KU LEUVEN

Numerical methods in Energy Sciences - H9x34A
Finite elements and electromagnetic fields

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Overview
Electromagnetism (EM) & Finite elements (FEs)

- electromagnetic field **models**
  - statics: electrostatics, electrokinetics, magnetostatics
  - quasi-statics: electrodynamics, magnetodynamics
    (time harmonic/frequency domain vs transient/time domain)
  - full-wave, wave propagation
- models of electromagnetic **sources**
  - inductors/coils (stranded & massive ⬄ eddy current effects in conductors)
  - permanent magnets
- **non-linear** material laws (saturation, hysteresis)
- **formulations** (in terms of potentials, fields)

Overview (cont'd)
Electromagnetism (EM) & Finite elements (FEs)

- computation of **global quantities**
  - current and voltage
  - lumped circuit elements (resistance, inductance, capacitance)
  - flux linkage, losses (eddy current losses, iron losses), magn. energy
  - force computation (magnetic/electric), torque
- **coupled models**, interaction with other physics
  - field-circuit coupling via local and global quantities
  - electro- or magneto-mechanical models, motion of rigid bodies (linear/rotational)
  - electro-thermal models

Literature

- **G. Meunier, The Finite Element Method for Electromagnetic Modeling, 2008.**
- J.P.A. Bastos, N. Sadowski, Electromagnetic Modeling by Finite Element Methods, 2003.
- P.P. Silvester, R.L. Ferrari, Finite Elements for Electrical Engineers, 1996.
- S. Humphries, Finite-element Methods for Electromagnetics (http://www.fieldp.com/femethods.html), 1997.

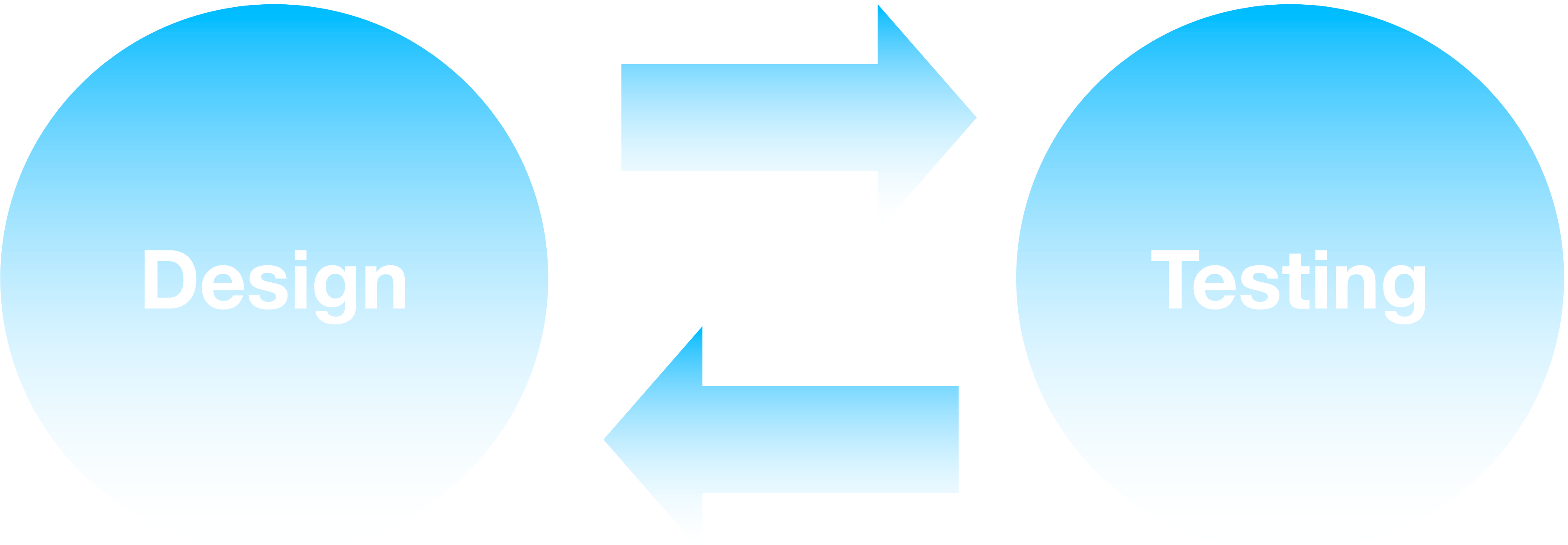
- **R. Hiptmair, Maxwell's Equations: Continuous and Discrete, In Computational electromagnetism, Springer, 2014** (https://link.springer.com/chapter/10.1007/978-3-319-19306-9_1).
- F. Assous, P. Ciarlet, S. Labrunie, Mathematical Foundations of Computational Electromagnetism, volume 198, in Applied Mathematical Sciences, Springer 2018.
- P. Monk, Finite Element Methods for Maxwell's equations, 2006.

If you need to freshen up electromagnetic theory:

- **The Feynman Lectures on Physics, Volume II** – electromagnetism and matter (https://www.feynmanlectures.caltech.edu/II_toc.html)
- N. Ida, Engineering Electromagnetism, Springer 2015 (https://www.springer.com/gp/book/9783030155568).

About modern product development

Development cycle

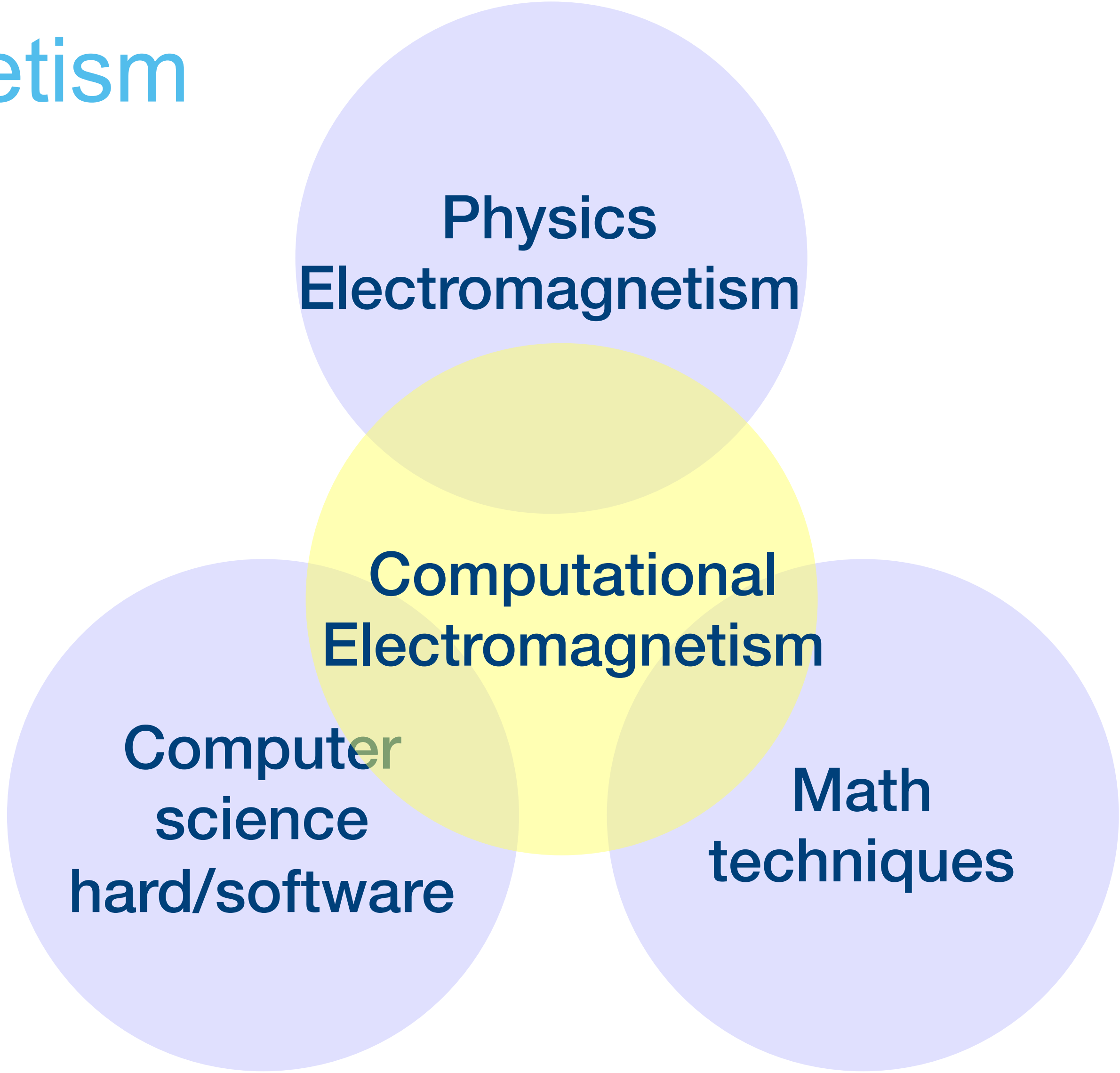


how is testing performed?

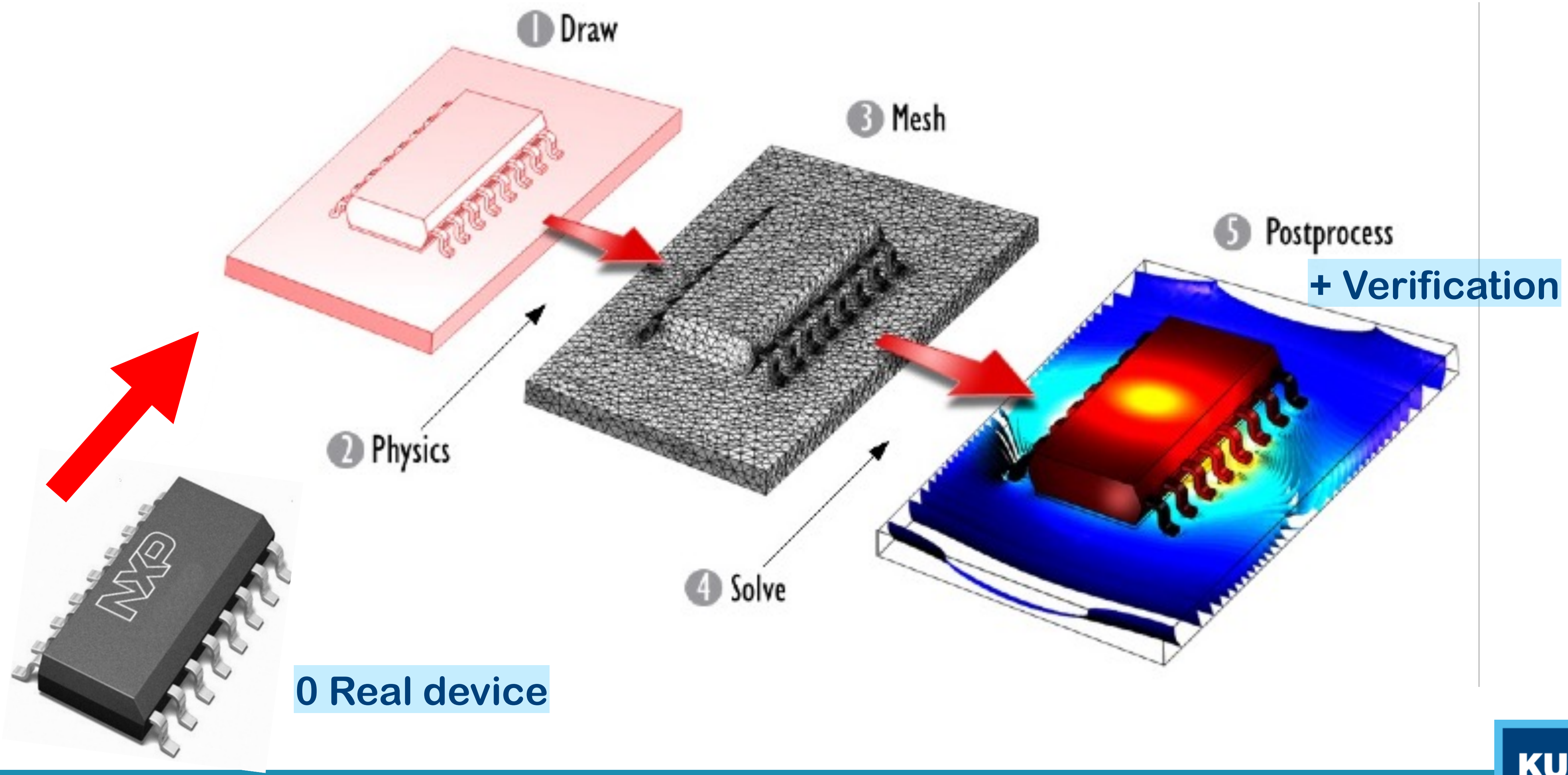
physical vs. numerical

Computational electromagnetism

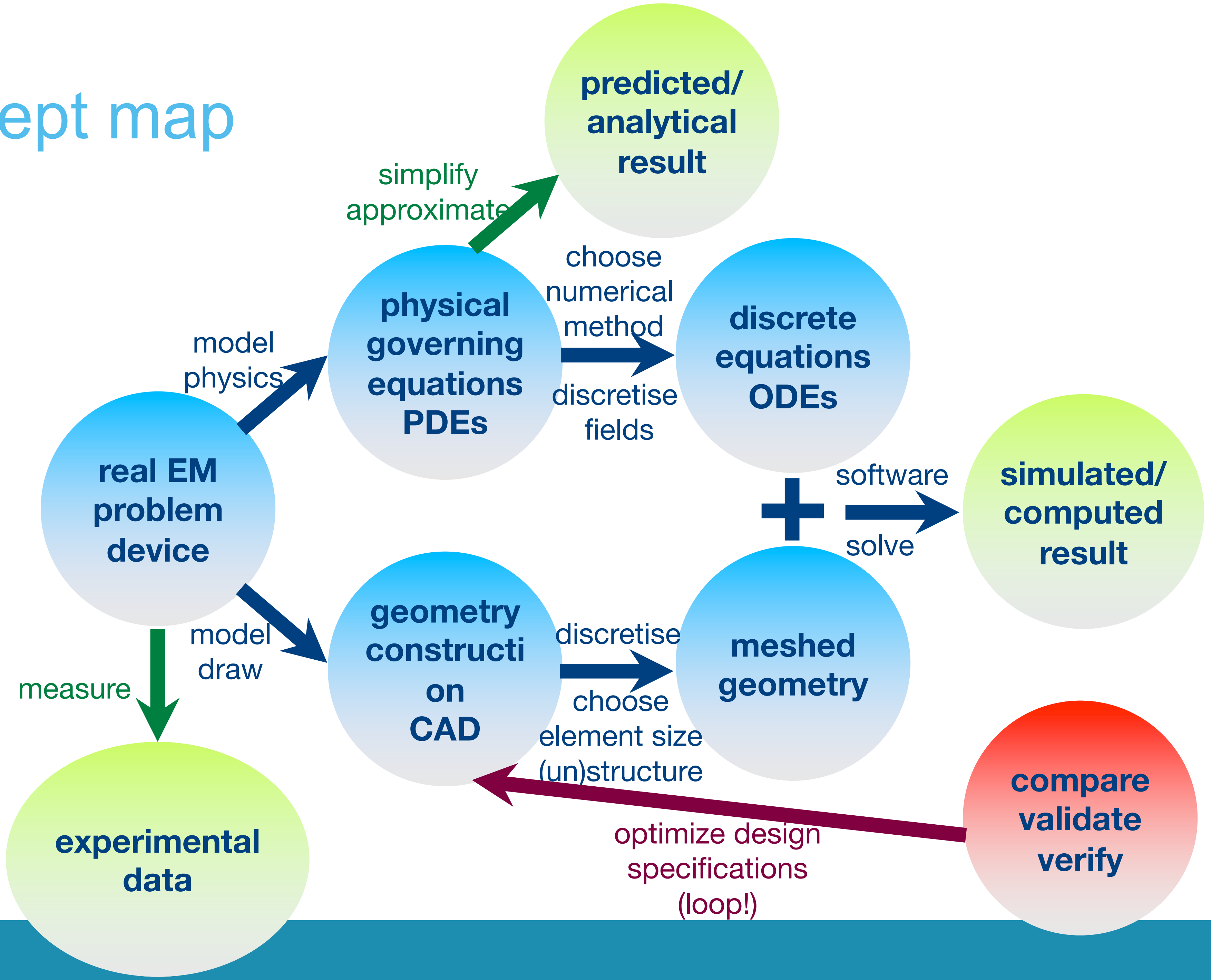
- find the solution to real-world electromagnetic problems by numerical simulation
- aim at analyzing and designing electrical devices
  - complex geometry
  - different materials, possibly nonlinear
  - dynamical behavior - time dependence, non-static field sources
  - multi-physics - phenomena of different physical nature (electromagnetic, thermal, mechanical, ...)



Stages in a typical FE analysis



Concept map



predicted/ analytical result
simplify
approximate
choose numerical method
discretise fields
physical governing equations PDEs
discrete equations ODEs
model physics
real EM problem device
software
solve
simulated/ computed result
model draw
measure
geometry constructi on CAD
discretise
choose element size (un)structure
meshed geometry
experimental data
compare validate verify
optimize design specifications (loop!)

