Continuous mathematical structure

Square integrable vector fields
Useful definitions

The solutions of Maxwell's equations belong to spaces of square integrable scalar and vector fields $L^2(\Omega)$ and $\boldsymbol{L}^2(\Omega)$. They are defined by

$$L^2(\Omega) = \left\{ u : \int_\Omega u^2(\boldsymbol{x})\, d\boldsymbol{x} < \infty \right\} \qquad \boldsymbol{L}^2(\Omega) = \left\{ \boldsymbol{u} : \int_\Omega \|\boldsymbol{u}(\boldsymbol{x})\|^2\, d\boldsymbol{x} < \infty \right\}$$

The scalar product of $u, v \in L^2(\Omega)$ and $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$ is defined by

$$(u, v)_\Omega = \int_\Omega u(\boldsymbol{x})\, v(\boldsymbol{x})\, d\boldsymbol{x} \quad \text{and} \quad (\boldsymbol{u}, \boldsymbol{v})_\Omega = \int_\Omega \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x}$$

The norm of $u \in L^2(\Omega)$ and $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$ is defined by

$$\|u\|_{L^2(\Omega)} = (u, u)_\Omega^{1/2} = \left[\int_\Omega u^2(\boldsymbol{x})\, d\boldsymbol{x} \right]^{1/2} \qquad \|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} = (\boldsymbol{u}, \boldsymbol{u})_\Omega^{1/2} = \left[\int_\Omega \|\boldsymbol{u}(\boldsymbol{x})\|^2\, d\boldsymbol{x} \right]^{1/2}$$

Hilbert and Sobolev spaces

The spaces $L^2(\Omega)$ and $\boldsymbol{L}^2(\Omega)$ are *Hilbert spaces* and can welcome physical fields, characterised by a finite energy.

The subspaces of $L^2(\Omega)$ and $\boldsymbol{L}^2(\Omega)$ for which all first order partial derivatives are also square integrable are known as the *Sobolev spaces* of scalar fields $H^1(\Omega)$ and vector fields $\boldsymbol{H}^1(\Omega)$, respectively. They are defined as

$$H^1(\Omega) = \left\{u \in L^2(\Omega) : \partial_x u, \partial_y u, \partial_z u \in L^2(\Omega)\right\}$$

$$\boldsymbol{H}^1(\Omega) = \left\{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega) : \partial_x \boldsymbol{u}, \partial_y \boldsymbol{u}, \partial_z \boldsymbol{u} \in \boldsymbol{L}^2(\Omega)\right\}$$

$$H^p(\Omega) = \left\{u \in H^{p-1}(\Omega) : \partial_x u, \partial_y u, \partial_z u \in H^{p-1}(\Omega)\right\}$$

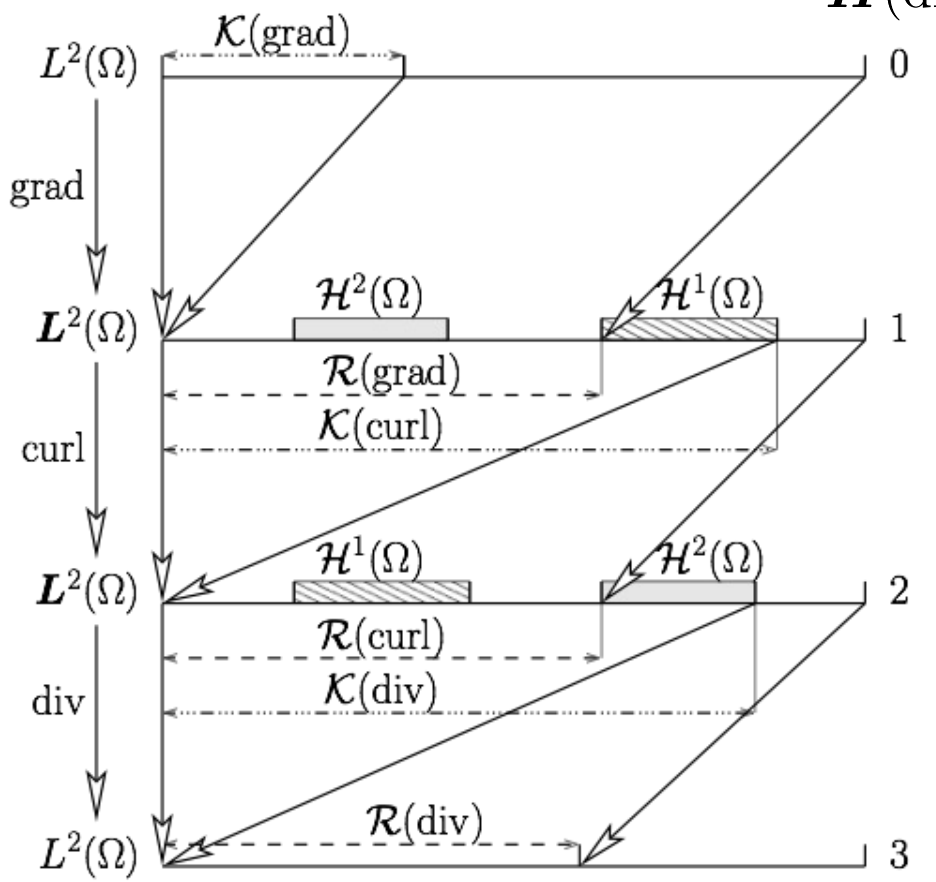
$$\boldsymbol{H}^p(\Omega) = \left\{\boldsymbol{u} \in \boldsymbol{H}^{p-1}(\Omega) : \partial_x \boldsymbol{u}, \partial_y \boldsymbol{u}, \partial_z \boldsymbol{u} \in \boldsymbol{H}^{p-1}(\Omega)\right\}$$

de Rham complex

$$H(\mathrm{grad};\Omega) = \{u \in L^2(\Omega) : \mathrm{grad}\, u \in \boldsymbol{L}^2(\Omega)\}$$

