Math 471: Midterm Exam – Wed 10/18/2006

Consider the following heat problem in dimensionless variables:

$$\begin{aligned} &u_t(x,t) = u_{xx}(x,t) + q, && 0 < x < 1, \quad t > 0, \\ &u_x(0,t) = 0, \quad u(1,t) = 1, && t > 0, \\ &u(x,0) = u_0, && 0 < x < 1, \end{aligned}$$

where $q > 0$ and $u_0 > 0$ are constants. This is the heat equation with a source, where the rod is insulated at $x = 0$ and kept at 1 degree at $x = 1$.

1. Derive the steady-state (equilibrium) solution $u^{ss}(x)$. **(10 pts)**

Solution: The steady-state $u^{ss}(x)$ satisfies

$$\begin{aligned} &0 = u^{ss}_{xx}(x) + q, && 0 < x < 1 \\ &u^{ss}_x(0) = 0, \quad u^{ss}(1) = 1. \end{aligned}$$

Integrating the ODE gives $u^{ss}(x) = -q\frac{x^2}{2} + c_1 x + c_2$. Applying the BCs leads to $c_1 = 0$ and $c_2 = 1 + \frac{q}{2}$, hence:

$$u^{ss}(x) = 1 + \frac{q}{2}(1 - x^2).$$

2. Using the steady-state solution, transform the given heat problem for $u(x,t)$ into a problem for a function $v(x,t)$:

$$\begin{aligned} &v_t(x,t) = v_{xx}(x,t), && 0 < x < 1, \quad t > 0, \\ &v_x(0,t) = 0, \quad v(1,t) = 0, && t > 0, \\ &v(x,0) = f(x), && 0 < x < 1, \end{aligned}$$

where $f(x)$ is to be determined by the transformation. **(10 pts)**

Solution: Writing $v(x,t) = u(x,t) - u^{ss}(x)$, *we have*

$$\begin{aligned} v_t &= u_t \\ v_{xx} &= u_{xx} - u^{ss}_{xx} = u_{xx} + q \end{aligned}$$

and hence the PDE becomes

$$v_t = v_{xx}.$$

The BCs become

$$\begin{aligned} v_x(0,t) &= u_x(0,t) - u^{ss}_x(0) = 0 - 0 = 0 \\ v(1,t) &= u(1,t) - u^{ss}(1) = 1 - 1 = 0. \end{aligned}$$

The IC becomes

$$v(x,0) = u(x,0) - u^{ss}(x) = u_0 - 1 - \frac{q}{2}(1 - x^2) = f(x).$$

3. Prove that the problem in terms of $v(x,t)$ has a unique solution. **(20 pts)**

Solution: Consider 2 solutions $v_1(x,t)$ and $v_2(x,t)$ and define $h(x,t) = v_1(x,t) - v_2(x,t)$. Then $h(x,t)$ satisfies

$$\begin{aligned} &h_t(x,t) = h_{xx}(x,t), && 0 < x < 1, \quad t > 0, \\ &h_x(0,t) = 0, \quad h(1,t) = 0, && t > 0, \\ &h(x,0) = 0, && 0 < x < 1. \end{aligned}$$

Define the function

$$H(t) = \int_0^1 h^2(x,t)\,dx.$$

Differentiating in time, we get

$$\frac{dH}{dt} = \int_0^1 2hh_t\,dx = \int_0^1 2hh_{xx}\,dx = 2hh_x|_0^1 - 2\int_0^1 h_x^2\,dx = -2\int_0^1 h_x^2\,dx \leq 0.$$

Also, $H(0) = \int_0^1 0\,dx = 0$ and $H(t) \geq 0$. Thus $H(t) = 0$ for all time, which implies $h(x,t) = 0$ for all space and time, and thus $v_1(x,t) = v_2(x,t)$.

4. Find the solution $v(x,t)$ in the form of an infinite series $v(x,t) = \sum_{n=1}^{\infty} v_n(x,t)$. You may use (without proof) the following integrals in your derivation ($n, m \in \mathbb{Z}$):

$$\int_0^1 \cos\left(\frac{2m-1}{2}\pi x\right)\cos\left(\frac{2n-1}{2}\pi x\right)\,dx = \begin{cases} \frac{1}{2} & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

$$\int_0^1 \cos\left(\frac{2m-1}{2}\pi x\right)\,dx = \frac{2(-1)^{m+1}}{(2m-1)\pi}$$

$$\int_0^1 (1-x^2)\cos\left(\frac{2m-1}{2}\pi x\right)\,dx = \frac{16(-1)^{m+1}}{(2m-1)^3\pi^3}$$

(40 pts)

Solution: We separate variables $u(x,t) = X(x)T(t)$ and substitute in the PDE to obtain

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda$$

where λ is a constant. The Sturm-Liouville problem for $X(x)$ is

$$X'' + \lambda X = 0, \quad X'(0) = 0 = X(1),$$

whose solution is

$$X_n(x) = B_n \cos\left(\sqrt{\lambda_n} x\right), \quad \lambda_n = (2n-1)^2\pi^2/4, \quad n = 1, 2, \ldots$$