Common mistakes when using FE software

- Doing analysis for the sake of it — keep always in mind the **end requirements**
- Lack of **verification** (measurements, analytical solution/approximation, another numerical method)
- Wrong element: **inefficient FE** type (linear, higher order) or model (3D vs 2D planar/axisymmetric)
- Ignoring
  - geometry or boundary condition approximations
  - discretization errors (in space and time: mesh, time step)
- **Bad post-processing** — check local, not only global (average) quantities
- Non-standard FE analysis to be derived from standard procedures when possible
- Inadequate **archiving** (teamwork), lack of documentation

Electromagnetic theory - review

Maxwell (1831-1879)
Ampere (1775-1836)
Gauss (1777-1855)
Faraday (1791-1867)
Lenz (1804-1865)

Maxwell's equations

$$\operatorname{curl} \boldsymbol{h} - \partial_t \boldsymbol{d} = \boldsymbol{j}$$
Maxwell–Ampère's equation

$$\operatorname{curl} \boldsymbol{e} + \partial_t \boldsymbol{b} = 0$$
Faraday's equation

$$\operatorname{div} \boldsymbol{b} = 0$$
$$\operatorname{div} \boldsymbol{d} = q$$
Conservation laws

$$\boldsymbol{b} = \mu \boldsymbol{h}$$
$$\boldsymbol{j} = \sigma \boldsymbol{e}$$
$$\boldsymbol{d} = \epsilon \boldsymbol{e}$$

+ constitutive laws expression the EM behavior of materials

linear, isotropic

$\boldsymbol{h}$ magnetic field [A/m]
$\boldsymbol{b}$ magnetic flux density or induction [T]
$\boldsymbol{j}$ current density [$\mathrm{A/m^2}$]

$\boldsymbol{e}$ electric field [V/m]
$\boldsymbol{d}$ displacement current density [$\mathrm{C/m^2}$]
q charge density [$\mathrm{C/m^3}$]

$$\boldsymbol{h}(x,y,z,t) = h_x(x,y,z,t)\hat{\boldsymbol{x}} + h_y(x,y,z,t)\hat{\boldsymbol{y}} + h_z(x,y,z,t)\hat{\boldsymbol{z}}$$

$\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$: unit vectors along x, y, z axes

$$\operatorname{grad} f_0 \equiv \nabla f_0 = (\partial_x, \partial_y, \partial_z) f_0$$
$$\operatorname{curl} \boldsymbol{f}_1 \equiv \nabla \times \boldsymbol{f}_1 \equiv (\partial_x, \partial_y, \partial_z) \times \boldsymbol{f}_1$$
$$\operatorname{div} \boldsymbol{f}_2 \equiv \nabla \cdot \boldsymbol{f}_2 \equiv (\partial_x, \partial_y, \partial_z) \cdot \boldsymbol{f}_2$$

Maxwell's equations (cont'd)
Differential forms

v		$\text{0-form} = \text{scalar, continuous field}$
	grad ↓	
$\boldsymbol{h}, \boldsymbol{e}$		$\text{1-form} = \text{vector field of continuous tangential component}$
	curl ↓	
$\boldsymbol{b}, \boldsymbol{j}, \boldsymbol{d}$		$\text{2-form} = \text{vector field of continuous normal component}$
	div ↓	
q		$\text{3-form} = \text{scalar field}$

Operation d (grad, curl, div) is the exterior derivative.
Applying d to a (k-1)-form gives a (k-form)

$$\operatorname{curl} \boldsymbol{h} - \partial_t \boldsymbol{d} = \boldsymbol{j}$$
$$\operatorname{curl} \boldsymbol{e} + \partial_t \boldsymbol{b} = 0$$
$$\operatorname{div} \boldsymbol{b} = 0$$
$$\operatorname{div} \boldsymbol{d} = q$$

Notation: differential operators

$$\operatorname{grad} f_0 \equiv \nabla f_0 = (\partial_x, \partial_y, \partial_z) f_0$$
$$\operatorname{curl} \boldsymbol{f}_1 \equiv \nabla \times \boldsymbol{f}_1 \equiv (\partial_x, \partial_y, \partial_z) \times \boldsymbol{f}_1$$
$$\operatorname{div} \boldsymbol{f}_2 \equiv \nabla \cdot \boldsymbol{f}_2 \equiv (\partial_x, \partial_y, \partial_z) \cdot \boldsymbol{f}_2$$

Maxwell's equations (cont'd)

$$\boldsymbol{n} \times (\boldsymbol{h}_2 - \boldsymbol{h}_1)|_{\partial S} = \boldsymbol{j}_s$$
$$\boldsymbol{n} \times (\boldsymbol{e}_2 - \boldsymbol{e}_1)|_{\partial S} = 0$$
$$\boldsymbol{n} \cdot (\boldsymbol{b}_2 - \boldsymbol{b}_1)|_{\partial S} = 0$$
$$\boldsymbol{n} \cdot (\boldsymbol{d}_2 - \boldsymbol{d}_1)|_{\partial S} = \rho_s$$

Transmission conditions

Differential forms

$$\operatorname{curl} \boldsymbol{h} - \partial_t \boldsymbol{d} = \boldsymbol{j}$$
$$\operatorname{curl} \boldsymbol{e} + \partial_t \boldsymbol{b} = 0$$

Stokes theorem →

$$\operatorname{div} \boldsymbol{b} = 0$$
$$\operatorname{div} \boldsymbol{d} = q$$

Divergence theorem →

Integral forms

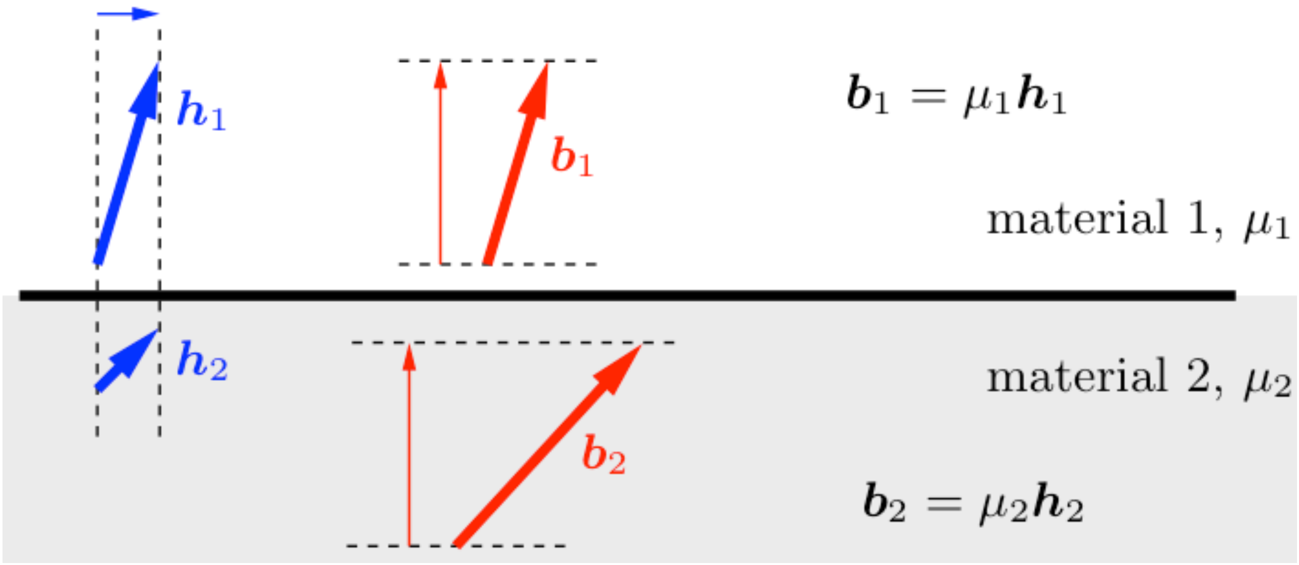
$$\oint_{\partial S} \boldsymbol{h} \cdot \mathrm{d}\boldsymbol{l} = \int_S (\boldsymbol{j} + \partial_t \boldsymbol{d}) \cdot \boldsymbol{n} \, \mathrm{d}S \quad [A]$$
$$\oint_{\partial S} \boldsymbol{e} \cdot \mathrm{d}\boldsymbol{l} = -\int_S \partial_t \boldsymbol{b} \cdot \boldsymbol{n} \, \mathrm{d}S \quad [V]$$
$$\oint_{\partial V} \boldsymbol{b} \cdot \boldsymbol{n} \, \mathrm{d}S = 0$$
$$\oint_{\partial V} \boldsymbol{d} \cdot \boldsymbol{n} \, \mathrm{d}S = \int_V q \, \mathrm{d}V \quad [C]$$

Stokes theorem $\int_S \operatorname{curl} \boldsymbol{a} \, \mathrm{dS} = \oint_{\mathrm{l} \equiv \partial \mathrm{S}} \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l}$

$\mathrm{d}\boldsymbol{l} = \boldsymbol{\tau} \mathrm{d}l$

Divergence theorem
Gauss law $\int_V \operatorname{div} \boldsymbol{a} \, \mathrm{d}\Omega = \oint_{S \equiv \partial V} \boldsymbol{a} \cdot \boldsymbol{n} \, \mathrm{d}S$

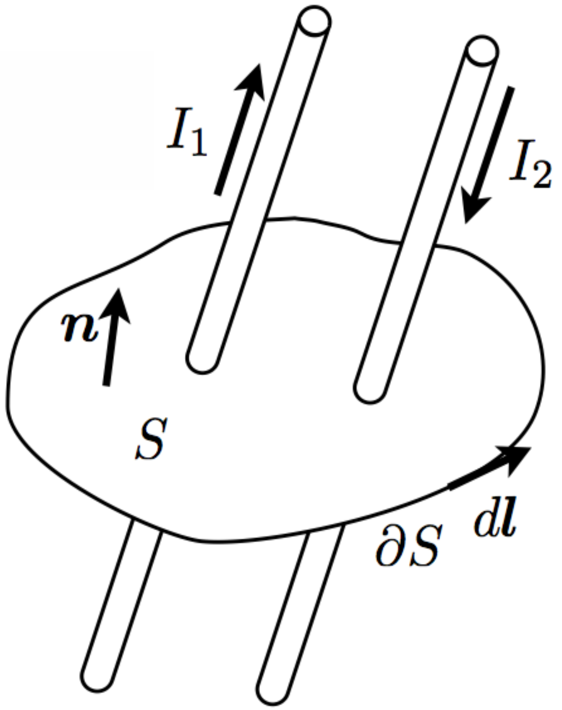
Maxwell's equations (cont'd)
Interface or transmission conditions



$$\boldsymbol{n} \times (\boldsymbol{h}_2 - \boldsymbol{h}_1)|_{\partial S} = \boldsymbol{j}_s$$
$$\boldsymbol{n} \times (\boldsymbol{e}_2 - \boldsymbol{e}_1)|_{\partial S} = 0$$
$$\boldsymbol{n} \cdot (\boldsymbol{b}_2 - \boldsymbol{b}_1)|_{\partial S} = 0$$
$$\boldsymbol{n} \cdot (\boldsymbol{d}_2 - \boldsymbol{d}_1)|_{\partial S} = \rho_s$$

Integral form—Ampère’s law

$$\operatorname{curl} \boldsymbol{h} = \boldsymbol{j} \qquad \Rightarrow \oint_{\partial S} \boldsymbol{h} \cdot d\boldsymbol{l} = I$$



Magnetomotive force (m.m.f.) [A]

The circulation of the magnetic field along a closed curve equals the algebraic sum of currents crossing the underlying surface

Conservation of current

$$\operatorname{div} \boldsymbol{j} = -\partial_t \rho_v$$

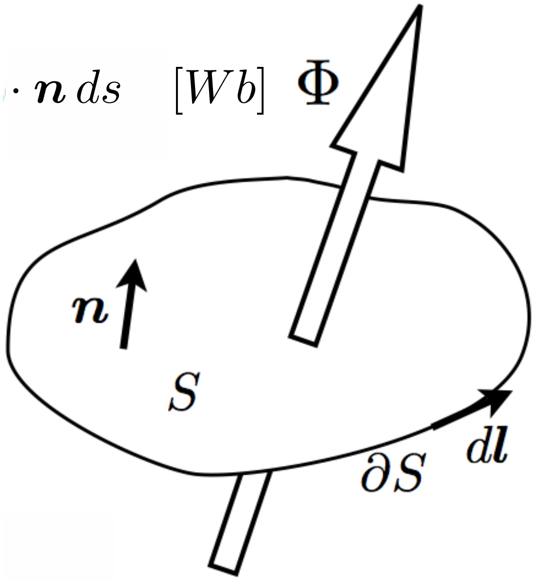
$$\operatorname{div} \boldsymbol{j} = 0 \text{ if } \rho_{\mathrm{v}} = \mathrm{cte} \Rightarrow \oint_{\partial \mathrm{V}} \boldsymbol{j} \cdot \boldsymbol{n} \, \mathrm{ds} = 0$$

The sum of currents arriving to a node is zero

Integral form—Faraday's law

$$\Phi = \int_S \boldsymbol{b} \cdot \boldsymbol{n}\, ds \quad [Wb]$$

$$\mathrm{curl}\, \boldsymbol{e} = -\partial_t \boldsymbol{b} \quad \Rightarrow \oint_{\partial S} \boldsymbol{e} \cdot d\boldsymbol{l} = -\partial_t \Phi$$



Electromotive force (e.m.f.)

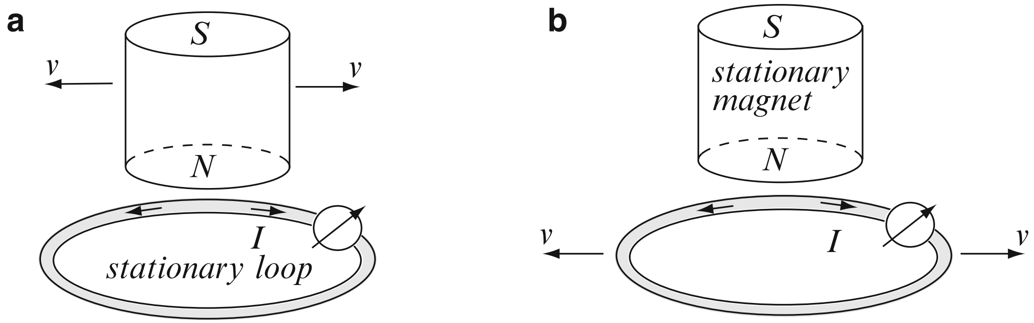
Any variation of magnetic flux through a circuit gives rise to an electromotive force

$$e.m.f. = -\partial_t \Phi \quad [\mathrm{V}]$$

$$e.m.f. = -N\partial_t \Phi \quad [\mathrm{V}]$$

For a circuit moving at speed v:

$$e.m.f. = \oint_{\partial S} \boldsymbol{f}/q \cdot d\boldsymbol{l} = \oint_{\partial S} (\boldsymbol{e} + \boldsymbol{v} \times \boldsymbol{b}) \cdot d\boldsymbol{l} = -\int_S \partial_t \boldsymbol{b} \cdot \boldsymbol{n}\, ds \quad [\mathrm{V}]$$

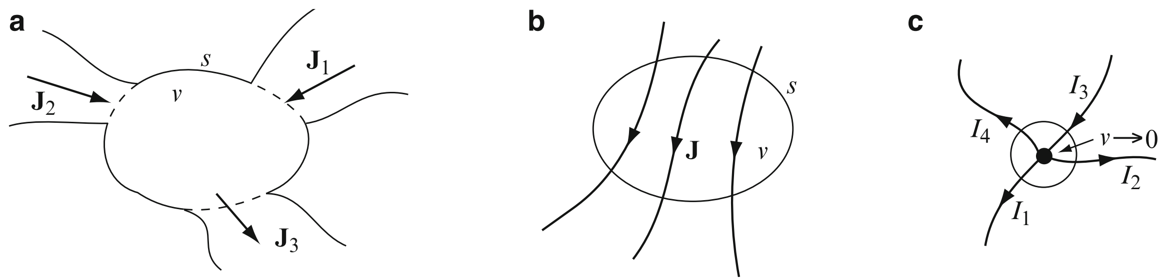


Conservation of flux density

$$\mathrm{div}\, \boldsymbol{b} = 0 \Rightarrow \oint_{\partial V} \boldsymbol{b} \cdot \boldsymbol{n}\, ds = 0$$

Magnetic flux lines are closed

Integral forms—Conservation laws



Conservation of current

$$\operatorname{div} \boldsymbol{j} = 0 \Rightarrow \oint_{\partial V} \boldsymbol{j} \cdot \boldsymbol{n} \, ds = 0$$

The sum of currents arriving to a node is zero

Conservation of flux density

$$\operatorname{div} \boldsymbol{b} = 0 \Rightarrow \oint_{\partial V} \boldsymbol{b} \cdot \boldsymbol{n} \, ds = 0$$

Magnetic flux lines are closed

Gauss law (conservation of charge)

$$\operatorname{div} \boldsymbol{d} = q \Rightarrow \oint_{\partial V} \boldsymbol{d} \cdot \boldsymbol{n} \, ds = \int_V q \, dv \quad [\mathrm{C}]$$

Total charge contained inside a volume

Electromagnetic power

Poynting vector [W/m²] $\boldsymbol{s} = \boldsymbol{e} \times \boldsymbol{h}$

Poynting theorem – Power exchanged with a volume (interior normal) =

$$P = \oint_{\partial V} \boldsymbol{s} \cdot \boldsymbol{n} \, \mathrm{d}s = - \int_V \operatorname{div} \boldsymbol{s} \, \mathrm{d}v = \int_V p \, \mathrm{d}v$$

$$= \int_V \boldsymbol{e} \cdot \boldsymbol{j} \, \mathrm{d}v + \int_V \boldsymbol{h} \cdot \partial_t \boldsymbol{b} \, \mathrm{d}v + \int_V \boldsymbol{e} \cdot \partial_t \boldsymbol{d} \, \mathrm{d}v$$

Total dissipated power due to ohmic losses

Rate of decrease due to total electromagnetic energy stored in V

Power density

$$p = -\operatorname{div} \boldsymbol{e} \times \boldsymbol{h} = -\boldsymbol{h} \cdot \operatorname{curl} \boldsymbol{e} + \boldsymbol{e} \cdot \operatorname{curl} \boldsymbol{h}$$

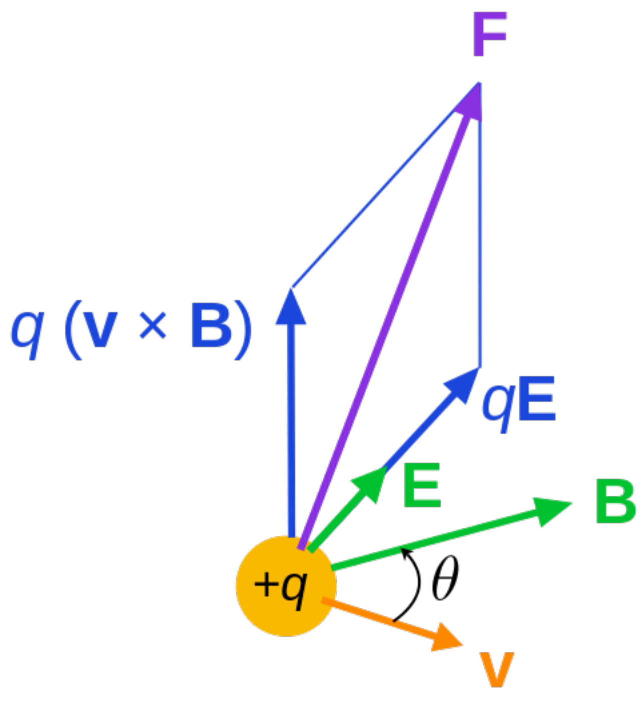
$$= \boldsymbol{e} \cdot \boldsymbol{j} + \boldsymbol{h} \cdot \partial_t \boldsymbol{b} + \boldsymbol{e} \cdot \partial_t \boldsymbol{d}$$

conduction magnetic electric

Lorentz-Coulomb force law

The Lorentz-Coulomb force is given by the interaction of electromagnetic fields with a punctual charge moving at a certain speed v

$$\boldsymbol{f} = \boldsymbol{f}_e + \boldsymbol{f}_m = q\,\boldsymbol{e} + q\,\boldsymbol{v} \times \boldsymbol{b} \quad [\mathrm{N}]$$



The magnetic force acting on a current-carrying wire is a variation of Lorentz formula, often called Laplace force:

$$\boldsymbol{f} = \boldsymbol{j} \times \boldsymbol{b} = \mathrm{curl}\,\boldsymbol{h} \times \boldsymbol{b} \quad [\mathrm{N}]$$

Material constitutive relations

$$b = \mathcal{B}(e, h) = \mu h \ (+b_r)$$ Magnetic relation

$$d = \mathcal{D}(e, h) = \epsilon e \ (+d_{src})$$ Electric relation

$$j = \mathcal{J}(e, h) = \sigma e \ (+j_{src})$$ Ohm's law

Characteristics of materials

- constants (linear isotropic materials)
- functions of electromagnetic fields (nonlinear materials)
- tensors (anisotropic materials)
- functions of other physical fields (temperature...)