$$\boldsymbol{H}(\mathrm{curl};\Omega) = \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega) : \mathrm{curl}\, \boldsymbol{u} \in \boldsymbol{L}^2(\Omega)\}$$

$$\boldsymbol{H}(\mathrm{div};\Omega) = \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega) : \mathrm{div}\, \boldsymbol{u} \in L^2(\Omega)\}$$



$$H^1(\Omega) \xrightarrow{\mathrm{grad}} \boldsymbol{H}(\mathrm{curl};\Omega) \xrightarrow{\mathrm{curl}} \boldsymbol{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega)$$

$$\mathcal{H}^1(\Omega) = \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega) : \mathrm{curl}\, \boldsymbol{u} = 0, \mathrm{div}\, \boldsymbol{u} = 0, \boldsymbol{n} \cdot \boldsymbol{u}|_\Gamma = 0\}$$

$$\mathcal{H}^2(\Omega) = \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega) : \mathrm{curl}\, \boldsymbol{u} = 0, \mathrm{div}\, \boldsymbol{u} = 0, \boldsymbol{n} \times \boldsymbol{u}|_\Gamma = 0\}$$

Maxwell's equations

$$\text{grad}\, f_0 \equiv \nabla f_0 = (\partial_x, \partial_y, \partial_z) f_0$$
$$\text{curl}\, \boldsymbol{f}_1 \equiv \nabla \times \boldsymbol{f}_1 \equiv (\partial_x, \partial_y, \partial_z) \times \boldsymbol{f}_1$$
$$\text{div}\, \boldsymbol{f}_2 \equiv \nabla \cdot \boldsymbol{f}_2 \equiv (\partial_x, \partial_y, \partial_z) \cdot \boldsymbol{f}_2$$

$$\text{curl}\, \boldsymbol{h} - \partial_t \boldsymbol{d} = \boldsymbol{j}$$
$$\text{curl}\, \boldsymbol{e} + \partial_t \boldsymbol{b} = 0$$
$$\text{div}\, \boldsymbol{b} = 0$$
$$\text{div}\, \boldsymbol{d} = q$$

$$\boldsymbol{b} = \mathcal{B}(\boldsymbol{e}, \boldsymbol{h}) = \mu \boldsymbol{h} \ (+\boldsymbol{b}_r)$$
$$\boldsymbol{d} = \mathcal{D}(\boldsymbol{e}, \boldsymbol{h}) = \epsilon \boldsymbol{e} \ (+\boldsymbol{d}_{src})$$
$$\boldsymbol{j} = \mathcal{J}(\boldsymbol{e}, \boldsymbol{h}) = \sigma \boldsymbol{e} \ (+\boldsymbol{j}_{src})$$

v — 0-form = scalar, continuous field

↓ grad

$\boldsymbol{h}, \boldsymbol{e}$ — 1-form = vector field of continuous tangential component

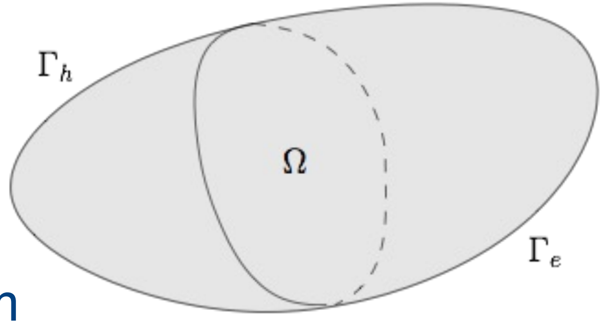
↓ curl

$\boldsymbol{b}, \boldsymbol{j}, \boldsymbol{d}$ — 2-form = vector field of continuous normal component

↓ div

q — 3-form = scalar field

Maxwell's house —Tonti diagram



$$\Gamma = \Gamma_e \cup \Gamma_h$$

0-form → 1-form → 2-form → 3-form

$$\varphi \quad H_h^1(\Omega) \longleftrightarrow L^2(\Omega)$$

$\mathrm{grad}_h \downarrow \qquad \uparrow \mathrm{div}_e$

$$\boldsymbol{h},\ \boldsymbol{t} \quad \boldsymbol{H}_h(\mathrm{curl};\Omega) \overset{\mu}{\longleftrightarrow} \boldsymbol{H}_e(\mathrm{div};\Omega) \quad \boldsymbol{b}$$

$\mathrm{curl}_h \downarrow \qquad \uparrow \mathrm{curl}_e$

$$\boldsymbol{d},\ \boldsymbol{j} \quad \boldsymbol{H}_h(\mathrm{div};\Omega) \overset{\epsilon,\ \sigma}{\longleftrightarrow} \boldsymbol{H}_e(\mathrm{curl};\Omega) \quad \boldsymbol{e},\ \boldsymbol{a}$$

$\mathrm{div}_h \downarrow \qquad \uparrow \mathrm{grad}_e$

$$q \quad L^2(\Omega) \longleftrightarrow H_e^1(\Omega) \quad v$$

3-form ↑ 2-form ↑ 1-form ↑ 0-form

boundary conditions accounted for in subspaces

boundary split in two parts

square integrable scalar & vector fields: field + field with differential operator

$$H_u^{10}(\Omega) = \{u \in L^2(\Omega) : \mathrm{grad}\, u \in \boldsymbol{L}^2(\Omega), u|_{\Gamma_u} = 0\}$$

$$\boldsymbol{H}_{\boldsymbol{u}}^0(\mathrm{curl};\Omega) = \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega) : \mathrm{curl}\, \boldsymbol{u} \in \boldsymbol{L}^2(\Omega), \boldsymbol{n} \times \boldsymbol{u}|_{\Gamma_u} = 0\}$$

$$\boldsymbol{H}_{\boldsymbol{u}}^0(\mathrm{div};\Omega) = \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega) : \mathrm{div}\, \boldsymbol{u} \in L^2(\Omega), \boldsymbol{n} \cdot \boldsymbol{u}|_{\Gamma_u} = 0\}$$