The equations for $T(t)$ are

$$T' + \lambda_n T = 0, \quad n = 1, 2, \ldots$$

and thus

$$T_n(t) = C_n e^{-(2n-1)^2\pi^2 t/4}, \quad n = 1, 2, \ldots,$$

which leads to the solutions $v_n(x,t)$ to the PDE (with BCs):

$$v_n(x,t) = X_n(x)T_n(t) = A_n \cos\left(\frac{2n-1}{2}\pi x\right) e^{-(2n-1)^2\pi^2 t/4}, \quad n = 1, 2, \ldots$$

Summing all $v_n(x,t)$ together gives

$$v(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{2n-1}{2}\pi x\right) e^{-(2n-1)^2\pi^2 t/4}.$$

Imposing the IC gives

$$v(x,0) = u_0 = \sum_{n=1}^{\infty} A_n \cos\left(\frac{2n-1}{2}\pi x\right).$$

Using the given orthogonality relation leads to

$$A_n = 2\int_0^1 f(x)\cos\left(\frac{2n-1}{2}\pi x\right)\,dx = 2\int_0^1 \left(u_0 - 1 - \frac{q}{2}(1-x^2)\right)\cos\left(\frac{2n-1}{2}\pi x\right)\,dx,$$

and thus, using the given integrals,

$$A_n = 2(u_0 - 1)\frac{2(-1)^{n+1}}{(2n-1)\pi} - q\frac{16(-1)^{n+1}}{(2n-1)^3\pi^3}, \quad n = 1, 2, \ldots$$

5. Without doing any computations, explain how you would prove that the series solution $v(x,t) = \sum_{n=1}^{\infty} v_n(x,t)$ converges uniformly for $0 < x < 1$. **(5 pts)**

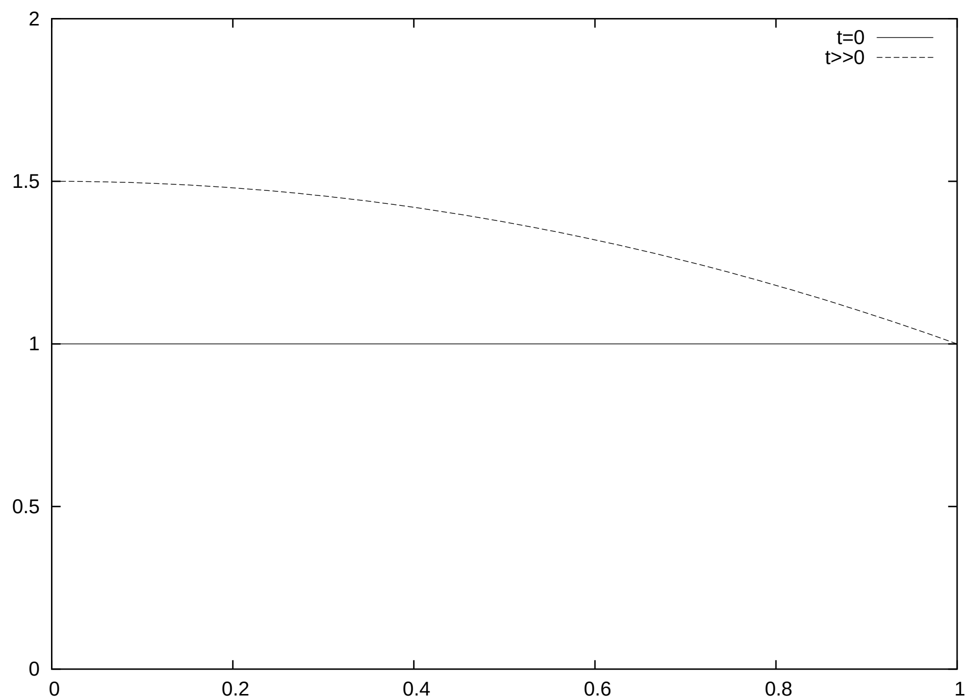
Solution: Bound each $v_n(x,t)$ on $[0,1]$ by M_n. If $\sum_{n=1}^{\infty} M_n$ converges absolutely (which can be shown using the ratio test), then $\sum_{n=1}^{\infty} v_n(x,t)$ converges uniformly (by the M-test theorem).

6. Find the solution $u(x, t)$. **(5 pts)**

Solution: $u(x, t) = u^{ss}(x) + v(x, t) = \dots$

7. Assuming that $u_0 = 1$ and $q = 1$, sketch the spatial (in x) temperature profile $u(x,t)$ for $t = 0$ and $t \to \infty$. **(5 pts)**

Solution:



8. Briefly (in a couple of sentences) explain how you would solve the same heat problem if the boundary condition on the right end of the rod was changed to $u(1,t) = \cos(t)$. **(5 pts)**

Solution: Look for a solution in the form $u(x,t) = v(x,t) + u^{ss}(x,t)$*, where* $v(x,t)$ *is a transient satisfying a PDE with homogeneous BCs and* $u^{ss}(x,t) = A(x)\cos(t + \phi(x))$ *is a quasi steady-state.*