μ magnetic permeability (H/m)
ϵ dielectric permittivity (F/m)
σ electric conductivity (S/m)

Sources

b_r remanent induction (magnets)
d_{src} source electric flux density
j_{src} source current density (inductors)

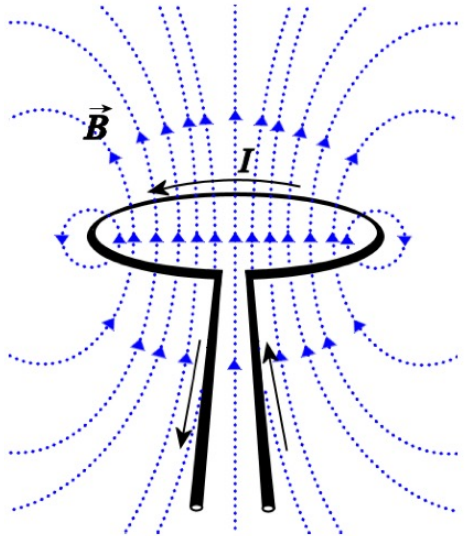
Magnetic constitutive relation

$$\boldsymbol{b} = \mu_0(\boldsymbol{h}+\boldsymbol{m}) = \mu_0(\boldsymbol{h}+\chi_m \boldsymbol{h}) = \mu_0\mu_r \boldsymbol{h}$$

$\mu_r = 1 + \chi_m$ relative magnetic permeability

$\mu = \mu_0\mu_r,\ \mu_0 = 4\pi 10^{-7}\ \mathrm{H/m}$ vacuum permeability

magnetic moment

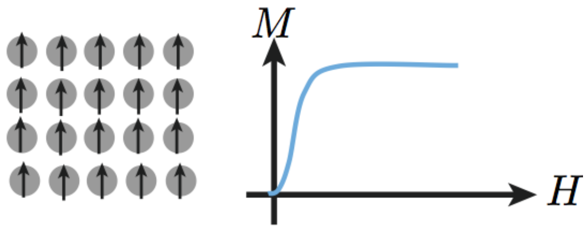


Diamagnetic and paramagnetic materials: $\mu_r \approx 1$

- linear (Ag, Cu, Al)

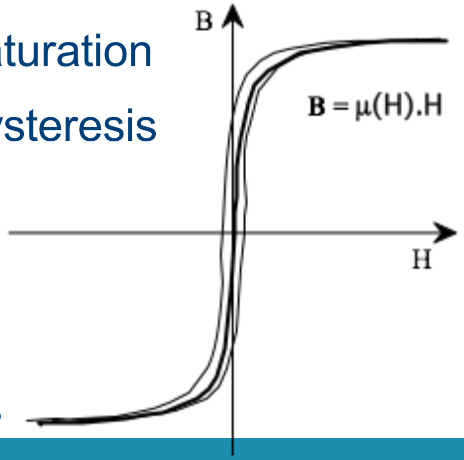
Ferromagnetic materials: $\mu_r \gg 1,\ \mu_r = \mu_r(h)$

- nonlinear (steel, iron)
- Ferromagnetic—paramagnetic transition for $T > T_{\text{Curie}}$

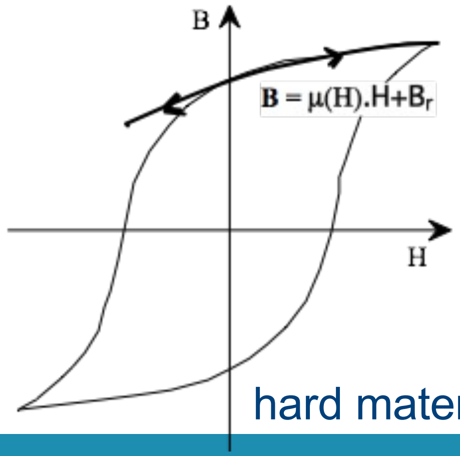


Saturation

Hysteresis



soft materials

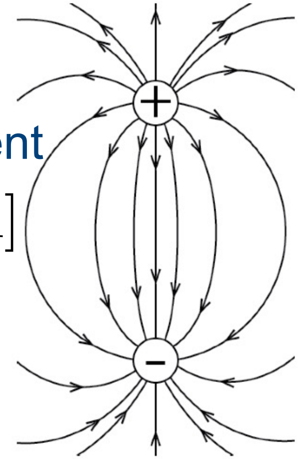


hard materials

Dissipated energy proportional to the area of the loops

Dielectric constitutive relation

field lines
dipole moment

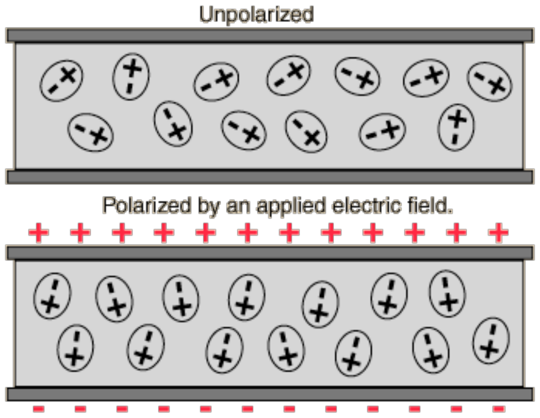


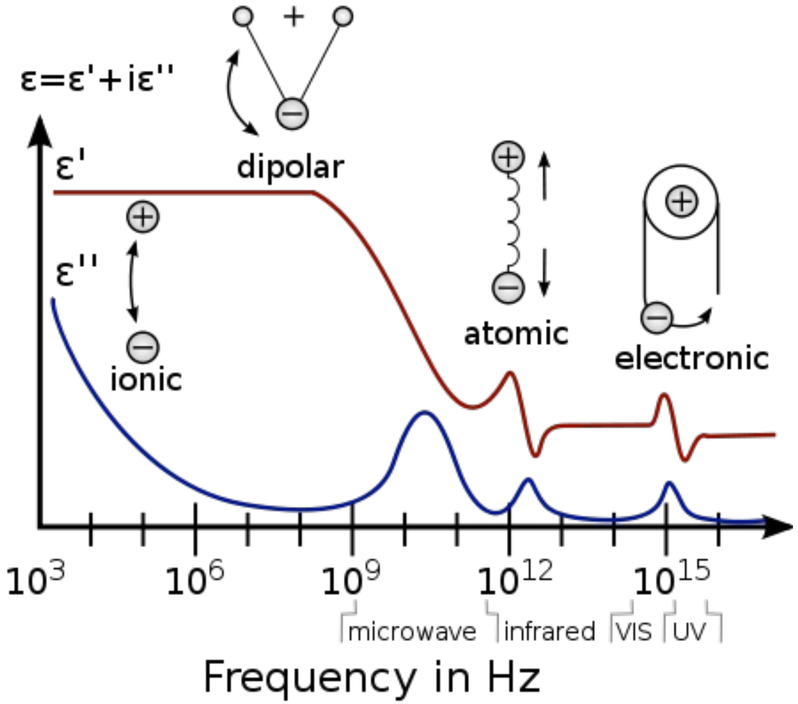
$$p = q\,d \quad [\mathrm{C}\cdot\mathrm{m}]$$

$$\boldsymbol{d} = \epsilon_0 \boldsymbol{e} + \boldsymbol{p} = \epsilon_0(\boldsymbol{e} + \chi_e \boldsymbol{e}) = \epsilon_0 \epsilon_r \boldsymbol{e} = \epsilon \boldsymbol{e}$$

$\epsilon_r = 1 + \chi_e$ relative dielectric permittivity

$\epsilon = \epsilon_0 \epsilon_r, \ \epsilon_0 = \dfrac{1}{\mu_0 c_0^2} = 8.854187\,10^{-12}\ \mathrm{F/m}$ vacuum permittivity





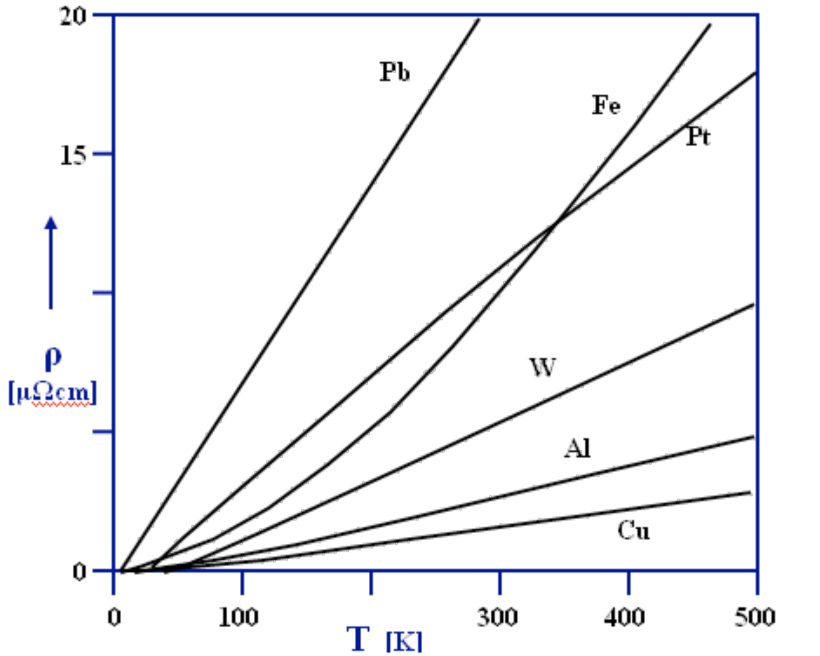
Air	1.0006
Teflon	2.1
Polyethylene	2.25
Paper	3.85
Glass	3.7 - 10
Concrete	4.5
Water	80

Ohm's law

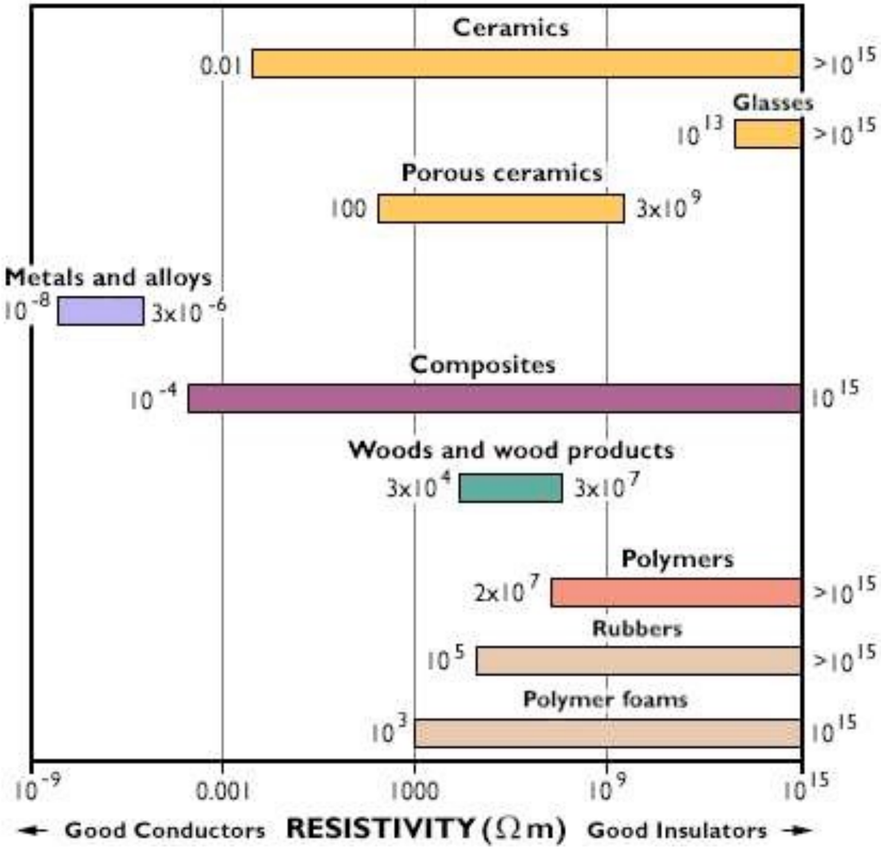
$$\boldsymbol{j} = \sigma \boldsymbol{e}\,(+\boldsymbol{j}_s)\,, \quad \text{resistivity } \rho = \frac{1}{\sigma}$$

$$\boldsymbol{j} = \sigma(\boldsymbol{e} + \boldsymbol{v} \times \boldsymbol{b})\,(+\boldsymbol{j}_s)$$ moving conductor

Simple models for temperature dependence: metals



$$\rho = \rho_0(1 + \alpha(T - T_0))$$



	ρ_0(T_0=20°C) (Ωm)	α (°C^{-1})
Aluminum	2.7 10^{-8}	4 10^{-3}
Copper	1.7 10^{-8}	3.9 10^{-3}
Iron	9.6 10^{-8}	6.5 10^{-3}

Potential fields

Potential field approach

- Maxwell's equations are rarely solved in terms of EM fields
- A potential is a scalar or vector function the derivative of which gives a field
- Fields are associated with forces; potentials are associated with energy
- In general, scalar and vector potentials are functions of position and time

- Maxwell's equations reformulated in terms of potentials
  - implicit properties that allow to simplify the PDEs/ODEs
  - reduced number of equations
  - reduced dimension: from vector unknowns to scalar unknowns

$$\boldsymbol{a} \implies \boldsymbol{b} = \operatorname{curl} \boldsymbol{a}$$
$$v \implies \boldsymbol{e} = -\operatorname{grad} v - \partial_t \boldsymbol{a}$$

$$\boldsymbol{u} \implies \boldsymbol{d} = \boldsymbol{d}_s + \operatorname{curl} \boldsymbol{u}$$
$$\boldsymbol{t} \implies \boldsymbol{j} = \operatorname{curl} \boldsymbol{t}$$

$$\omega \implies \boldsymbol{h} = \boldsymbol{t} - \operatorname{grad} \omega$$
$$\varphi \implies \boldsymbol{h} = \boldsymbol{h}_s - \operatorname{grad} \varphi$$

$$\text{div}\,(\text{curl}\,\boldsymbol{f}) = 0$$
$$\text{curl}\,(\text{grad}\,f) = 0$$
$$\text{curl}\,\text{curl}\,\boldsymbol{f} = \text{grad}\,(\text{div}\,\boldsymbol{f}) - \Delta\boldsymbol{f}$$
$$\text{div}\,\text{grad}\,f = \Delta f$$

reminder

Potential field approach (cont'd)

magnetic vector potential $\boldsymbol{a} \implies \boldsymbol{b} = \text{curl}\,\boldsymbol{a}$

electric scalar potential $v \implies \boldsymbol{e} = -\text{grad}\,v - \partial_t \boldsymbol{a}$

eqs. automatically satisfied

$$\text{curl}\,\boldsymbol{e} + \partial_t \boldsymbol{b} = 0$$
$$\text{div}\,\boldsymbol{b} = 0$$

from the other two equations (linear, isotropic materials)

$$\text{curl}\,\boldsymbol{b} = \text{curl}\,\text{curl}\,\boldsymbol{a} = \mu \boldsymbol{j} + \mu\epsilon\partial_t \boldsymbol{e}$$
$$\text{div}\,\boldsymbol{e} = -\text{div}\,\text{grad}\,v - \partial_t \text{div}\,\boldsymbol{a} = \frac{q}{\epsilon}$$

eqs. to solve in terms of the potentials

$$\Delta\boldsymbol{a} - \mu\epsilon\partial_t^2 \boldsymbol{a} - \text{grad}\,(\text{div}\,\boldsymbol{a} + \mu\epsilon\partial_t v) = -\mu\boldsymbol{j}$$
$$\Delta v + \partial_t \text{div}\,\boldsymbol{a} = -\frac{q}{\epsilon}$$

Simplified equations by imposing Lorentz gauge →

$$\Delta\boldsymbol{a} - \mu\epsilon\partial_t^2 \boldsymbol{a} = -\mu\boldsymbol{j}$$
$$\Delta v - \mu\epsilon\partial_t^2 v = -\frac{q}{\epsilon}$$

Gauge invariance
Time-independent sources

$$\operatorname{div}(\operatorname{curl} \boldsymbol{f}) = 0$$
$$\operatorname{curl}(\operatorname{grad} f) = 0$$
$$\operatorname{curl}\operatorname{curl} \boldsymbol{f} = \operatorname{grad}(\operatorname{div} \boldsymbol{f}) - \Delta \boldsymbol{f}$$
$$\operatorname{div}\operatorname{grad} f = \Delta f$$

Poisson's equation in the presence of source

$$\Delta v(\boldsymbol{r}) = -\frac{q(\boldsymbol{r})}{\epsilon} \qquad \Delta \boldsymbol{a}(\boldsymbol{r}) = -\mu \boldsymbol{j}(\boldsymbol{r})$$

Physics invariant under gauge transformation v_0 constant, ψ_0 scalar field

$$v \mapsto v + v_0 \qquad \boldsymbol{a} \mapsto \boldsymbol{a} + \operatorname{grad} \psi_0$$

One choice of gauge is

$$v(|\boldsymbol{r}| = \infty) = 0 \qquad \operatorname{div} \boldsymbol{a} = 0$$

Coulomb gauge

Potentials calculated directly from the sources => solutions!

$$v(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{q(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}V' \qquad \boldsymbol{a}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{j}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}V'$$

Gauge invariance
Time-dependent sources

$$\operatorname{div}(\operatorname{curl}\boldsymbol{f}) = 0$$
$$\operatorname{curl}(\operatorname{grad} f) = 0$$
$$\operatorname{curl}\operatorname{curl}\boldsymbol{f} = \operatorname{grad}(\operatorname{div}\boldsymbol{f}) - \Delta\boldsymbol{f}$$
$$\operatorname{div}\operatorname{grad} f = \Delta f$$

Equations to solve

$$\Delta v + \partial_t \operatorname{div}\boldsymbol{a} = -\frac{q}{\epsilon}$$

$$\Delta\boldsymbol{a} - \mu\epsilon\partial_t^2\boldsymbol{a} - \operatorname{grad}(\operatorname{div}\boldsymbol{a} + \mu\epsilon\partial_t v) = -\mu\boldsymbol{j}$$

Physics invariant under gauge transformation

$$v \mapsto v + v_0$$

$$\boldsymbol{a} \mapsto \boldsymbol{a} + \operatorname{grad}\psi_0$$

imposing Lorentz gauge $\operatorname{div}\boldsymbol{a} + \mu\epsilon\partial_t v = 0$

we get the simplified Maxwell's equations

$$\Delta v - \mu\epsilon\partial_t^2 v = -\frac{q}{\epsilon}$$

$$\Delta\boldsymbol{a} - \mu\epsilon\partial_t^2\boldsymbol{a} = -\mu\boldsymbol{j}$$

Magnetic vector potential
Gauge condition — uniqueness

$$\boldsymbol{b} = \operatorname{curl} \boldsymbol{a} \implies \operatorname{div} \boldsymbol{b} = 0$$

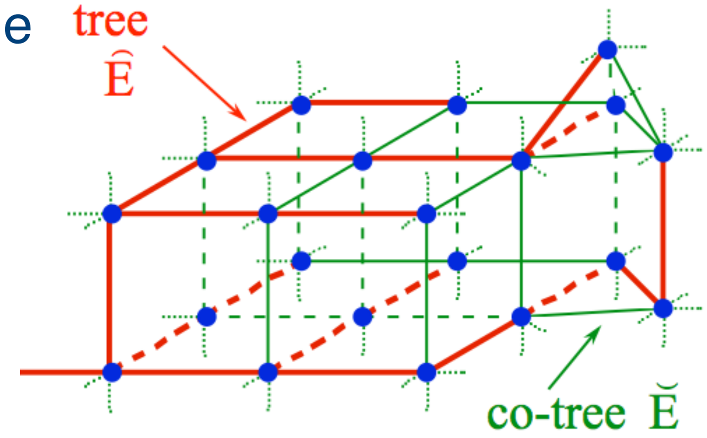
$$\operatorname{div} \boldsymbol{b} = 0 \implies \boldsymbol{b} = \operatorname{curl} \boldsymbol{a} + \operatorname{grad} \eta$$

Non-uniqueness of magnetic vector potential
We need to impose a **gauge condition**!