$u = e \text{ or } h$

Strong and weak formulations

notation

$(u,v)_\Omega = \int_\Omega u \cdot v \, d\Omega$ $\quad \langle u,v\rangle_\Gamma = \int_\Gamma u \cdot v \, d\Gamma$

Strong formulation

$\mathcal{L}u = f$ in Ω

$\mathcal{B}u = g$ on Γ

- ✓ $\mathcal{L}$ differential operator of order n
- ✓ $\mathcal{L}^*$ adjoint of $\mathcal{L}$
- ✓ $\mathcal{B}$ differential operator imposing BC
- ✓ f function in Ω, g function on $\Gamma = \partial\Omega$
- ✓ u unknown function
- ✓ Q_g linear function of v

Weak formulation

find u so that

$$(u, \mathcal{L}^* v)_\Omega - (f, v)_\Omega + \int_\Gamma Q_g(v)\, ds = 0\,, \ \forall v \in V(\Omega)$$

Continuous system $\Rightarrow \infty \times \infty$
Discrete system $\Rightarrow N \times N$
$\Rightarrow$ **numerical solution**

Green formulae

$$(\mathcal{L}u, v)_\Omega - (u, \mathcal{L}^* v)_\Omega = \int_\Gamma Q(u,v) ds$$

Q bilinear function of u and v

grad-div type

$$(\boldsymbol{v}, \operatorname{grad} u)_\Omega + (\operatorname{div} \boldsymbol{v}, u)_\Omega = \langle u, \hat{n} \cdot \boldsymbol{v}\rangle_\Gamma$$

curl-curl type

$$(\boldsymbol{v}, \operatorname{curl} \boldsymbol{w})_\Omega - (\operatorname{curl} \boldsymbol{v}, \boldsymbol{w})_\Omega = \langle \boldsymbol{v} \times \hat{n}, \boldsymbol{w}\rangle_\Gamma$$

Weak formulation – 1D example with no charges

Conservation law

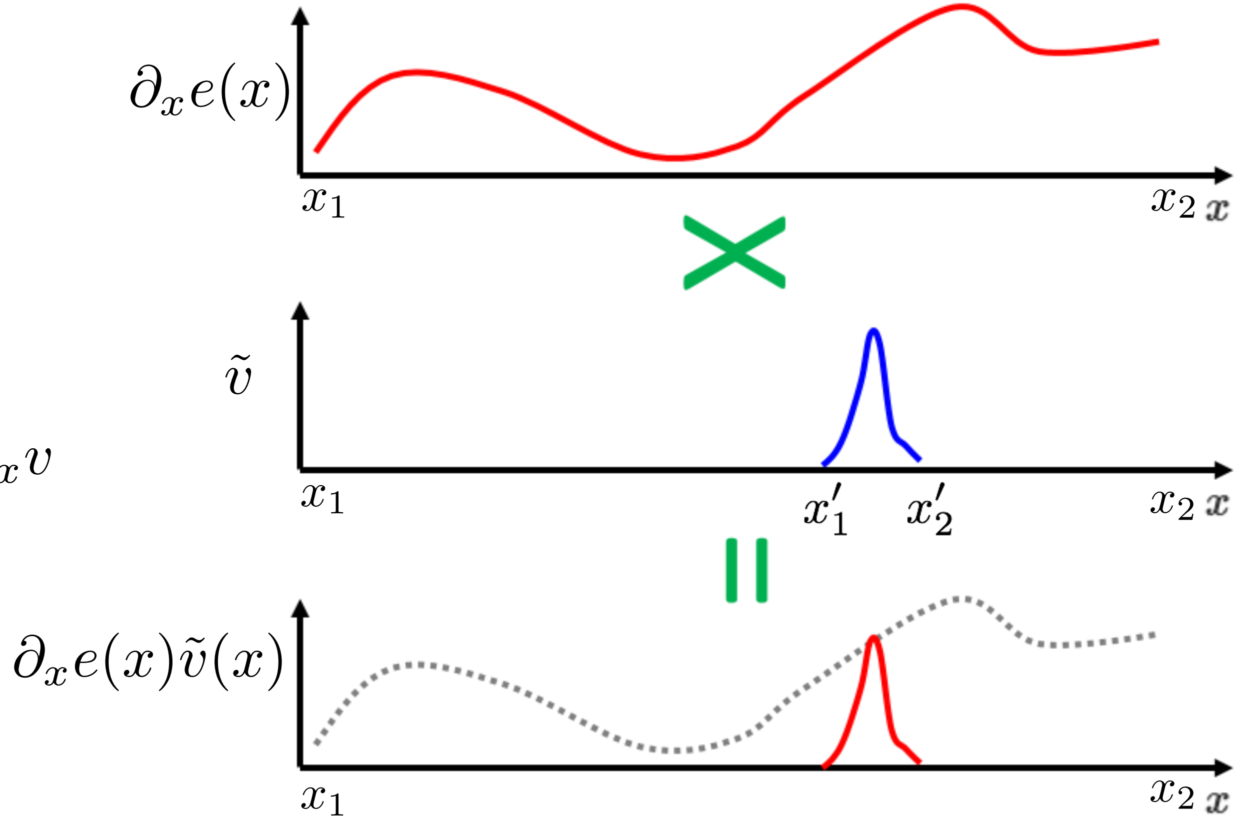
$$\mathrm{div}\, d = \mathrm{div}\, \epsilon e = q$$

$$\epsilon = 1\,, \quad q = 0$$

1D equation to solve

$$\partial_x e(x) = 0$$

$$e(x) = -\mathrm{grad}\, v = -\partial_x v$$



$$\int_{x_1}^{x_2} \partial_x e(x) \mathrm{d}x = 0$$

$$\int_{x_1'}^{x_2'} \partial_x e(x) \mathrm{d}x = 0$$

Weak form

$$\int_{x_1}^{x_2} \partial_x e(x)\, \tilde{v}(x)\, \mathrm{d}x = 0$$

Weak form after integration by parts

$$e(x_2)\tilde{v}(x_2) - e(x_1)\tilde{v}(x_1) - \int_{x_1}^{x_2} e(x)\, \partial_x \tilde{v}(x)\, \mathrm{d}x = 0$$

