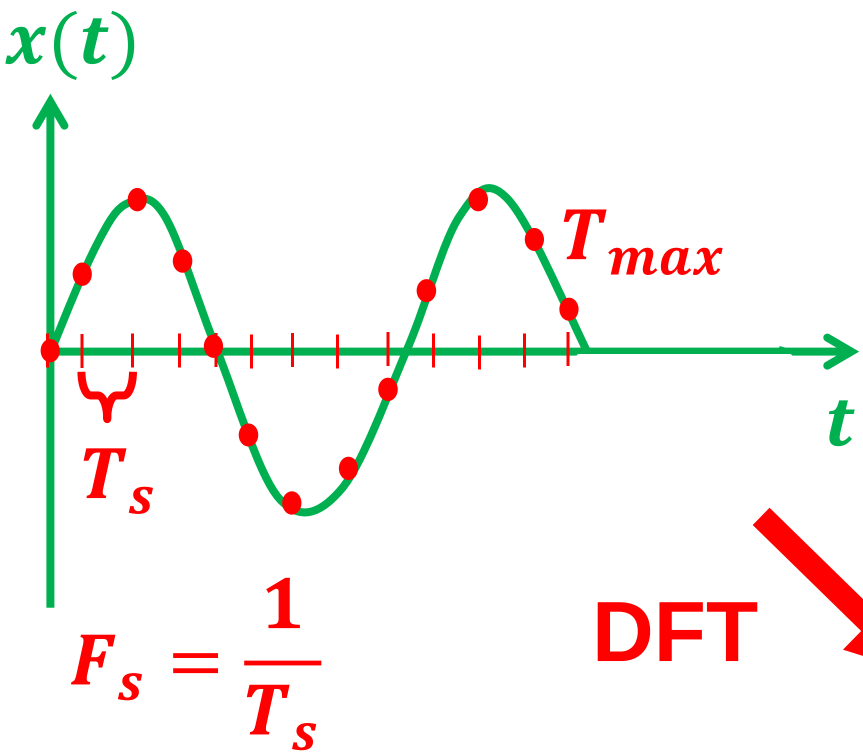

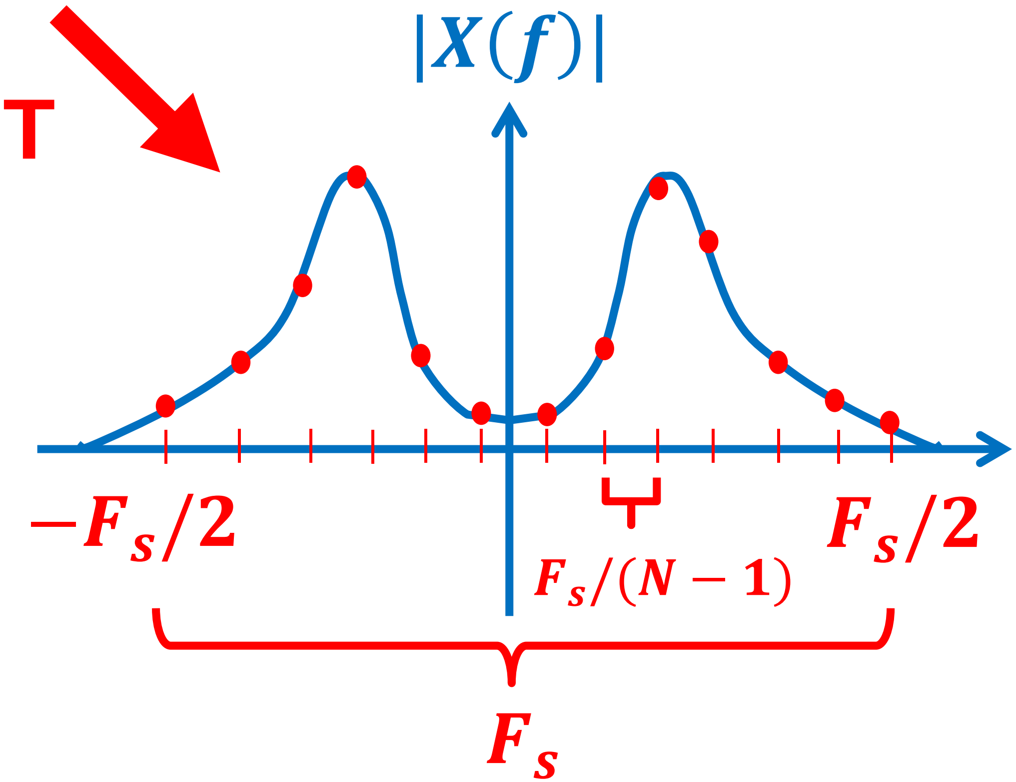

Time: N samples

Frequency: N samples

FFT is just an algorithm to compute DFT efficiently !