Coulomb gauge: $\operatorname{div} \boldsymbol{a} = 0$

Lorentz gauge: $\operatorname{div} \boldsymbol{a} + \mu\epsilon\partial_t v = 0$

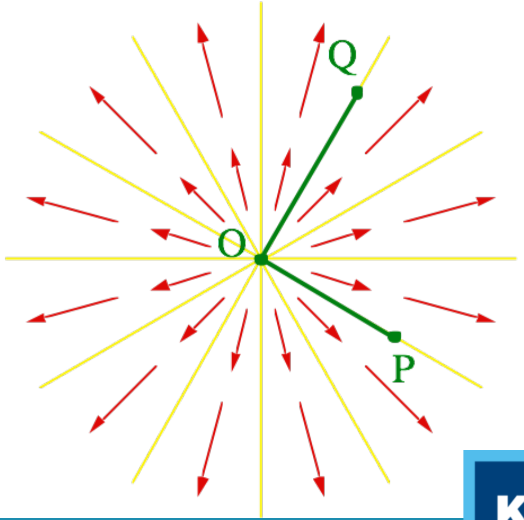
Tree—co-tree gauge



Another possibility: $\boldsymbol{a} \cdot \boldsymbol{w} = 0$

$\boldsymbol{w}$ is a vector field with no-close lines linking any pair of points in Ω

$$\boldsymbol{w}(\boldsymbol{r}) = \boldsymbol{r}$$



Model choice

Model choice – Approximating Maxwell's equations

- Maxwell's equations & constitutive relations in frequency domain, without sources (no conductivity inside conductors):

$$\Delta \boldsymbol{e} - \imath\omega\sigma\mu\, \boldsymbol{e} + \omega^2\epsilon\mu\, \boldsymbol{e} = 0$$
$$\Delta \boldsymbol{h} - \imath\omega\sigma\mu\, \boldsymbol{h} + \omega^2\epsilon\mu\, \boldsymbol{h} = 0$$

+ no RHS term

$$\Delta \boldsymbol{e} + k^2\, \boldsymbol{e} = 0$$
$$\Delta \boldsymbol{h} + k^2\, \boldsymbol{h} = 0$$

Helmholtz equations

- Using characteristic lengths
  - characteristic spatial dimension *L*
  - skin depth $\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$
  - wavelength

$$\lambda = \frac{2\pi}{k}, \text{wave number } k = \frac{\omega}{c}, \text{speed of light } c = \frac{1}{\sqrt{\epsilon\mu}}$$

- allows to write them in non-dimensional form:

$$\left(\frac{3}{L^2} - \frac{2\imath}{\delta^2} + \frac{4\pi^2}{\lambda^2}\right)\boldsymbol{e} = 0 \qquad \left(\frac{3}{L^2} - \frac{2\imath}{\delta^2} + \frac{4\pi^2}{\lambda^2}\right)\boldsymbol{h} = 0$$

Formal choice of model based on characteristic lengths

Non-dimensional numbers

$$
\begin{cases}
g_1 = \left(\frac{\lambda}{L}\right)^2 \\
g_2 = \left(\frac{\delta}{L}\right)^2 \\
g_3 = \left(\frac{\lambda}{\delta}\right)^2
\end{cases}
$$

Maxwell's equations

$$\left(\frac{3}{L^2} - \frac{2\imath}{\delta^2} + \frac{4\pi^2}{\lambda^2}\right)\boldsymbol{e} = 0$$

$$\left(\frac{3}{L^2} - \frac{2\imath}{\delta^2} + \frac{4\pi^2}{\lambda^2}\right)\boldsymbol{h} = 0$$

$g_1 \gg 1$ **uncoupled problem electric/magnetic**

$g_3 \ll 1$ electrostatics

$g_3 \gg 1$ electrokinetics

$g_3 \approx 1$ electrodynamics

$g_2 \gg 1$ magnetostatics

$g_2 \lesssim 1$ magnetodynamics

$g_1 \lesssim 1$ **full wave**

$g_1 \to 1$ HF asymptotics

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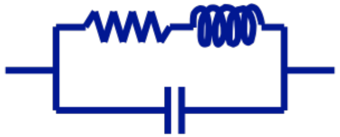
$$C = \frac{Q}{V} = \frac{\epsilon S}{d}$$

$$R = \frac{V}{I} = \frac{l}{\sigma S}$$

$$L = \frac{\Phi}{m.m.f.} = n^2 \frac{\mu_0 S}{l}$$

$$\delta = \sqrt{\frac{2}{\omega \mu \sigma}}$$

$$\lambda = \frac{2\pi}{k}$$

EM model		governing equations
electrostatics		$\mathrm{curl}\, \boldsymbol{e} = 0\,, \quad \mathrm{div}\, \boldsymbol{d} = q\,, \quad \boldsymbol{d} = \epsilon\, \boldsymbol{e}$
electrokinetics		$\mathrm{curl}\, \boldsymbol{e} = 0\,, \quad \mathrm{div}\, \boldsymbol{j} = 0\,, \quad \boldsymbol{j} = \sigma\, \boldsymbol{e}$
electrodynamics		$\mathrm{curl}\, \boldsymbol{e} = 0\,, \quad \mathrm{div}\, (\boldsymbol{j} + \partial_t \boldsymbol{d}) = 0$ $\boldsymbol{j} = \sigma\, \boldsymbol{e}\,, \quad \boldsymbol{d} = \epsilon \boldsymbol{e}$
magnetostatics		$\mathrm{curl}\, \boldsymbol{h} = \boldsymbol{j}\,, \quad \mathrm{div}\, \boldsymbol{b} = 0$ $\boldsymbol{b} = \mu\, \boldsymbol{h}$
magnetodynamics		$\mathrm{curl}\, \boldsymbol{h} = \boldsymbol{j}\,, \quad \mathrm{curl}\, \boldsymbol{e} = -\partial_t \boldsymbol{b}\,, \quad \mathrm{div}\, \boldsymbol{b} = 0$ $\boldsymbol{b} = \mu\, \boldsymbol{h}\,, \quad \boldsymbol{j} = \sigma\, \boldsymbol{e}$
full wave		$\mathrm{curl}\, \boldsymbol{h} = \boldsymbol{j} + \partial_t \boldsymbol{d}\,, \quad \mathrm{curl}\, \boldsymbol{e} = -\partial_t \boldsymbol{b}\,, \quad \mathrm{div}\, \boldsymbol{b} = 0$ $\boldsymbol{b} = \mu\, \boldsymbol{h}\,, \quad \boldsymbol{j} = \sigma\, \boldsymbol{e}\,, \quad \boldsymbol{d} = \epsilon\, \boldsymbol{e}$

Motivating examples

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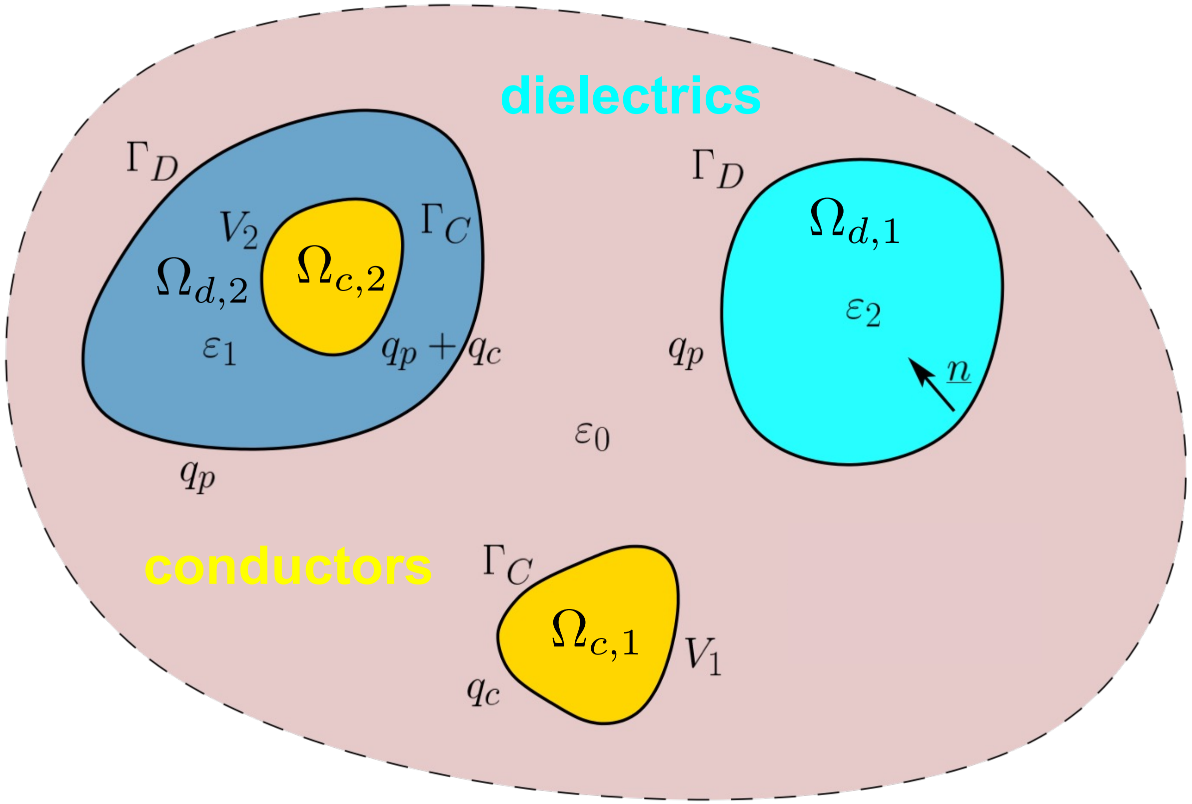
Electrostatics

Phenomena involving time-independent distributions of charges & fields $\partial_t \boldsymbol{b} = 0$

$$\operatorname{curl} \boldsymbol{e} = 0 \qquad \operatorname{div} \boldsymbol{d} = q \qquad \boldsymbol{d} = \epsilon\, \boldsymbol{e}$$

boundary conditions

$$\boldsymbol{n} \times \boldsymbol{e}|_{\Gamma_e} = 0 \qquad \boldsymbol{n} \cdot \boldsymbol{d}|_{\Gamma_d} = 0$$



electric scalar potential formulation in

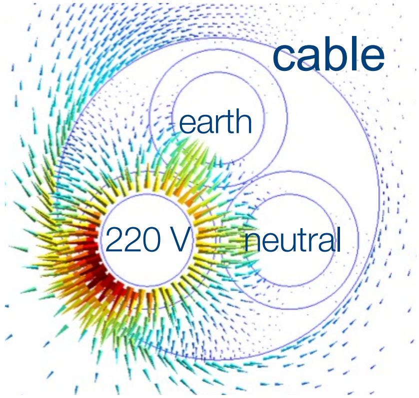
$$-\operatorname{div}(\epsilon \operatorname{grad} v) = q\,, \qquad \boldsymbol{e} = -\operatorname{grad} v$$

$\Downarrow$ no charges

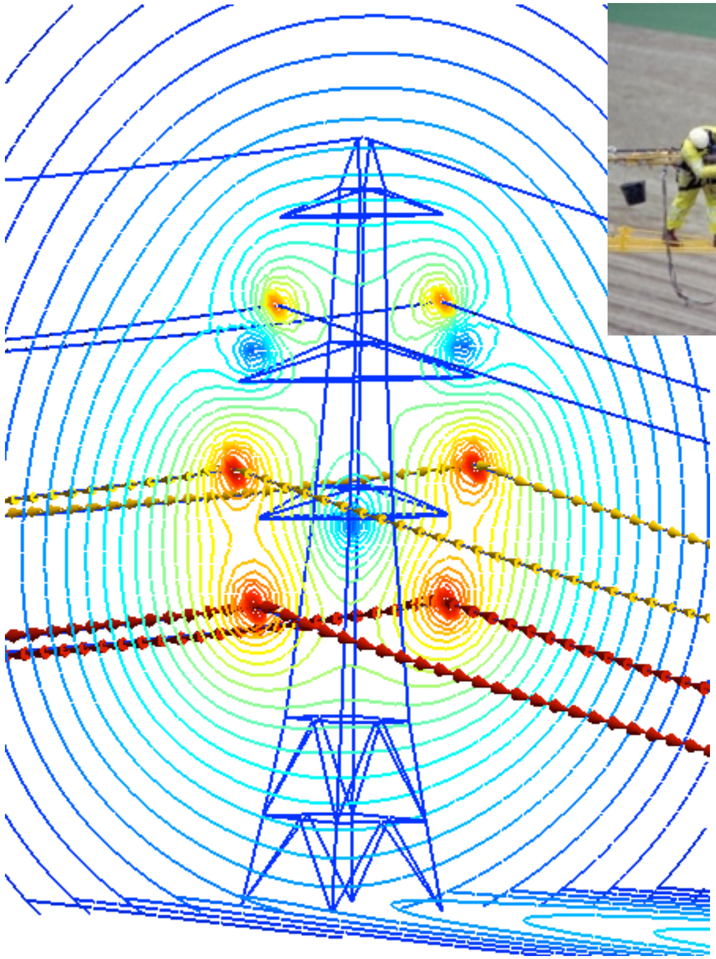
$$-\operatorname{div}\operatorname{grad} v = -\Delta v = 0$$ Laplace equation

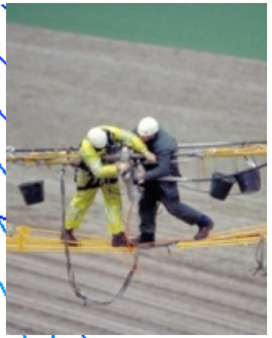
in each conducting region imposed potential at boundary $\Omega_{c,i},\; v = v_i \Rightarrow v|_{\Gamma_{c,i}} = v_i$

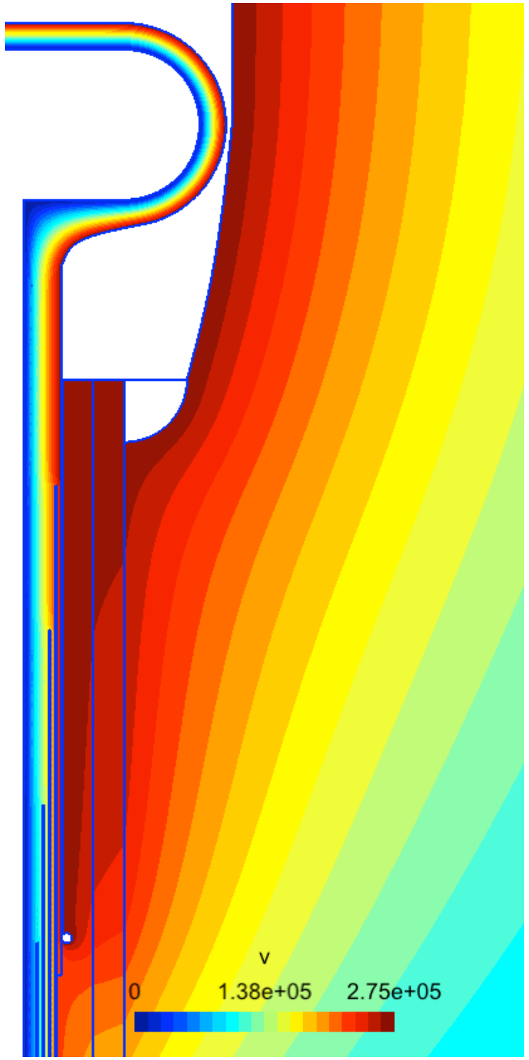
Electrostatics



field next to a 220 kV high voltage line

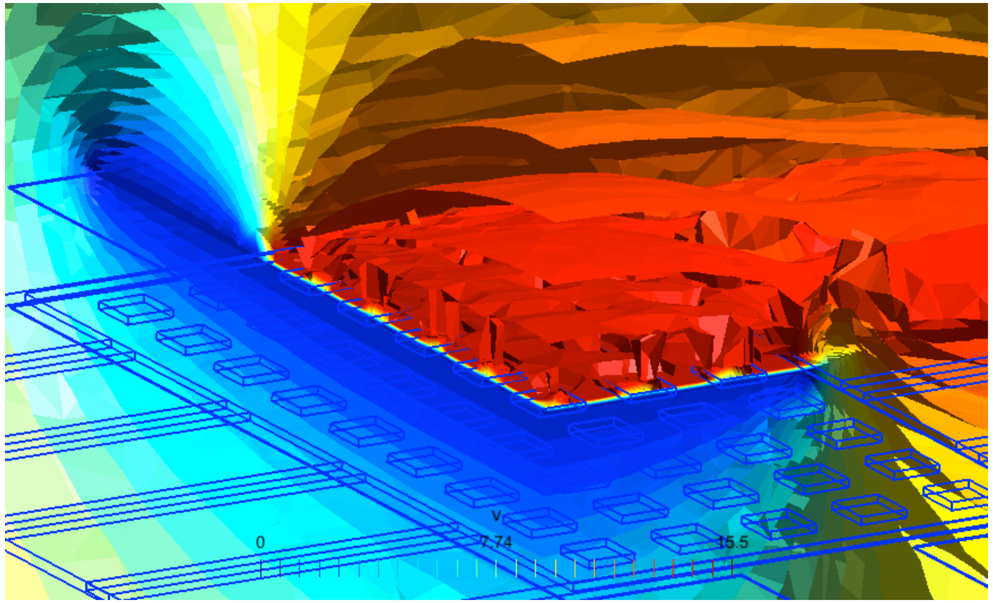


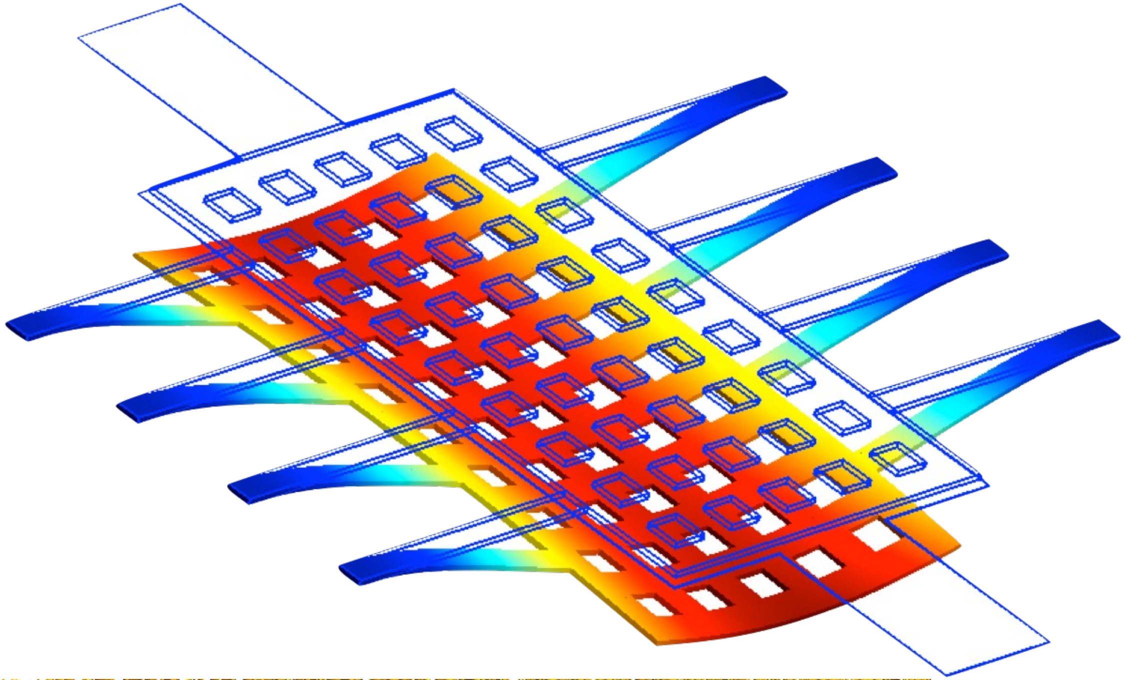




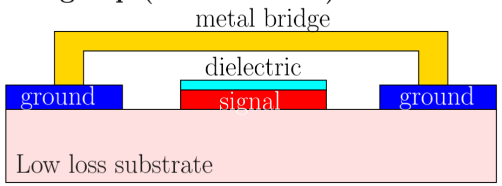
current transformer isolation

Electrostatics
shunt capacitive MEM switch

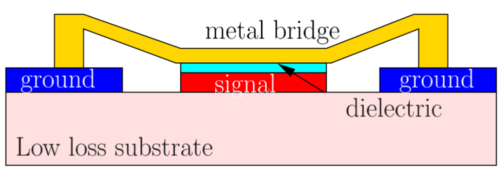


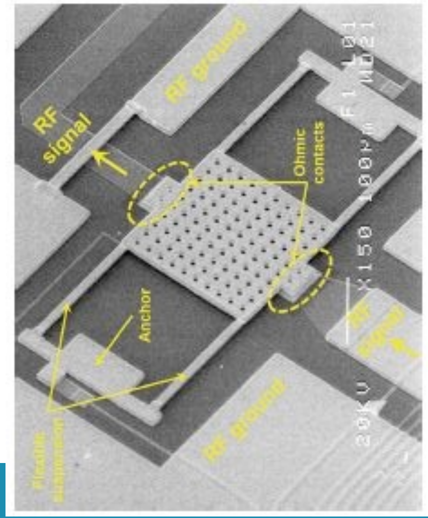


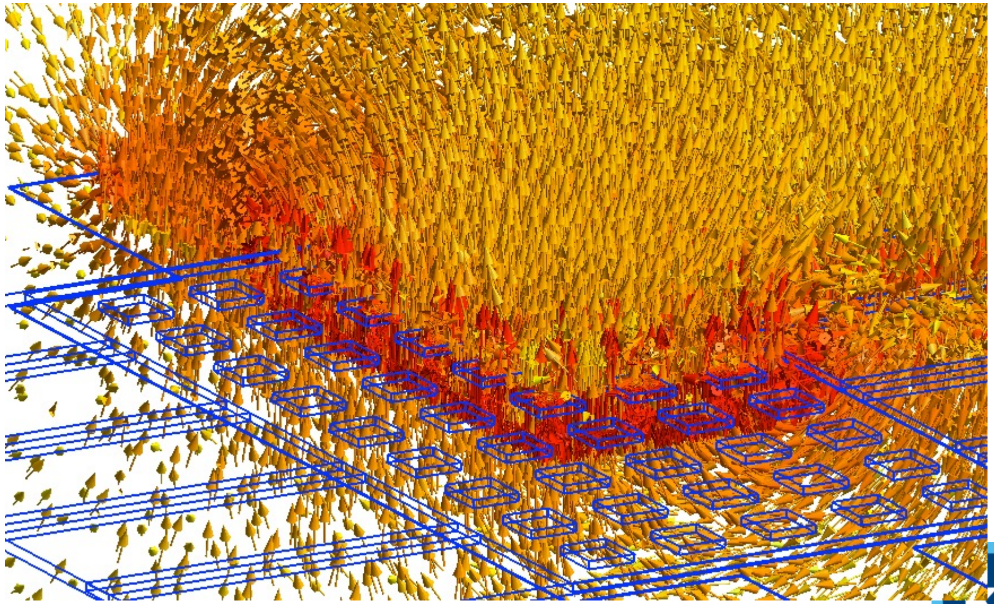
Bridge up (RF-ON state)