Strong and week formulations (cont'd)

e.g., strong electrostatic formulation

$$\boldsymbol{e} = -\operatorname{grad} v, \quad \boldsymbol{d} = \epsilon \boldsymbol{e}$$

$$-\Delta v = -\frac{\partial^2 v(x,y,z)}{\partial x^2} - \frac{\partial^2 v(x,y,z)}{\partial y^2} - \frac{\partial^2 v(x,y,z)}{\partial z^2} = \frac{q}{\epsilon} \quad \Rightarrow \begin{cases} \mathcal{L} = -\Delta \\ f = \dfrac{q}{\epsilon} \\ u = v \end{cases}$$

$$\boldsymbol{n} \times \boldsymbol{e}|_{\Gamma_e} = 0$$

governing equations and BCs

$$\boldsymbol{n} \cdot \boldsymbol{d}|_{\Gamma_d} = 0$$

e.g. weak electrostatic formulation

$$\int_\Omega -\Delta v \cdot w \, \mathrm{d}\Omega = \int_\Omega \Big(- \operatorname{div}(\operatorname{grad} v) \Big) \cdot w \, \mathrm{d}\Omega = \int_\Omega \frac{q}{\epsilon} w \, \mathrm{d}\Omega$$

$$\boldsymbol{v} \cdot \operatorname{grad} u + u \operatorname{div} \boldsymbol{v} = \operatorname{div}(u\boldsymbol{v})$$

Green formula: integrating by parts

$$\int_\Omega \Big(- \operatorname{div}(w \operatorname{grad} v) + \operatorname{grad} v \cdot \operatorname{grad} w \Big) \mathrm{d}\Omega = \int_\Omega \frac{q}{\epsilon} w \, \mathrm{d}\Omega$$

Constraints

$$\mathcal{L}u = f \text{ in } \Omega$$
$$\mathcal{B}u = g \text{ on } \Gamma$$

- Local constraints
  - boundary conditions (BCs) on local fields at the boundary of the domain
    - their choice influences the final solution
    - they can be exploited to reduce the computational domain
  - interface conditions (ICs): coupling of fields between subdomains

- Global constraints
  - flux or circulations of fields to be fixed (current, voltage, e.m.f., m.m.f., charge)
  - flux or circulations of fields to be connected (circuit coupling)

$$\mathcal{L}u = f \text{ in } \Omega$$
$$\mathcal{B}u = g \text{ on } \Gamma$$

Boundary conditions (BCs)

- Dirichlet BCs: fixing the unknown at the boundary to a given value

$$u|_\Gamma = u_0 \begin{cases} = 0\,, & \text{homogeneous BC} \\ \neq 0\,, & \text{inhomogeneous BC} \end{cases}$$

- Neumann BCs: fixing the normal derivative of the unknown at the boundary to a given value

$$\frac{\partial u(\boldsymbol{x})}{\partial n} = w(\boldsymbol{x}), \ \ \boldsymbol{x} \text{ on } \Gamma$$

- Mixed BCs: a combination of Dirichlet and Neumann BCs

$$\frac{\partial u(\boldsymbol{x})}{\partial n} + f_1(\boldsymbol{x})u = f_2(\boldsymbol{x})\,, \ \ \boldsymbol{x} \text{ on } \Gamma$$

$f_1(\boldsymbol{x})u$ and $f_2(\boldsymbol{x})$ explicitly known

$$\mathcal{L}u = f \text{ in } \Omega$$
$$\mathcal{B}u = g \text{ on } \Gamma$$

Boundary conditions (BCs) (cont'd)

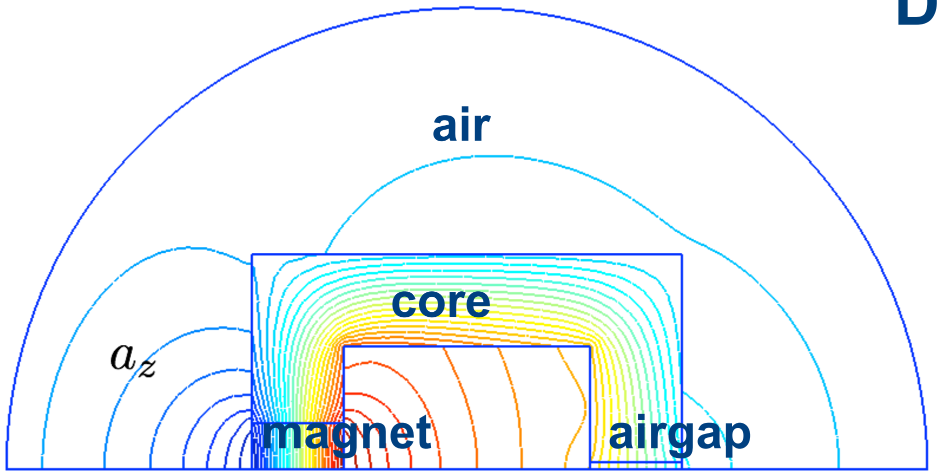
- Periodic BCs: fixing the unknown at the boundary to a given value

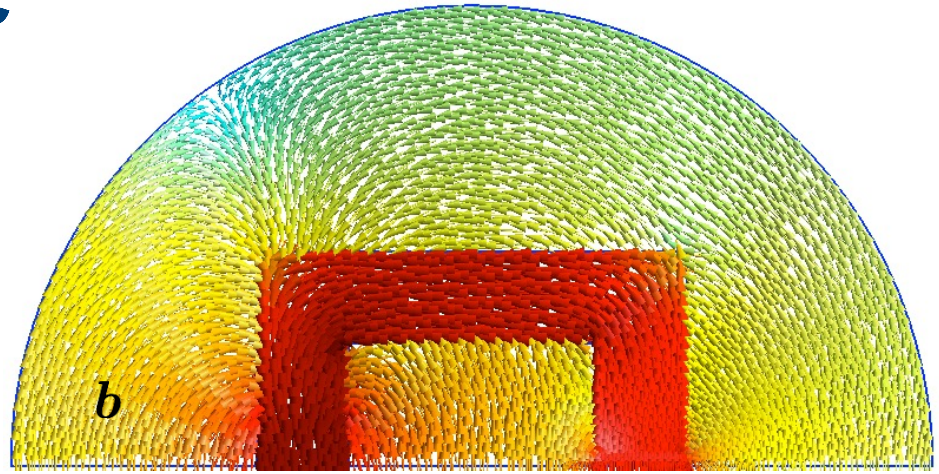
$$u(\boldsymbol{x}_0) + C_1 u(\boldsymbol{x}_1) = C_2\,, \quad \boldsymbol{x}_0 \text{ on } \Gamma_0\,,\ \boldsymbol{x}_1 \text{ on } \Gamma_1$$