In MATLAB:

```
X=fft(x)
or
X=fft(x,n)
```

Example 2.1: pure tone

```
% Sampling frequency
Fs=44100;
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N=N_sec*Fs;
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% Siganal length (in sample)
N=N_sec*Fs;
% Maximum time
Tmax=(N-1)*Ts;
% Time vector
t=0:Ts:Tmax;
```

Example 2.1: pure tone

```
% Signal
x=sin(2*pi*F0.*t);
% Play the sound
sound(x,Fs)
% Plot the sound (show from zero to 60 msec)
figure(1)
plot(t,x,'LineWidth',2.5)
xlim([0 0.06])
```

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% take FFT (without shift)
X1=fft(x);
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% plot result
figure(2)
plot(abs(X1),'LineWidth',2.5);
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You should always perform some post processing operations (shifting, scaling, etc.) to be able to present the results of fft !

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% take FFT (with shift)
X2=fftshift(fft(x));
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figure(3)
plot(F,abs(X2)/N,'LineWidth',2.5);
xlabel('Frequency (Hz)')
title('Double sided magnitude response')
```

Example 2.1: pure tone

fftshift

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% take FFT (with shift)
X2=fftshift(fft(x));
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```

Scaling

Example 2.1: pure tone

```
% take FFT (with shift)
X2=fftshift(fft(x));
% Frequency range
F=-Fs/2:Fs/(N-1):Fs/2;
figure(3)
plot(F,abs(X2)/N,'LineWidth',2.5);
xlabel('Frequency (Hz)')
title('Double sided magnitude response')
```

This is the correct method to graph a "double-sided" (negative and positive) frequency spectrum ☺

Example 2.2: single sided spectrum

```
% Sampling frequency
Fs=44100;
% Sampling period
Ts=1/Fs;
% Signal length (in second)
N_sec=5;
% Signal length (in sample)
N=N_sec*Fs;
% Maximum time
Tmax=(N-1)*Ts;
% Time vector
t=0:Ts:Tmax;
```

Example 2.2: single sided spectrum

```
% pure F0, F1 and F3
F0=600;
F1=1300;
F2=2000;
% Signal
x=sin(2*pi*F0.*t)+...
...0.5*sin(2*pi*F1.*t)+0.2*sin(2*pi*F2.*t);
% Play the sound
sound(x,Fs)
% Plot the sound (show from zero to 60 msec)
figure(1)
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xlim([0 0.06])
```

Example 2.2: single sided spectrum

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% Compute fft
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% Compute fft
X=fft(x);
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X2=abs(X/N);
% Pick the first half
X1=X2(1:N/2+1);
```

Example 2.2: single sided spectrum

```
% Compute fft
X=fft(x);
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X2=abs(X/N);
% Pick the first half
X1=X2(1:N/2+1);
% Multiply by 2 (except the DC part), to compensate
% the removed side from the spectrum.
X1(2:end-1) = 2*X1(2:end-1);
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% Multiply by 2 (except the DC part), to compensate
% the removed side from the spectrum.
X1(2:end-1) = 2*X1(2:end-1);
% Frequency range
F = Fs*(0:(N/2))/N;
```

Example 2.2: single sided spectrum

```
% Plot single-sided spectrum
plot(F,X1,'LineWidth',2.5)
title('Single-Sided Amplitude Spectrum')
xlabel('f (Hz)')
```

Most of the time we use “single-sided” amplitude or phase spectrum !

Example 2.3: siren

```
F0=1300;
F1=200;
F2=1400;
B0=100;
B1=100;
B2=500;
% Signal
x=sin(2*pi*F0.*t+B0*pi*t.^2)+...
sin(2*pi*F1.*t+B1*pi*t.^2)...
+sin(2*pi*F2.*t+B2*pi*t.^2);
```

Example 2.4: voice

```
% read audio file .wav
[x,Fs]=audioread('adult_female_speech.wav');
% play the sound
sound(x,Fs)
% Sampling period
Ts=1/Fs;
% Length of signal
N=length(x);
% Maximum time
Tmax=(N-1)*Ts;
% Time vector
t=0:Ts:Tmax;
…
```

Usable voice frequency band in telephony:

~ 300 Hz to 3400 Hz

How fast is it?

DTFT:

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega n}$$

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Sample ω at $\omega_k = (2\pi/N)k,\ \ k = 0, 1, 2, \cdots N$

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Sample ω **at** $\omega_k = (2\pi/N)k, \;\; k = 0, 1, 2, \cdots N$

DFT:

$$X[k] = X(e^{j\frac{2\pi}{N}k}) = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$$

How fast is it?

DTFT:

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega n}$$

Sample ω at $\omega_k = (2\pi/N)k, \;\; k = 0, 1, 2, \cdots N$

DFT:

$$X[k] = X(e^{j\frac{2\pi}{N}k}) = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$$

For each k:

N complex multiplications,

$N - 1$ complex adds

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$O(N^2)$ computations for direct DFT

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vs.

$O(N \log_2 N)$ computations for fft !