Bridge down (RF-OFF state)







Electrokinetics

$$\operatorname{curl} \boldsymbol{e} = 0$$

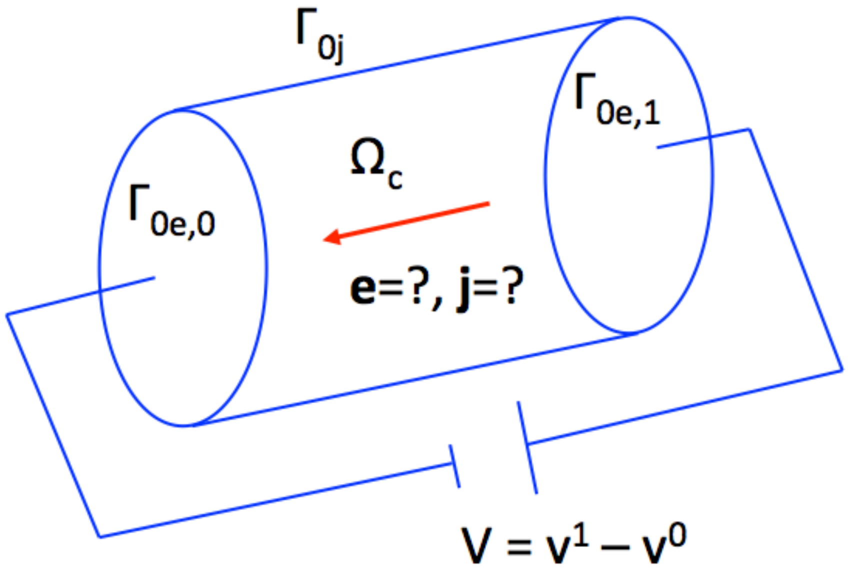
$$\operatorname{curl} \boldsymbol{h} = j \Rightarrow \operatorname{div} \boldsymbol{j} = 0$$

$$\boldsymbol{j} = \sigma\, \boldsymbol{e}$$

boundary conditions (BCs)

$$\boldsymbol{n} \times \boldsymbol{e}|_{\Gamma_e} = 0$$

$$\boldsymbol{n} \cdot \boldsymbol{j}|_{\Gamma_j} = 0$$



electric scalar potential formulation

$$-\operatorname{div}\,(\sigma \operatorname{grad} v) = 0\,, \quad \boldsymbol{e} = -\operatorname{grad} v$$

formulation for the conducting region Ω_c

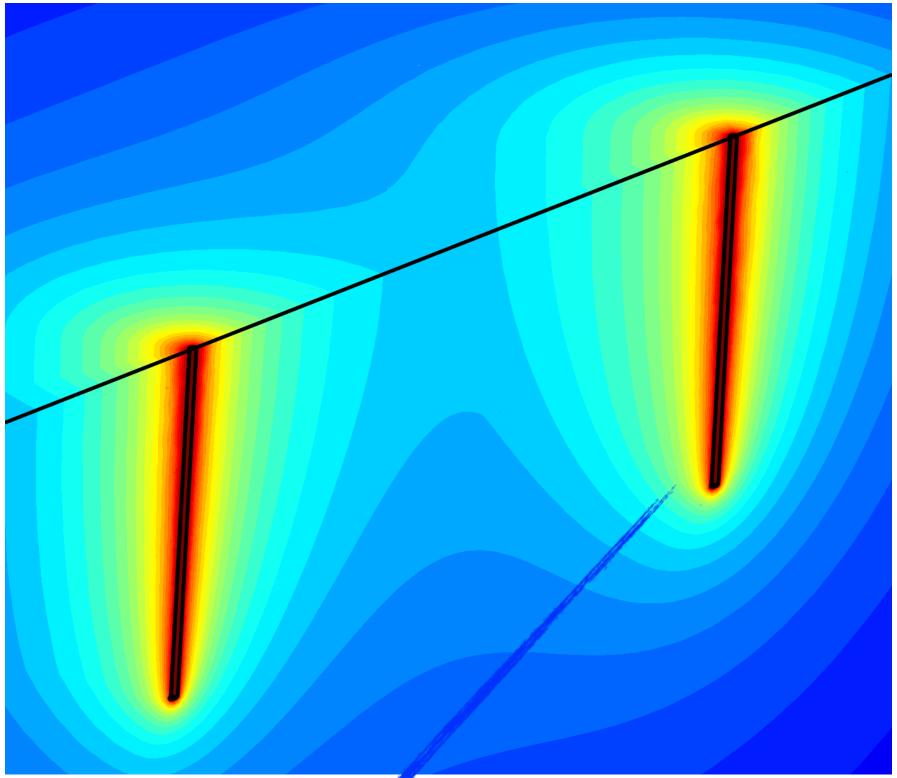
in each electrode
imposed potential

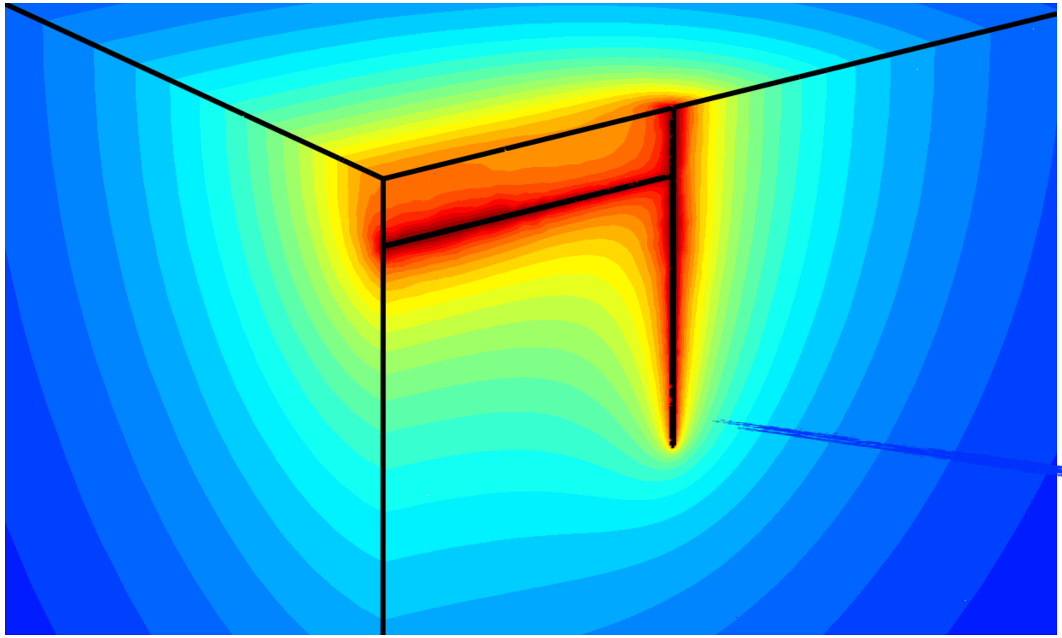
$$\Gamma_{e,i},\ v = v_i \Rightarrow v|_{\Gamma_{e,i}} = v_i$$

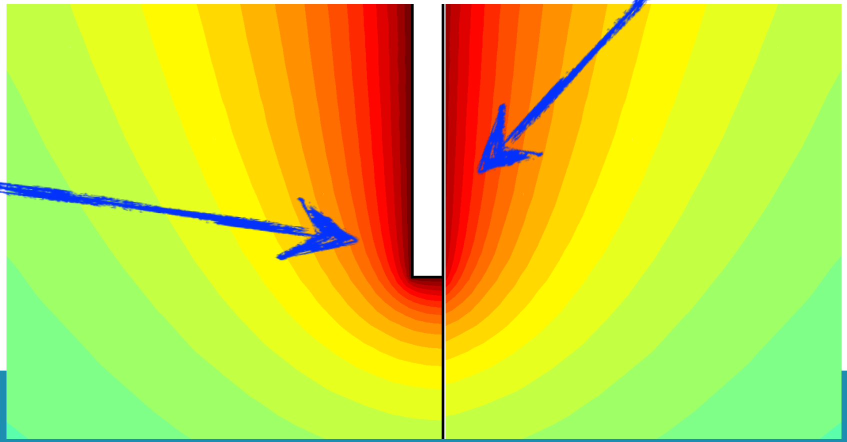
Electrokinetics

grounding systems: rods & cables

electric potential distribution







Electrodynamics

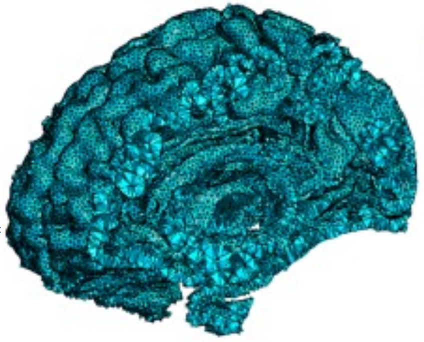
$$\begin{aligned} \operatorname{curl} \boldsymbol{e} &= 0 \\ \operatorname{curl} \boldsymbol{h} &= j + \partial_t \boldsymbol{d} \Rightarrow \operatorname{div} (\boldsymbol{j} + \partial_t \boldsymbol{d}) = 0 \\ \boldsymbol{j} &= \sigma\, \boldsymbol{e} \\ \boldsymbol{d} &= \epsilon\, \boldsymbol{e} \end{aligned}$$

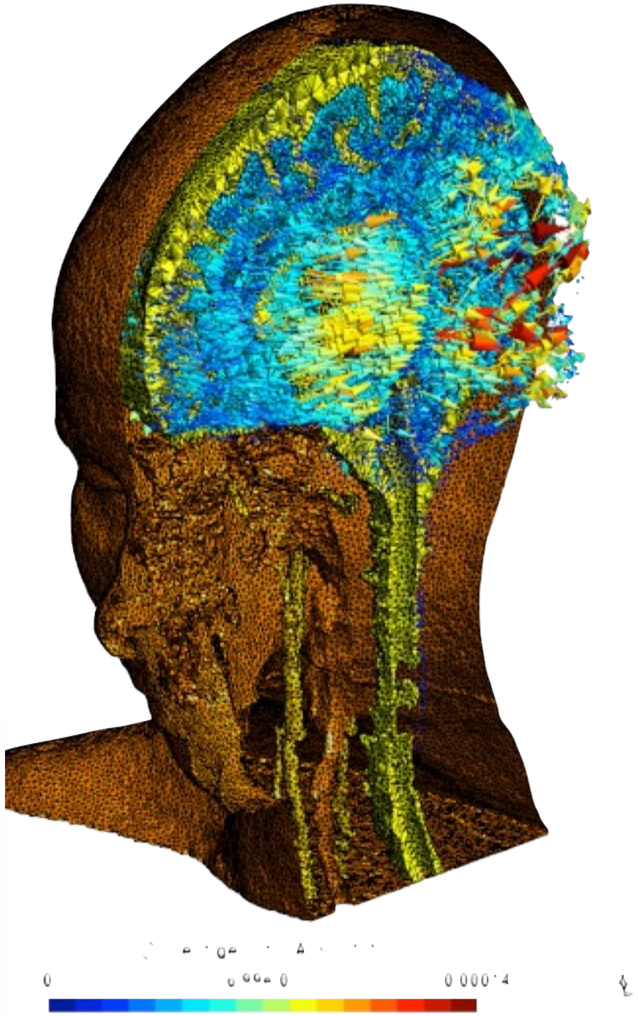
electric scalar potential formulation

$$-\operatorname{div} (\sigma \operatorname{grad} v + \epsilon \operatorname{grad} \partial_t v) = 0\,, \qquad \boldsymbol{e} = -\operatorname{grad} v$$

Electrodynamics

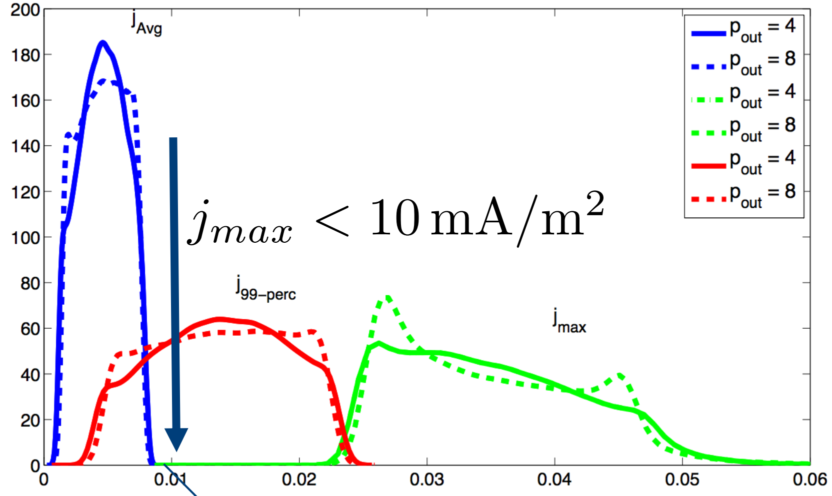
Tissue	Conductivity (S/m)
Muscle	0.2330
White matter	0.0533
Grey matter	**0.0753 – 0.5155**
Cerebellum	**0.0953 – 0.3020**
Eyes	1.5000
Spinal chord	0.0274
Lungs	0.0684
Heart	
Veins	
Liver	
Bones	
Cartilage	0.1714
Nerve	
CSF	
Midbrain	



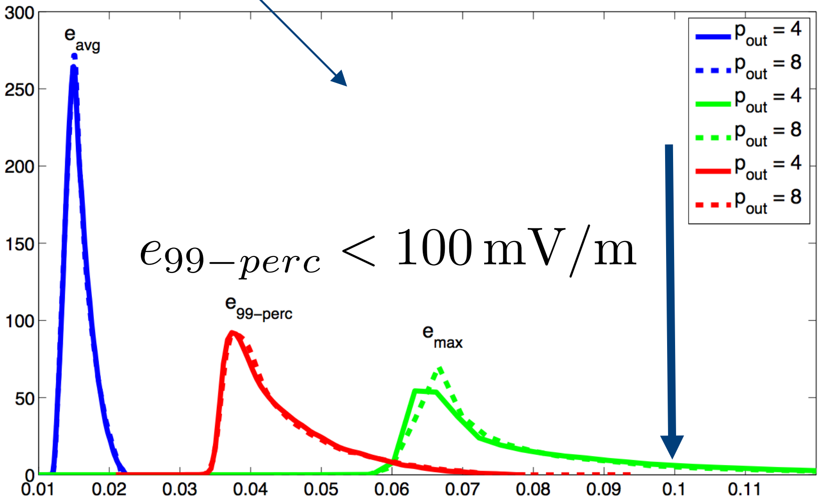


ELLA
Iso2mesh
45 different tissues
1e6 nodes
5.8e6 tetrahedra

density of probability



current density in the grey matter (A/m2)



electric field in the white matter (V/m)

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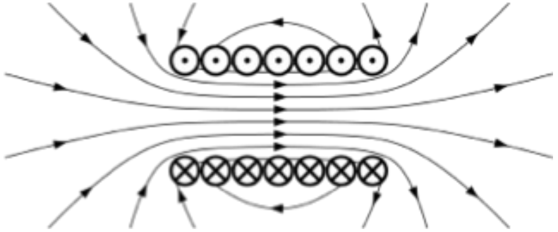
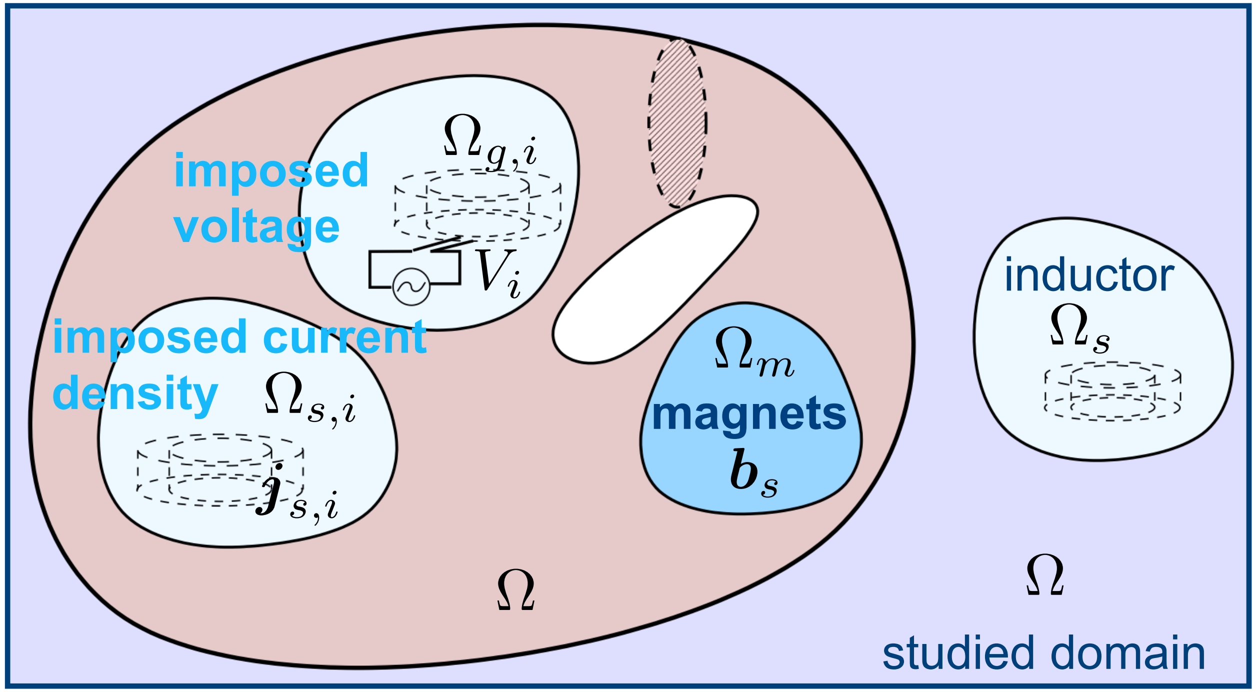
Magnetostatics



solenoid $L = \dfrac{\Phi}{m.m.f} = n^2 \dfrac{\mu_0 S}{l}$



magnetic vector potential formulation

$$\operatorname{curl} \nu \operatorname{curl} \boldsymbol{a} = \boldsymbol{j}_s\,, \quad \boldsymbol{b} = \operatorname{curl} \boldsymbol{a}$$

$$\operatorname{curl} \boldsymbol{h} = \boldsymbol{j}_s$$
$$\operatorname{div} \boldsymbol{b} = 0$$
$$\boldsymbol{b} = \mu\, \boldsymbol{h}(+\boldsymbol{b}_r)$$
$$\boldsymbol{h} = \nu\, \boldsymbol{b}(+\boldsymbol{h}_c)$$