$$\text{or } \frac{\partial u(\boldsymbol{x}_0)}{\partial n} + C_1' \frac{\partial u(\boldsymbol{x}_1)}{\partial n} = C_2'\,, \quad \boldsymbol{x}_0 \text{ on } \Gamma_0\,,\ \boldsymbol{x}_1 \text{ on } \Gamma_1$$

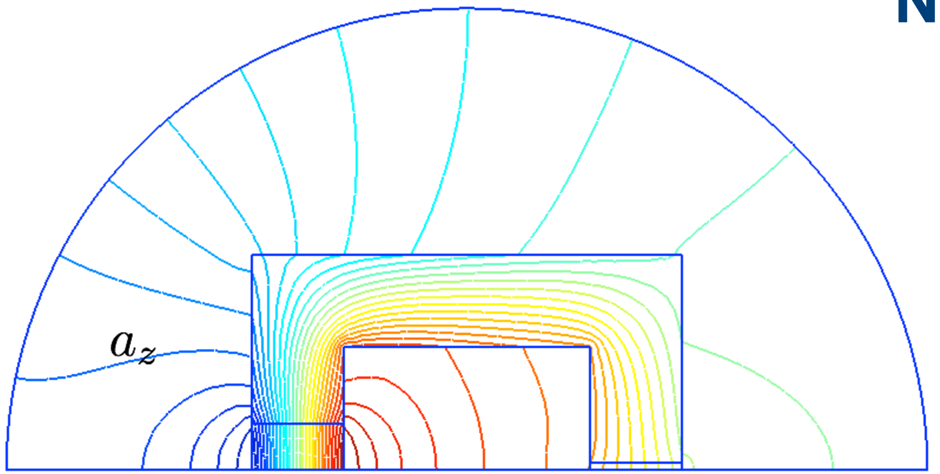
- Floating BCs: unknown fixed to a constant value that still must be determined, often related to global boundary conditions