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N	1000	10^6	10^9
N^2	10^6	10^{12}	10^{18}
$N \log_2 N$	10^4	20×10^6	30×10^9

How fast is it?

$O(N^2)$ computations for direct DFT

vs.

$O(N \log_2 N)$ computations for fft !

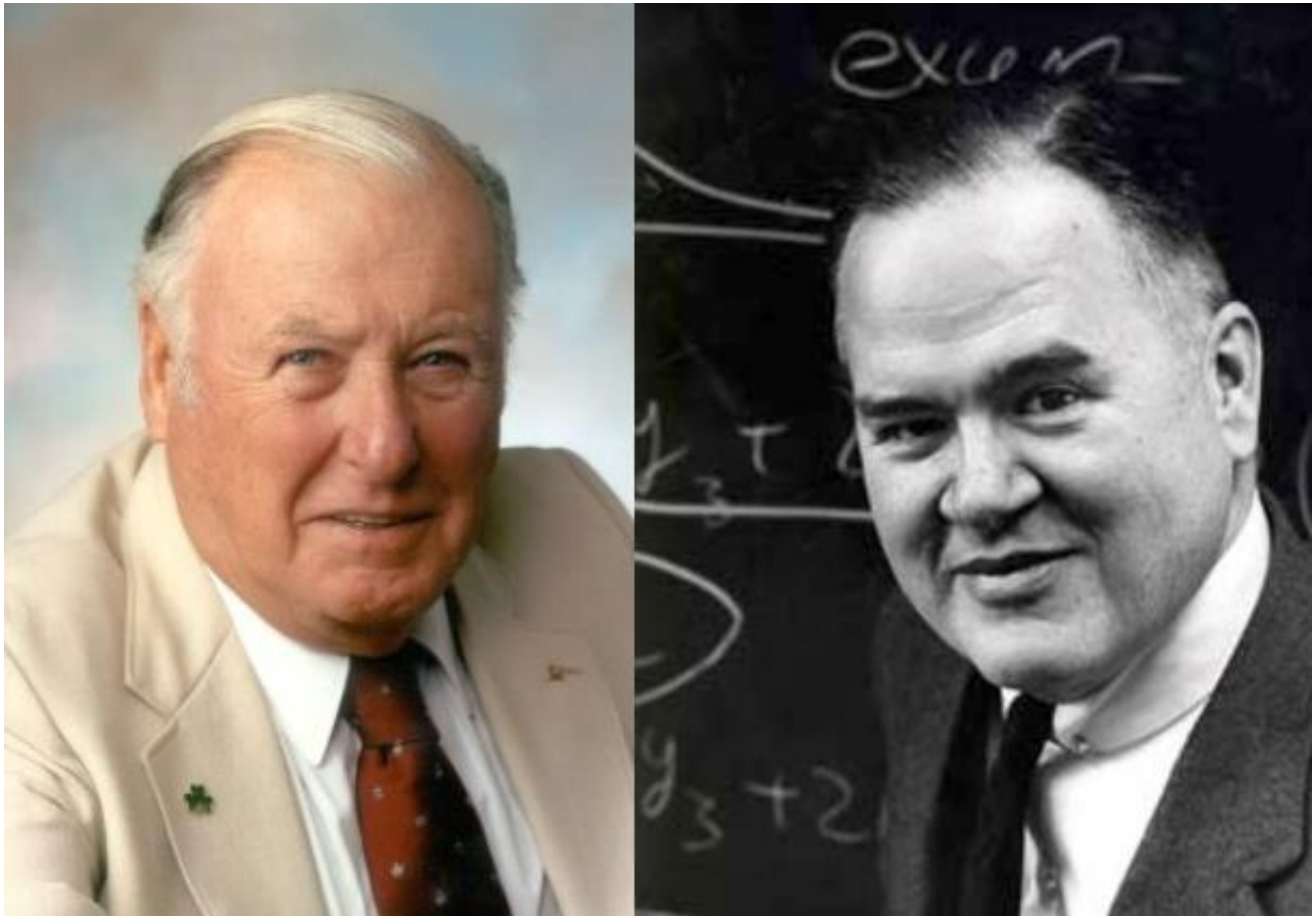
N	1000	10^6	10^9
N^2	10^6	10^{12}	10^{18}
$N \log_2 N$	10^4	20×10^6	30×10^9

$10^{18}\, ns \sim 31.2\ years$

vs.

$30 \times 10^9\, ns \sim 30\ seconds$

First time presented by …



Cooley and Tukey (1965)

First time presented by …

Gauss (1805)

Real-Time application

…

Useful links

- https://www.youtube.com/watch?v=iTMn0Kt18tg
- https://allsignalprocessing.com/fast-fourier-transform-fft-algorithm/
- https://allsignalprocessing.com/discrete-fourier-transform-sampling-the-dtft/
- https://nl.mathworks.com/help/matlab/ref/fft.html