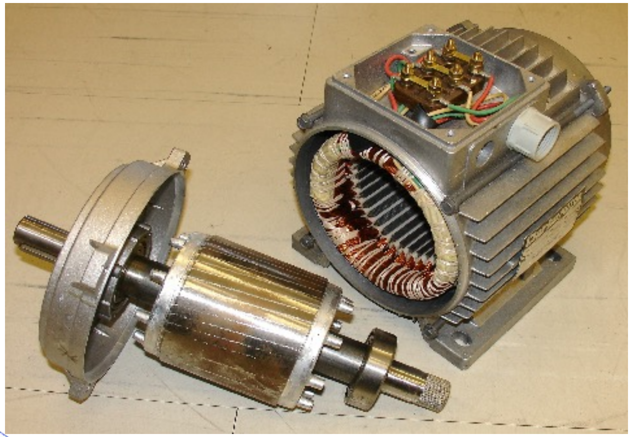
possible sources:

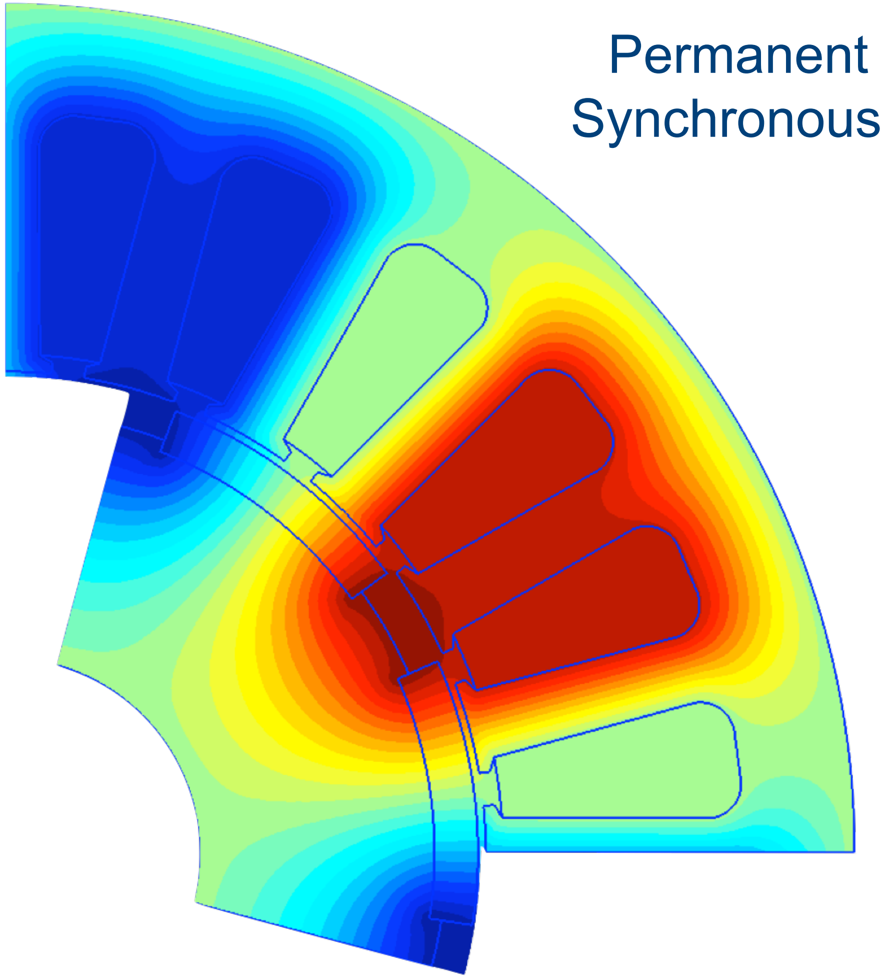
$\boldsymbol{j}_s$ imposed current density in inductor

$\boldsymbol{b}_r$ remanent induction if magnets

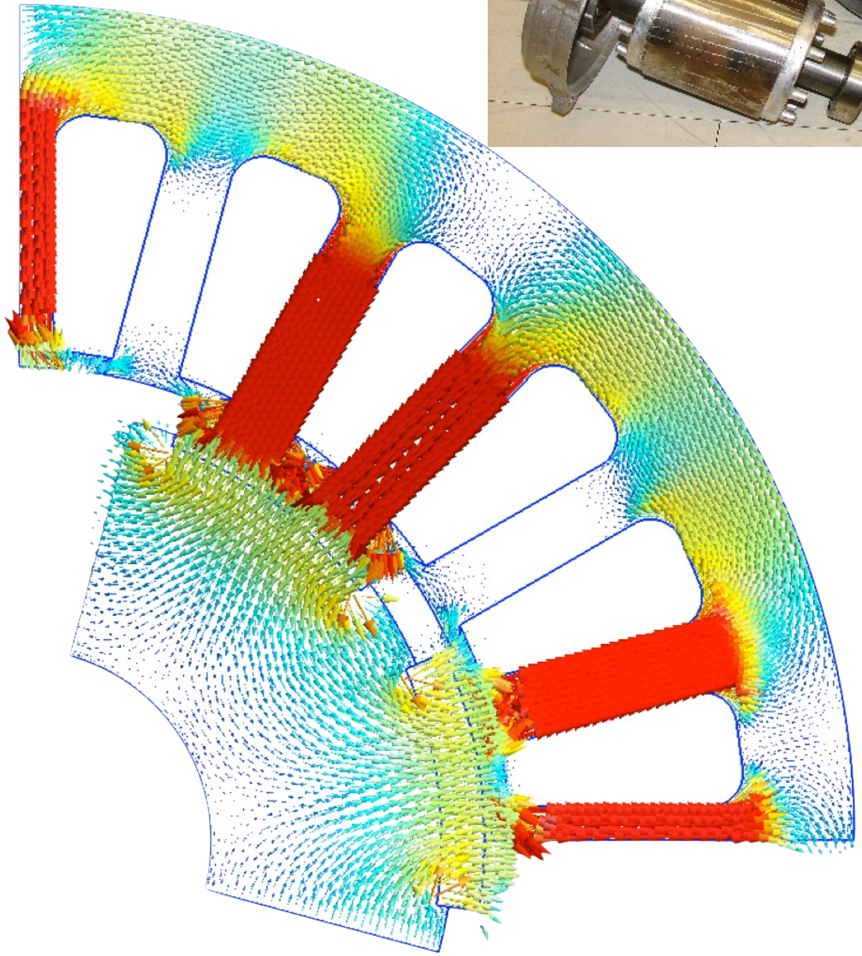
$\boldsymbol{h}_c$ coercive magnetic field if magnets

Magnetostatics



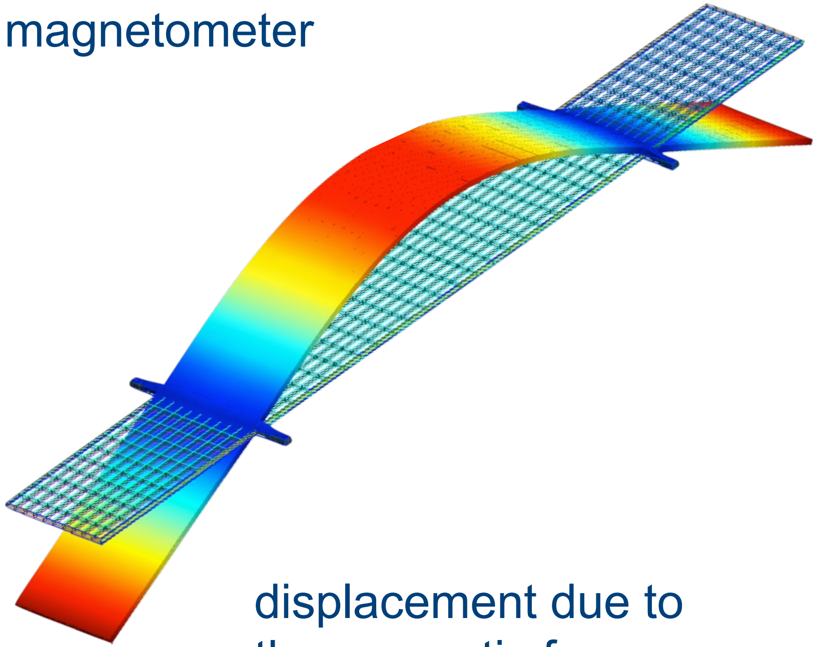


Permanent Magnet
Synchronous Machine

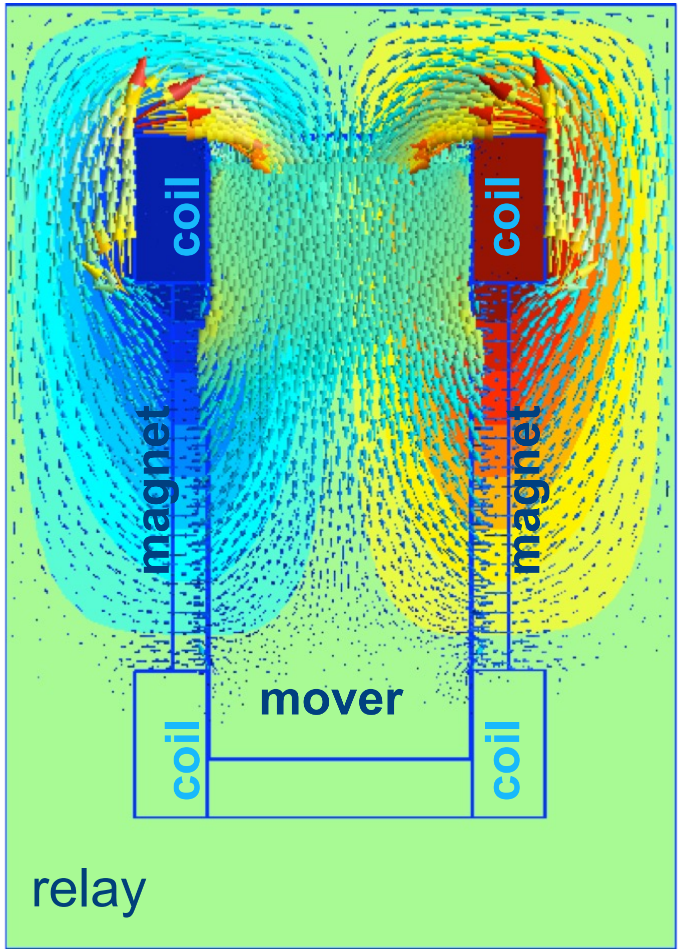


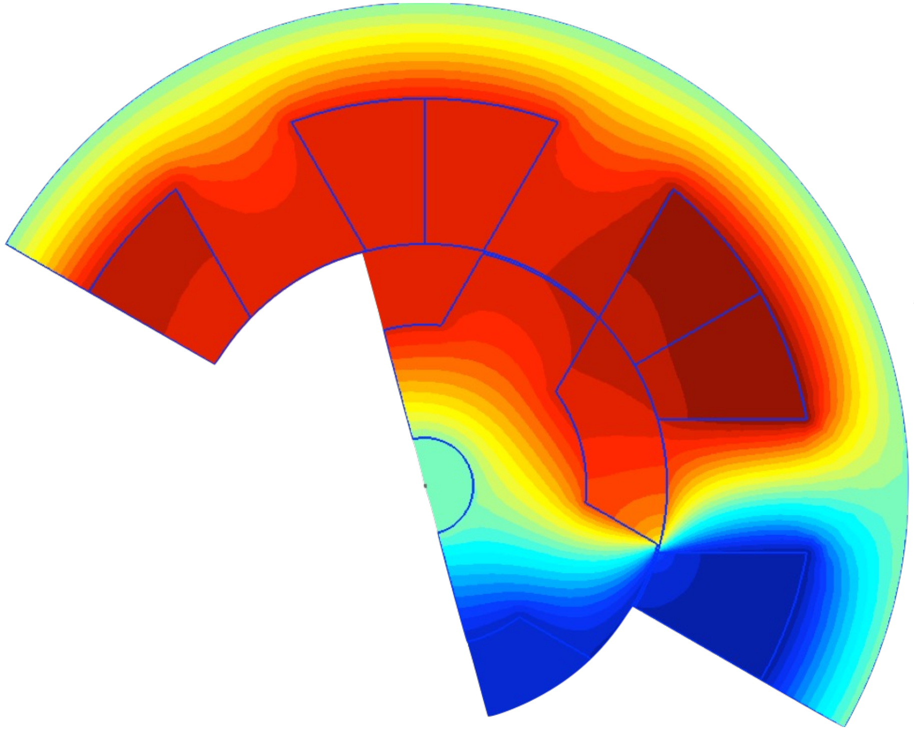
Magnetostatics

MEMS magnetometer



displacement due to the magnetic force

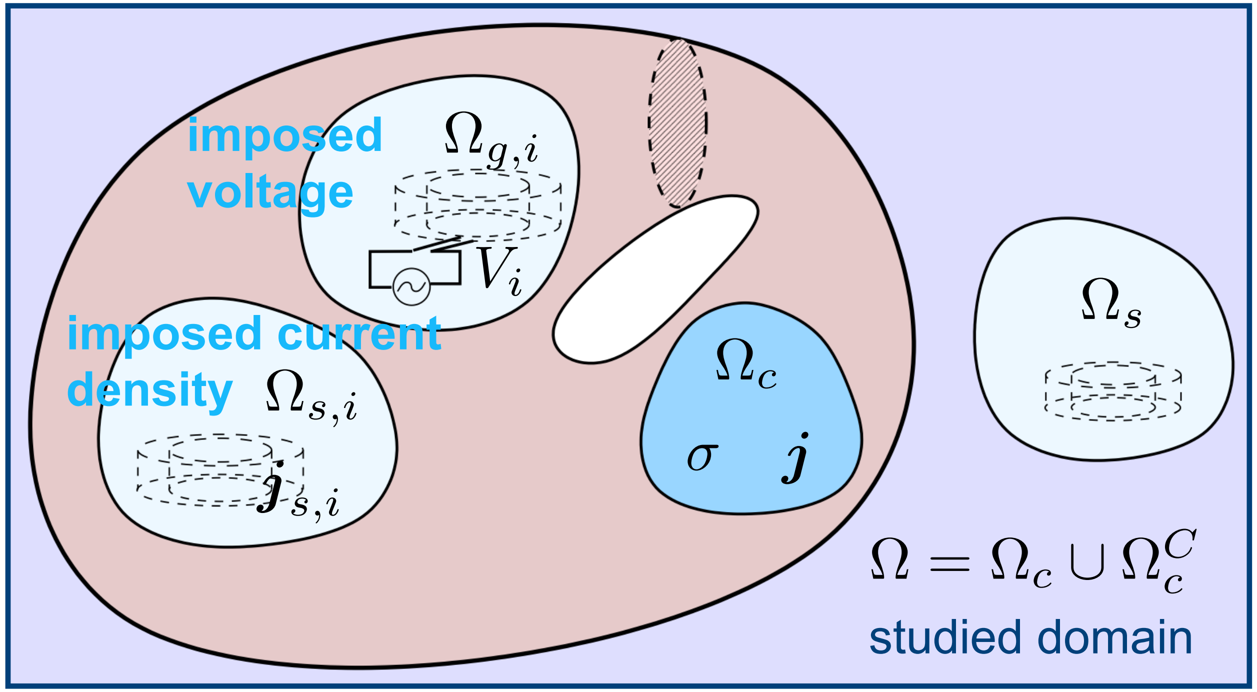




Switched reluctance machine

Magnetodynamics

$$\mathrm{curl}\,\boldsymbol{h} = \boldsymbol{j}$$
$$\mathrm{curl}\,\boldsymbol{e} = -\partial_t \boldsymbol{b}$$
$$\mathrm{div}\,\boldsymbol{b} = 0$$
$$\boldsymbol{b} = \mu\,\boldsymbol{h}(+\boldsymbol{b}_r)$$
$$\boldsymbol{j} = \sigma\,\boldsymbol{e} + \boldsymbol{j}_s$$



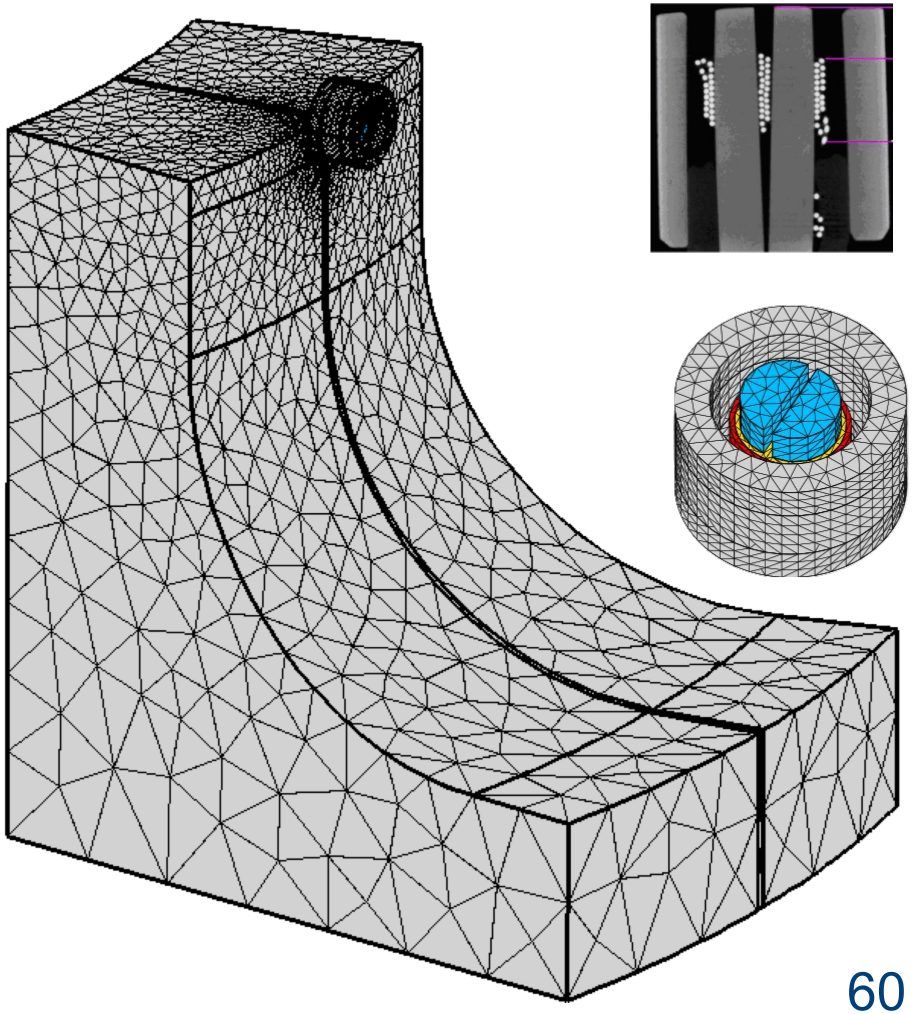
magnetic vector potential(-electric scalar potential) formulation