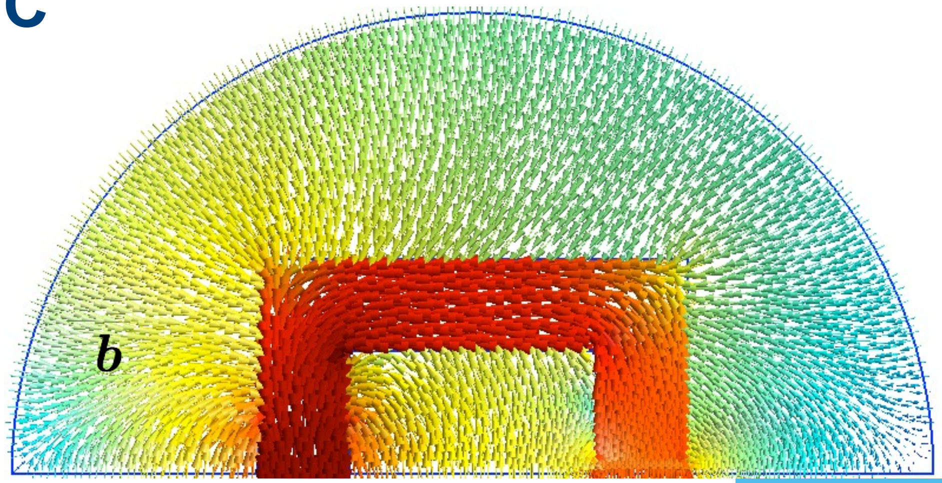
Dirichlet BC



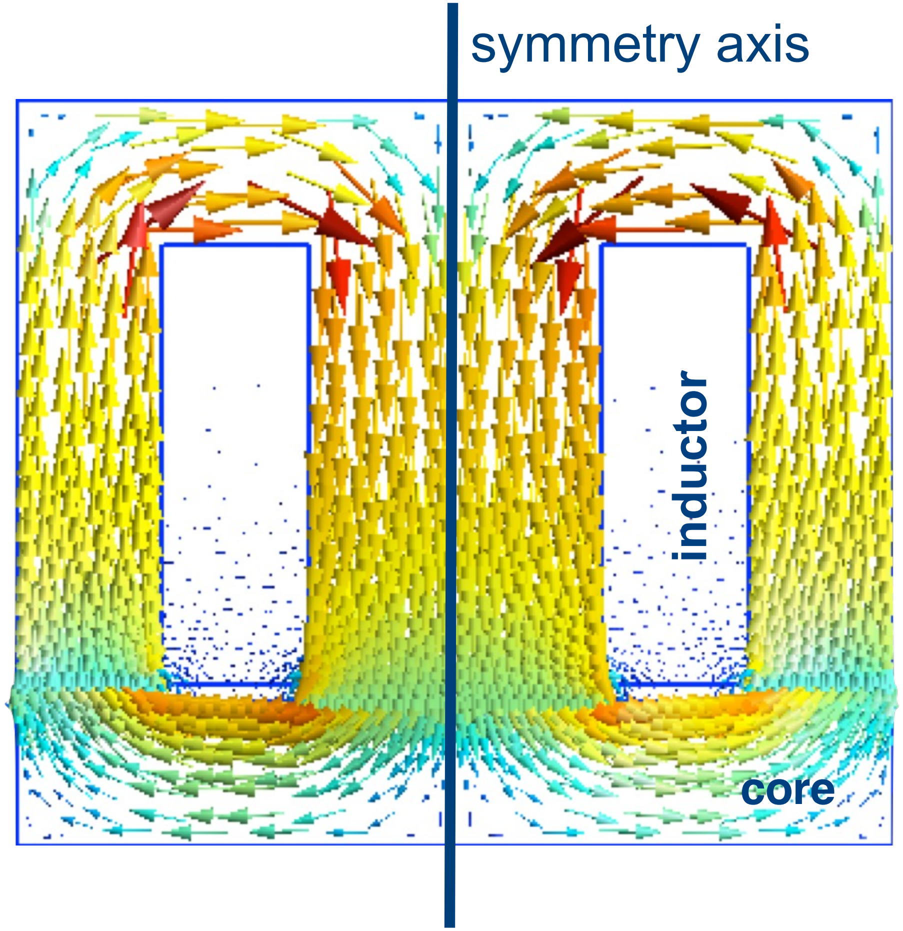


Neumann BC



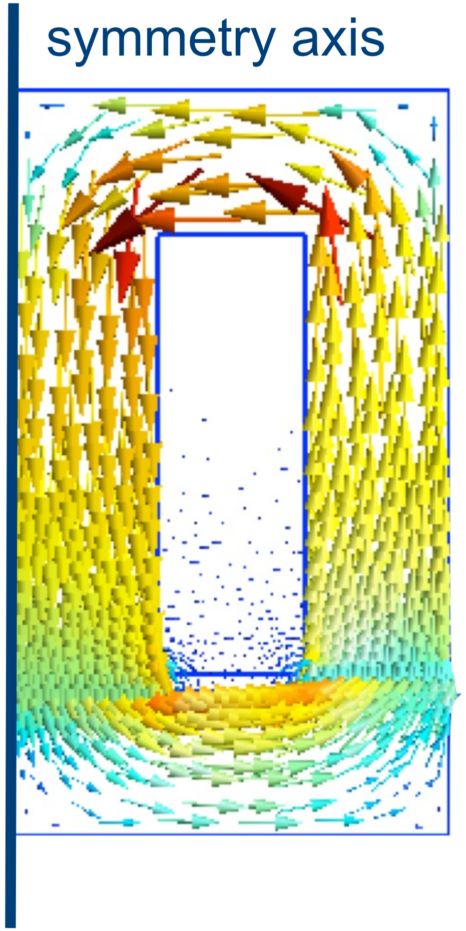


BCs — symmetry



normal component of induction = zero

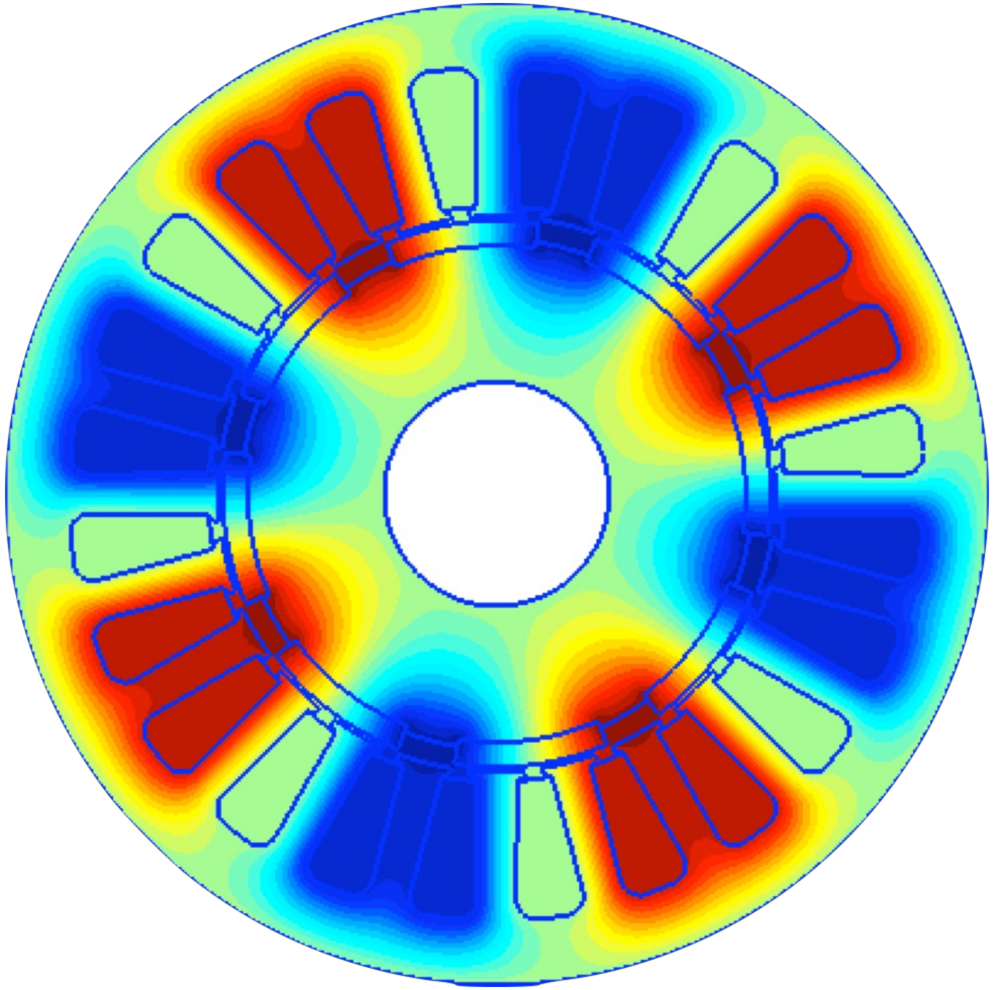
half model!



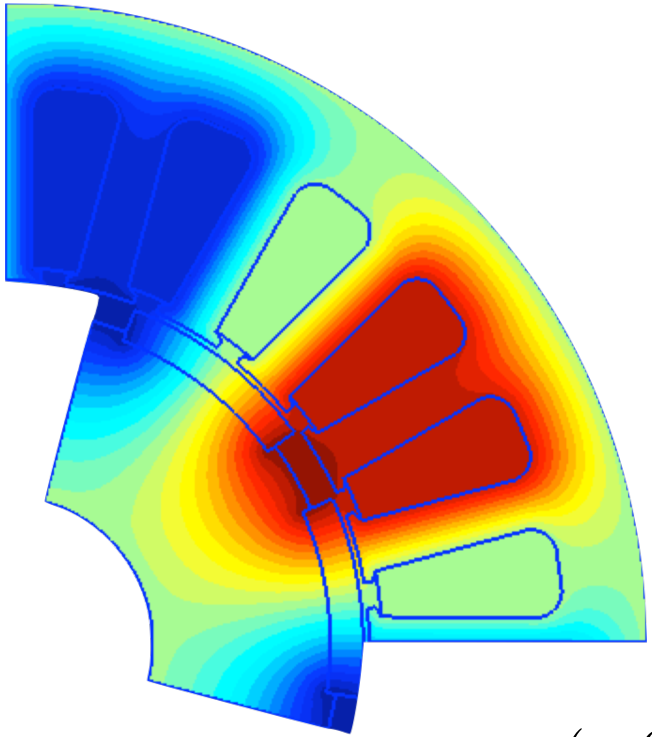
$$\boldsymbol{b} = \operatorname{curl} \boldsymbol{a}$$

$$\boldsymbol{b} = \hat{z} \times \operatorname{grad} a_z$$

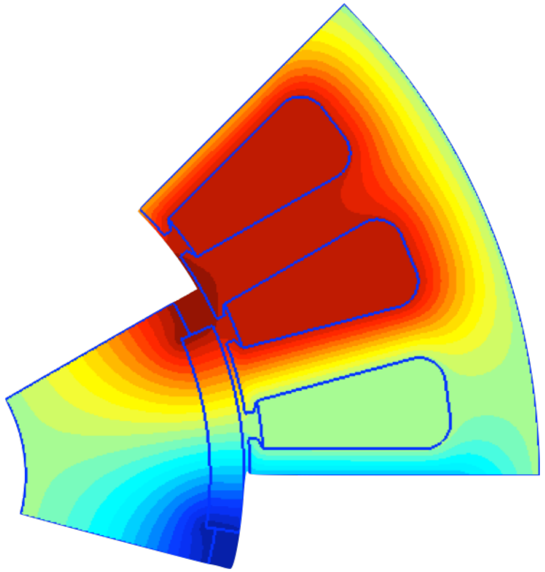
Periodic BCs



1/4 model
periodic BCs



1/8 model
anti-periodic BCs



$$a_z(r,\theta) = -a_z(r,\theta + (2k-1)\frac{\pi}{p})$$

$$a_z(r,\theta) = a_z(r,\theta + 2k\frac{\pi}{p})$$

$p =$ number of poles $\quad k = 1, 2, 3, \ldots$

Discrete mathematical structure
Whitney elements

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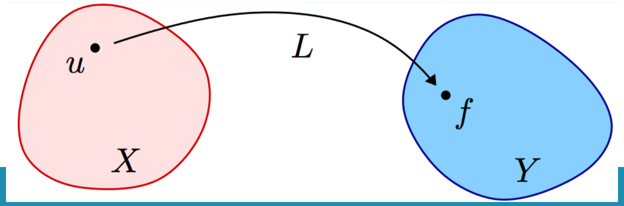
Discrete mathematical structure

Replace the continuous spaces (infinite dimension) by discrete spaces (finite dimension)

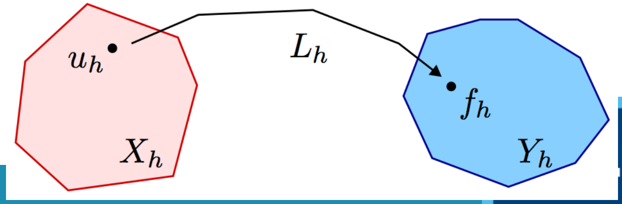
$$
\begin{array}{ccc}
H^1_h(\Omega) & \longleftrightarrow & L^2(\Omega) \\
\mathrm{grad}_h \big\downarrow & & \big\uparrow \mathrm{div}_e \\
\boldsymbol{H}_h(\mathrm{curl};\Omega) & \longleftrightarrow & \boldsymbol{H}_e(\mathrm{div};\Omega) \\
\mathrm{curl}_h \big\downarrow & & \big\uparrow \mathrm{curl}_e \\
\boldsymbol{H}_h(\mathrm{div};\Omega) & \longleftrightarrow & \boldsymbol{H}_e(\mathrm{curl};\Omega) \\
\mathrm{div}_h \big\downarrow & & \big\uparrow \mathrm{grad}_e \\
L^2(\Omega) & \longleftrightarrow & H^1_e(\Omega)
\end{array}
$$