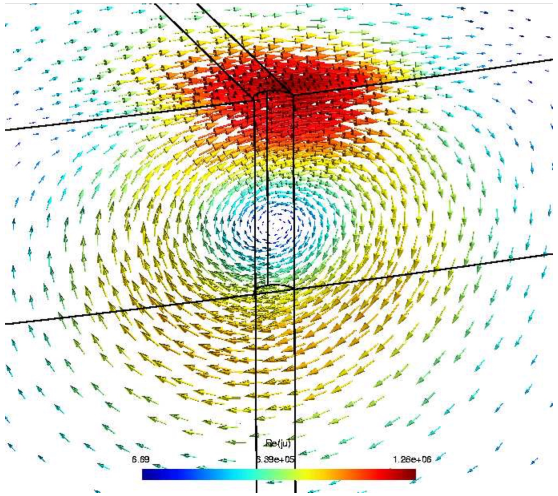
$$\mathrm{curl}\,\nu\,\mathrm{curl}\,\boldsymbol{a} + \sigma\,(\partial_t \boldsymbol{a} + \mathrm{grad}\,v) = \boldsymbol{j}_s\,, \quad \boldsymbol{b} = \mathrm{curl}\,\boldsymbol{a}$$

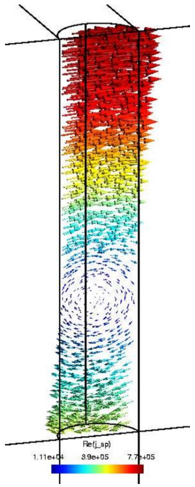
$$\mathrm{div}\Big(-\sigma(\partial_t \boldsymbol{a} + \mathrm{grad}\,v)\Big) = 0 \quad \text{in } \Omega^C \quad \boldsymbol{e} = -\mathrm{grad}\,v - \partial_t \boldsymbol{a}$$

reluctivity $\nu = \dfrac{1}{\mu}$

Magnetodynamics
eddy-current non-destructive testing (NDT)

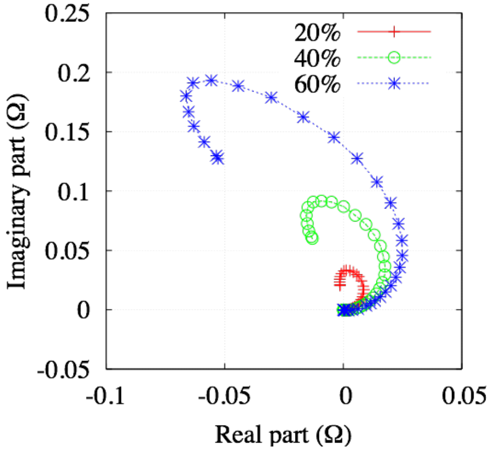


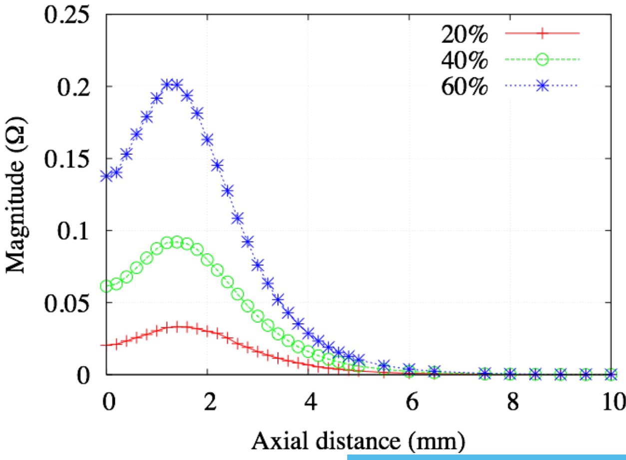




current density

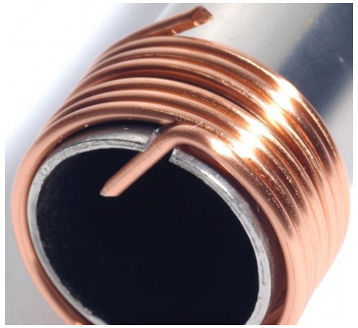
impedance variation

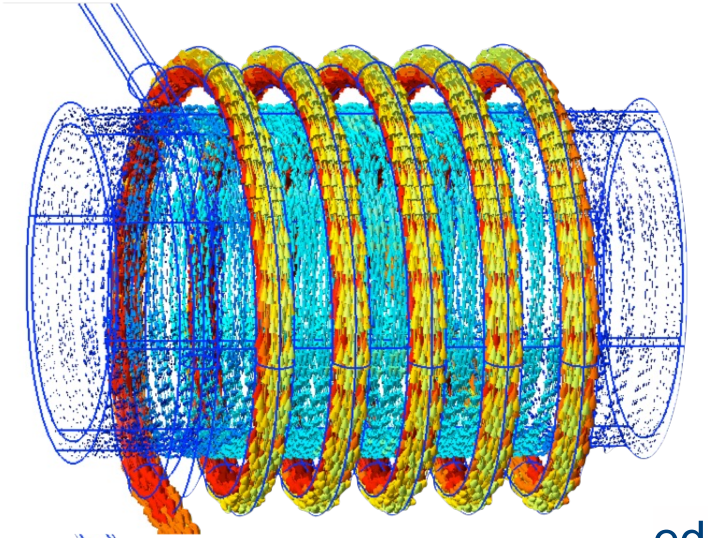


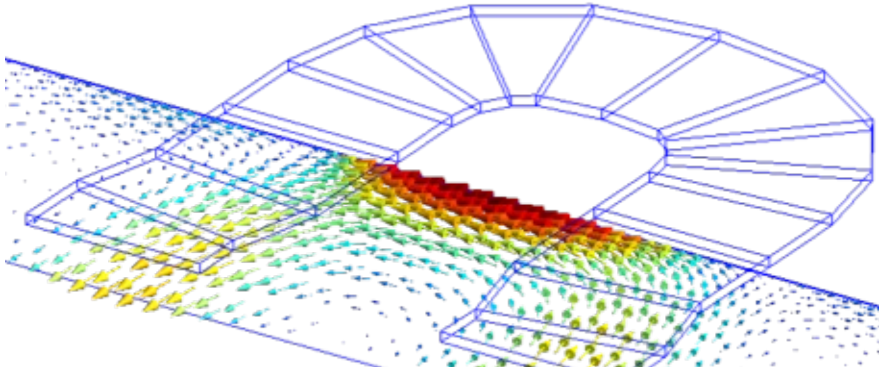


60 speedup factor for 100 positions

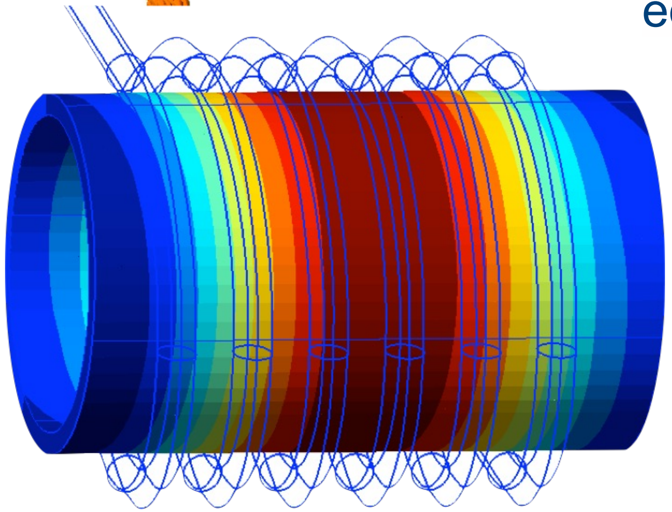
Magnetodynamics induction heating

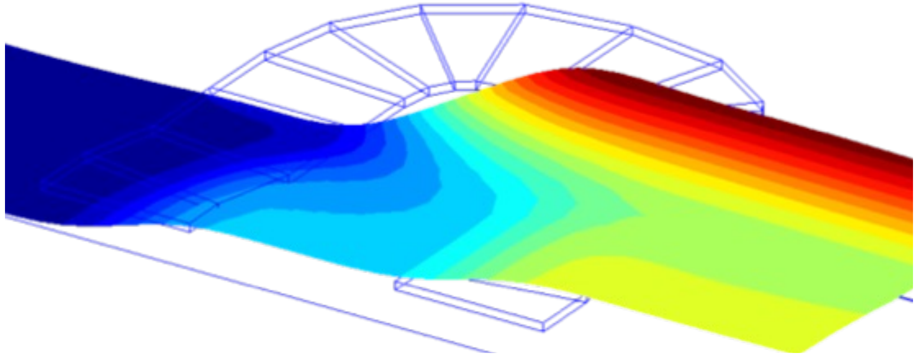






eddy current distribution





temperature distribution

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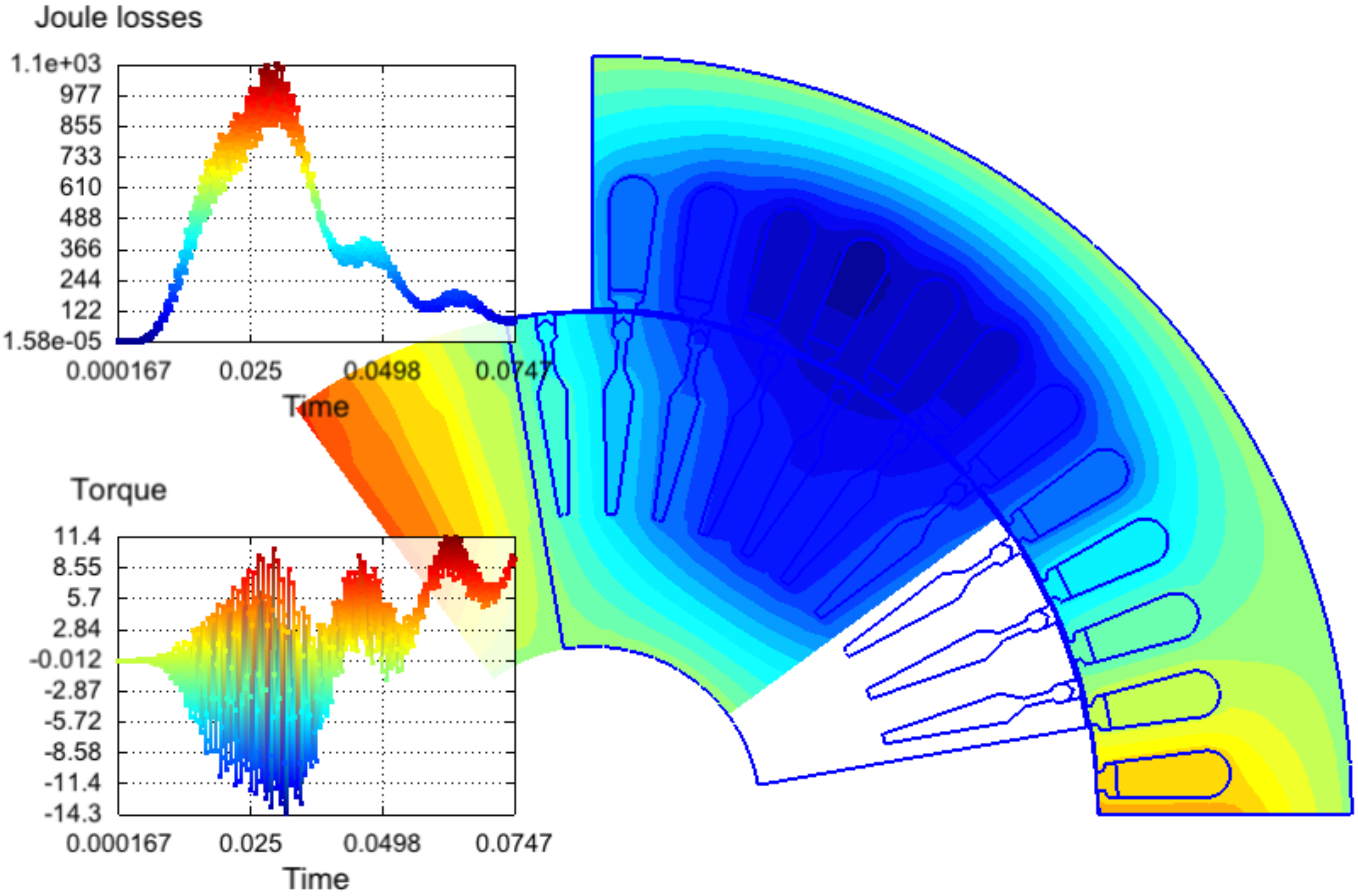
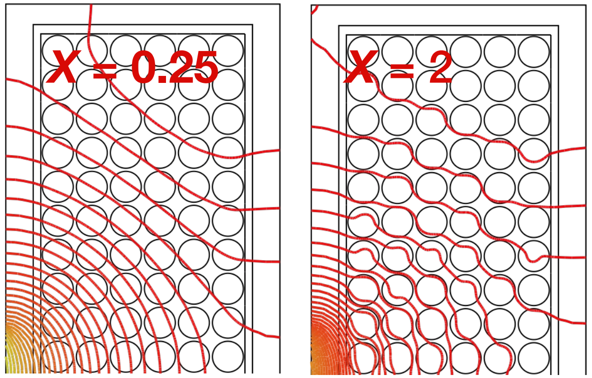
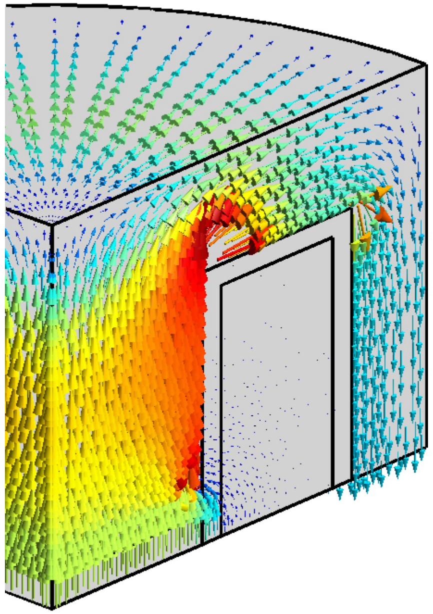
Magnetodynamics

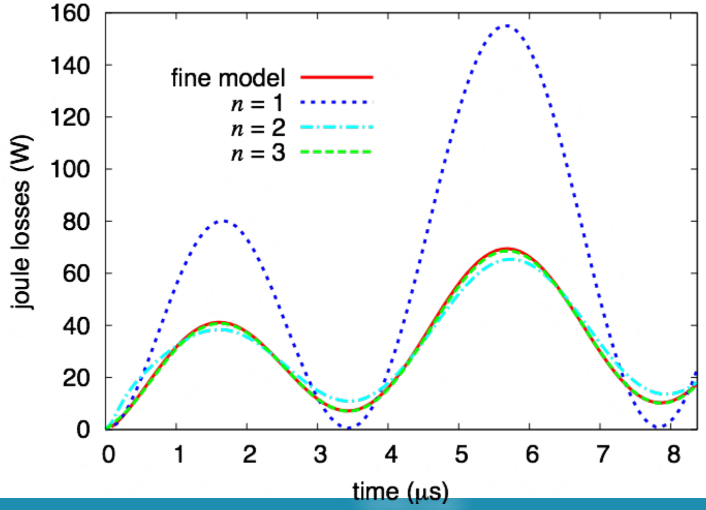


Magnetodynamics

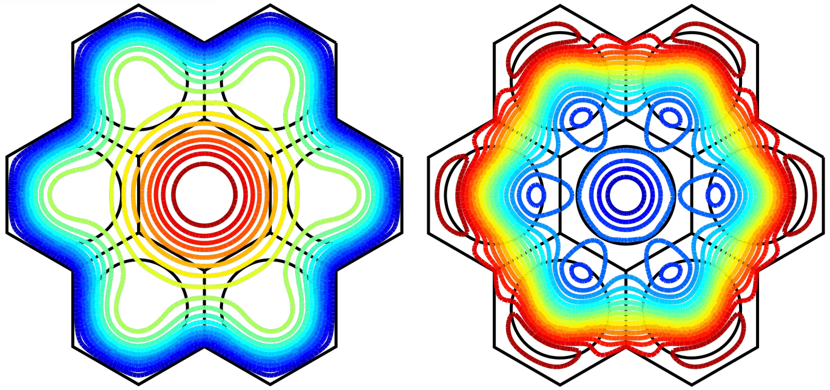
magnetic core
with a 3 mm central air-gap



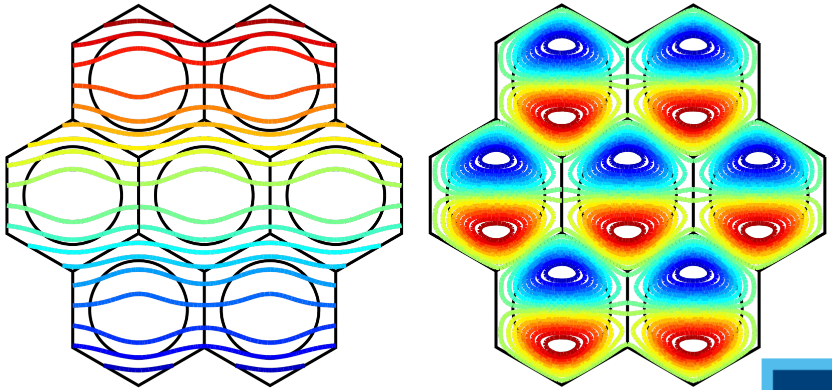




skin effect



proximity effect



Full wave – Maxwell's are fully coupled

$$\operatorname{curl} \boldsymbol{h} = \boldsymbol{j} + \partial_t \boldsymbol{d}$$
$$\operatorname{curl} \boldsymbol{e} = -\partial_t \boldsymbol{b}$$
$$\boldsymbol{b} = \mu \, \boldsymbol{h}$$
$$\boldsymbol{d} = \epsilon \, \boldsymbol{e}$$
$$\boldsymbol{j} = \sigma \, \boldsymbol{e}$$

total electric field = scattered field + incident field

$$\boldsymbol{e} = \boldsymbol{e}_s + \boldsymbol{e}_{inc}$$
$$\boldsymbol{e}_{\text{surf}} = (\boldsymbol{n} \times \boldsymbol{e}) \times \boldsymbol{n}$$

+ Silver-Müller radiation condition at infinity (outgoing waves)

$$\operatorname{curl} \boldsymbol{e} \times \boldsymbol{n} - \imath k \big((\boldsymbol{n} \times \boldsymbol{e}) \times \boldsymbol{n} \big) = \operatorname{curl} \boldsymbol{e}_{inc} \times \boldsymbol{n} - \imath k \big((\boldsymbol{n} \times \boldsymbol{e}_{inc}) \times \boldsymbol{n} \big) \,, \ \text{on } \Gamma$$

electric or magnetic field formulation

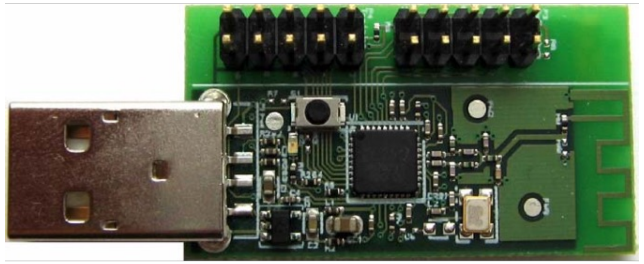
$$\operatorname{curl} \operatorname{curl} \boldsymbol{e} + \sigma\mu \, \partial_t \boldsymbol{e} + \epsilon\mu \, \partial_t^2 \boldsymbol{e} = 0$$
$$\operatorname{curl} \operatorname{curl} \boldsymbol{h} + \sigma\mu \, \partial_t \boldsymbol{h} + \epsilon\mu \, \partial_t^2 \boldsymbol{h} = 0$$

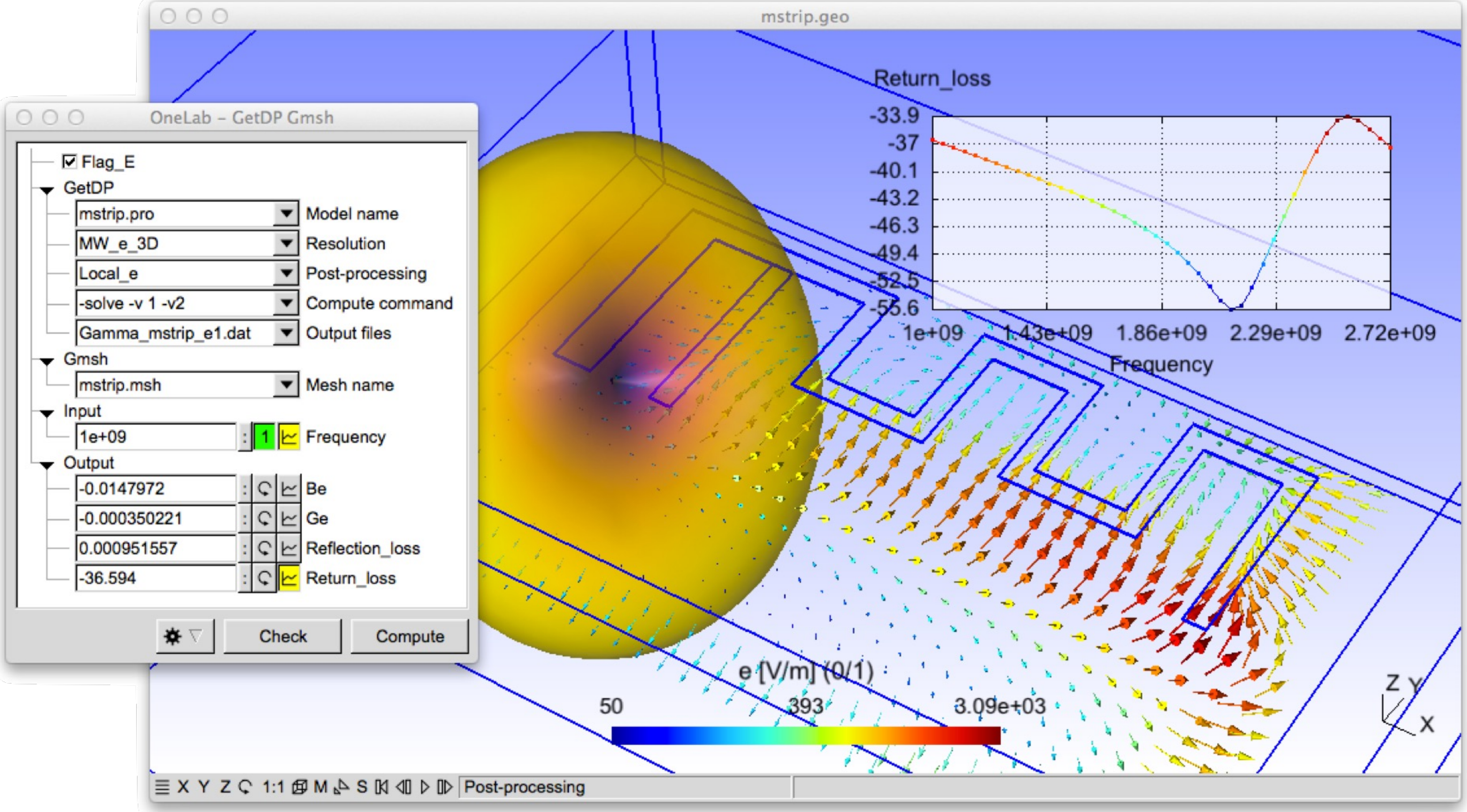
frequency domain (harmonic state or complex formalism)

$$\Delta \boldsymbol{e} - \imath\omega\sigma\mu \, \boldsymbol{e} + \omega^2 \epsilon\mu \, \boldsymbol{e} = 0$$
$$\Delta \boldsymbol{h} - \imath\omega\sigma\mu \, \boldsymbol{h} + \omega^2 \epsilon\mu \, \boldsymbol{h} = 0$$

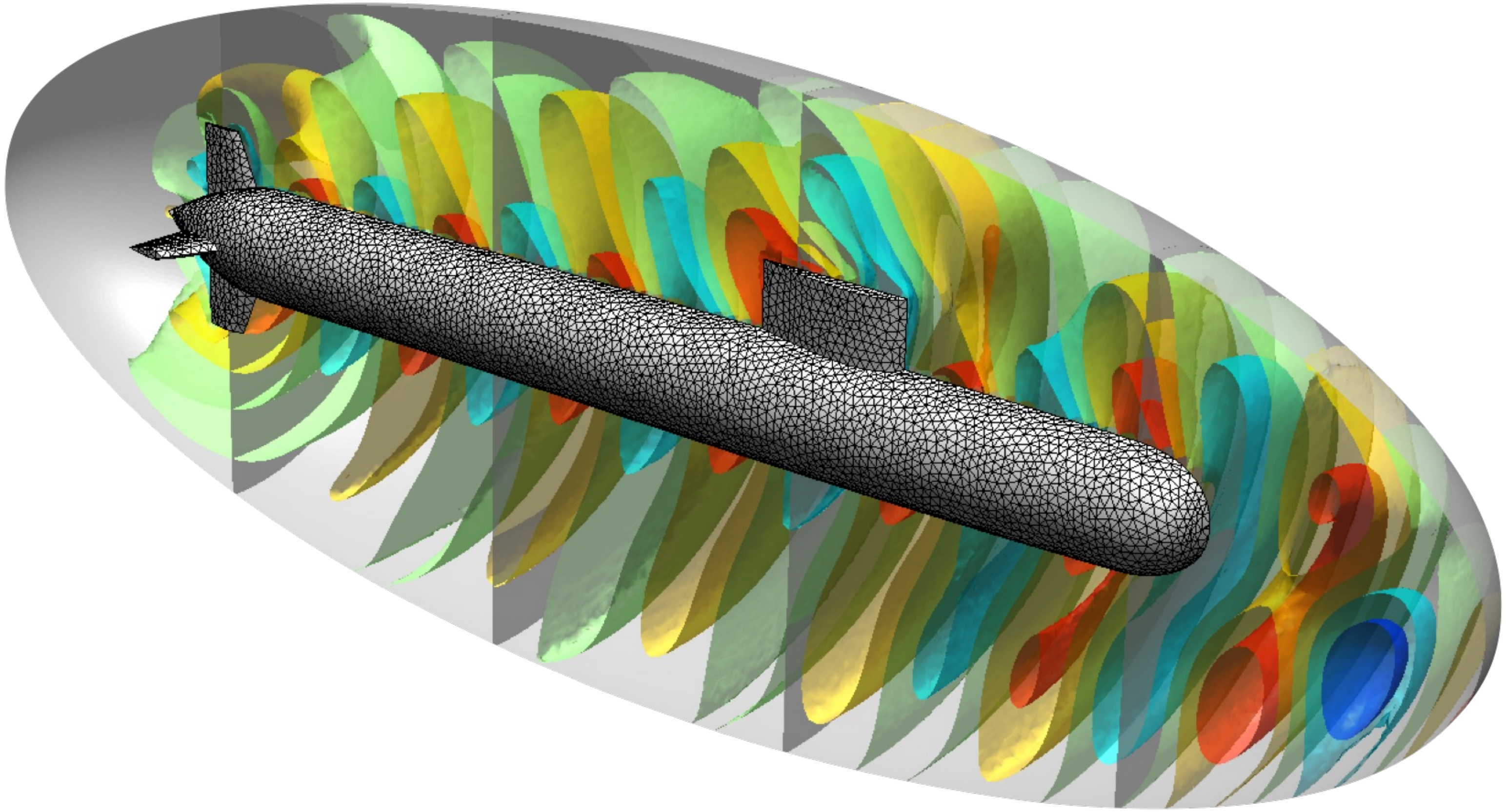
$$\omega = 2\pi f$$

Full wave
microstrip antenna

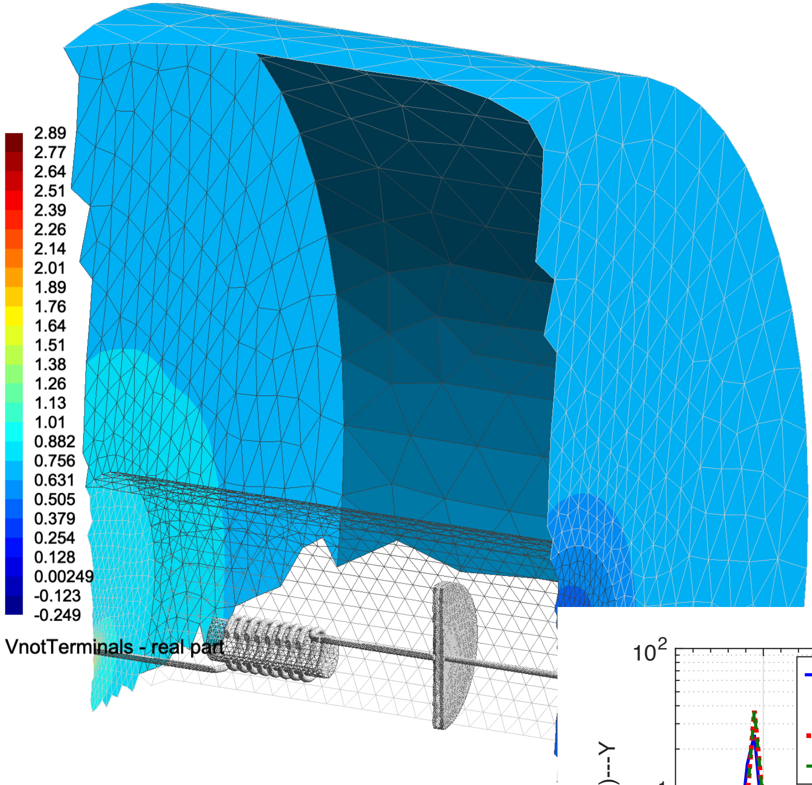


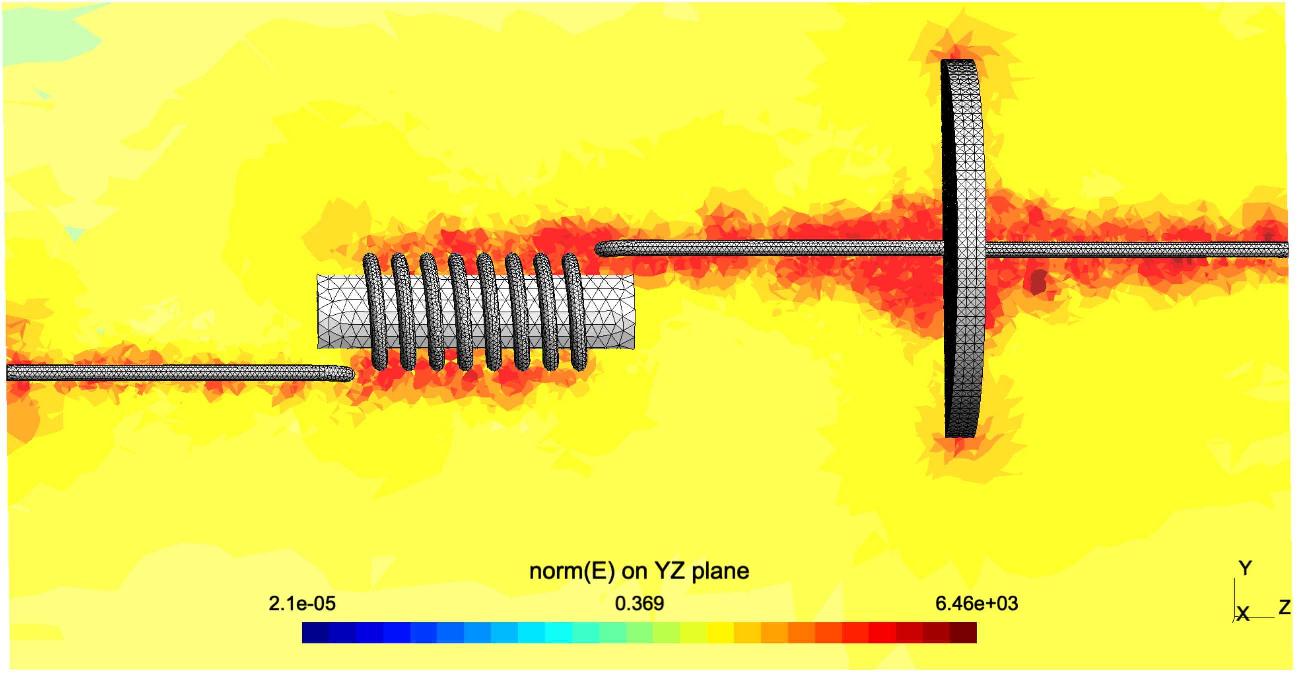


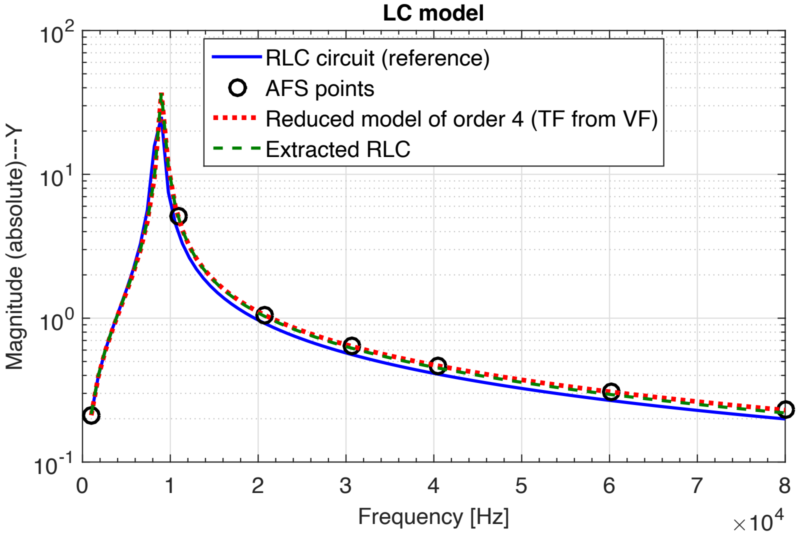
Full wave scattering by a submarine

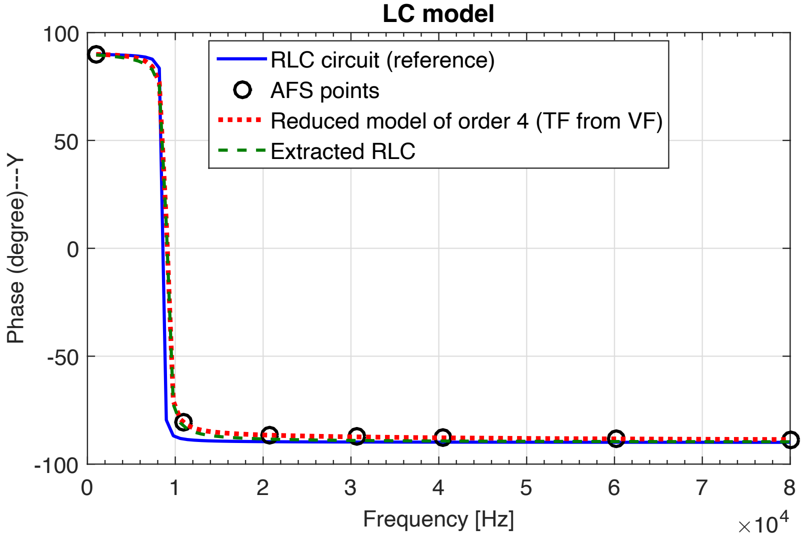


Full wave









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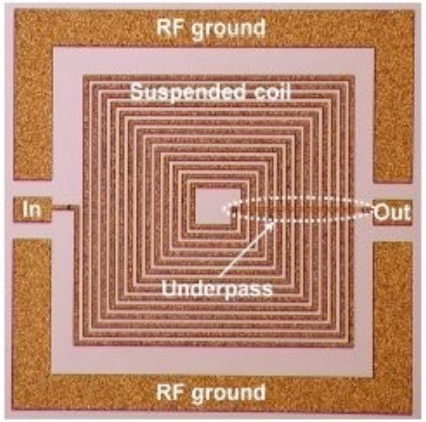
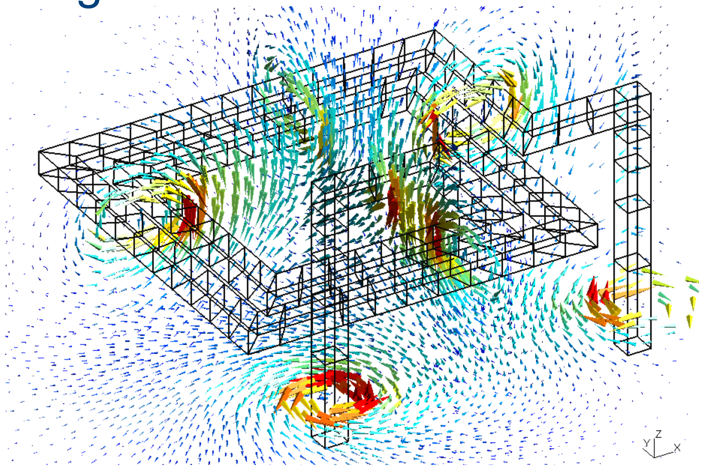
Ciuprina et al. *Journal of Mathematics in Industry, 2022,* Electric circuit element boundary conditions in the finite element method for full-wave passive electromagnetic devices

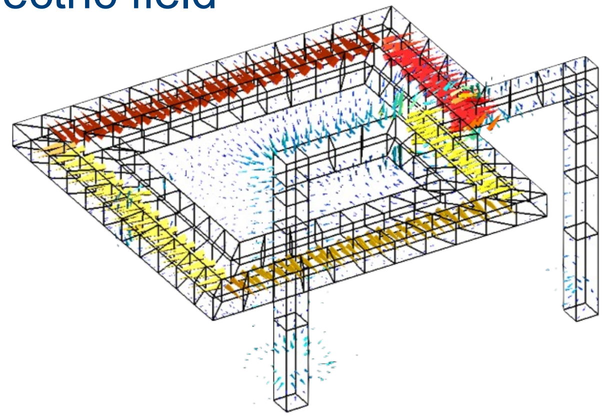
Full wave
inductive & capacitive effects in inductors



Magnetic induction



Electric field



Resistance, inductance and capacitance versus frequency