local function spaces on geometrical element $\mathcal{G}$

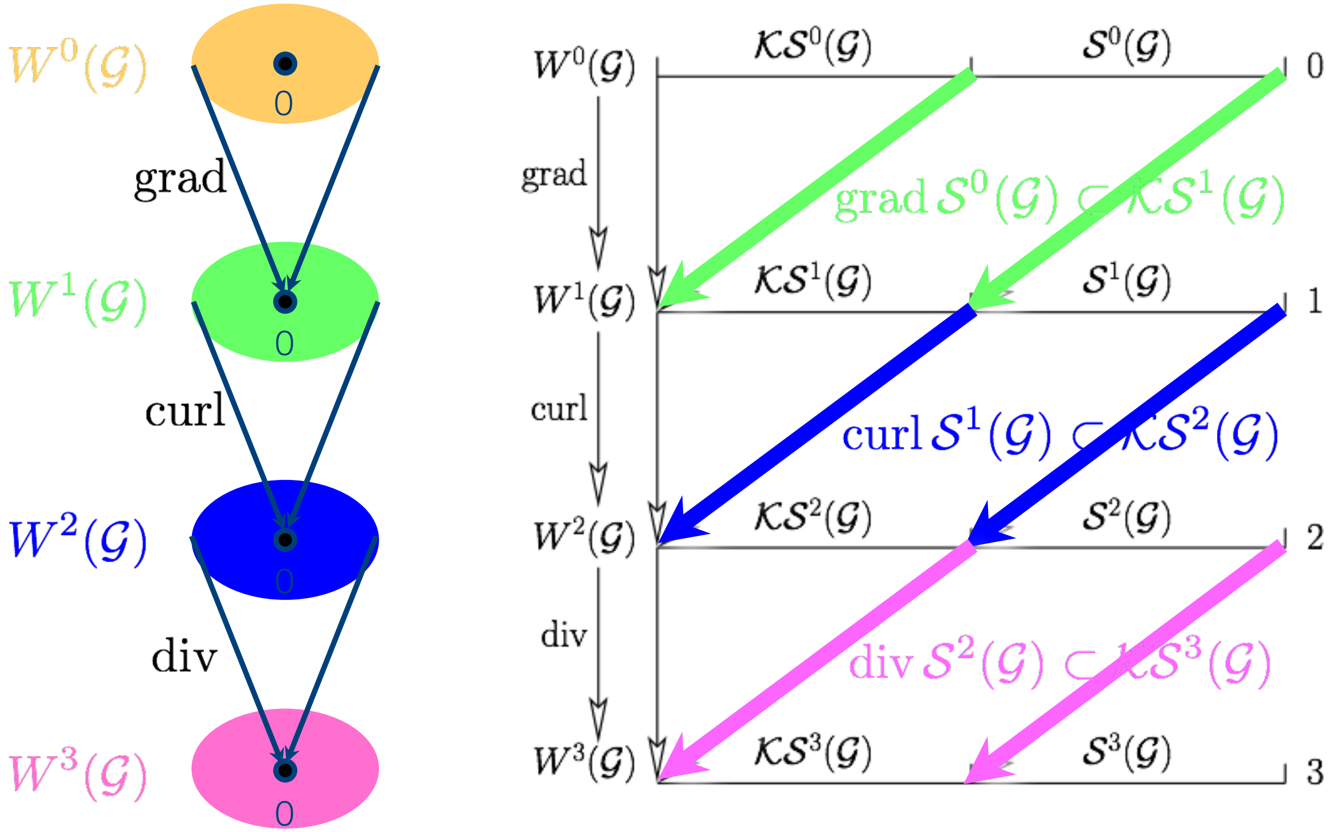
$$
\begin{array}{ccc}
W^0(\mathcal{G}) & \longleftrightarrow & W^3(\mathcal{G}) \\
\mathrm{grad} \big\downarrow & & \big\uparrow \mathrm{div} \\
W^1(\mathcal{G}) & \longleftrightarrow & W^2(\mathcal{G}) \\
\mathrm{curl} \big\downarrow & & \big\uparrow \mathrm{curl} \\
W^2(\mathcal{G}) & \longleftrightarrow & W^1(\mathcal{G}) \\
\mathrm{div} \big\downarrow & & \big\uparrow \mathrm{grad} \\
W^3(\mathcal{G}) & \longleftrightarrow & W^0(\mathcal{G})
\end{array}
$$

$$W^0(\mathcal{G}) \subset H^1(\mathcal{G})$$
$$W^1(\mathcal{G}) \subset \boldsymbol{H}(\mathrm{curl};\mathcal{G})$$
$$W^2(\mathcal{G}) \subset \boldsymbol{H}(\mathrm{div};\mathcal{G})$$
$$W^3(\mathcal{G}) \subset L^2(\mathcal{G})$$



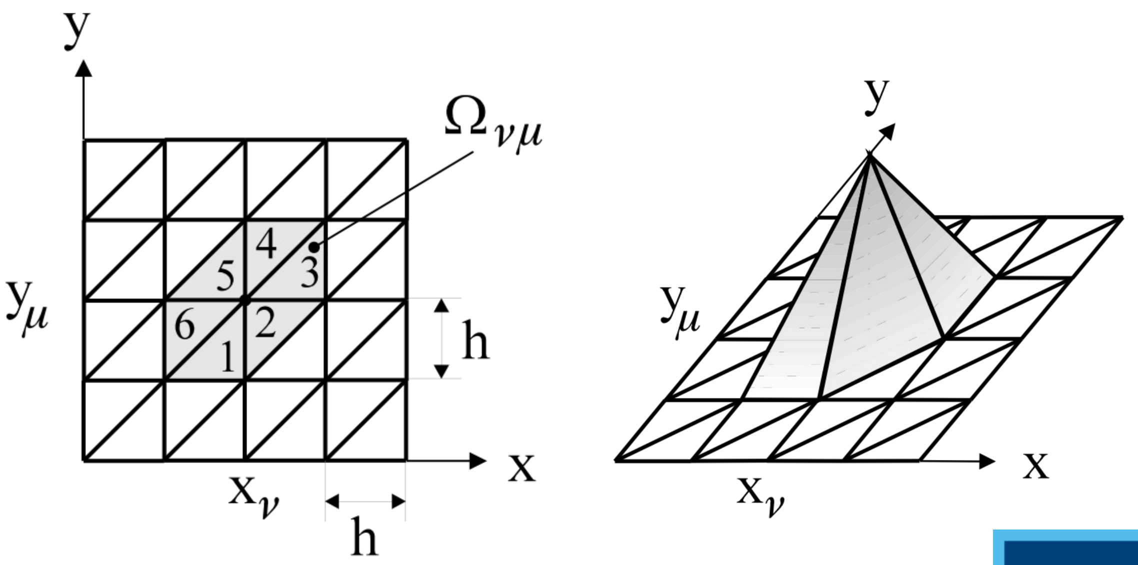
The Whitney complex

$W^p(\mathcal{G})$ is the finite dimensional subspace spanned by the p-Whitney elements on $\mathcal{G}$. They satisfy the property of conformity:



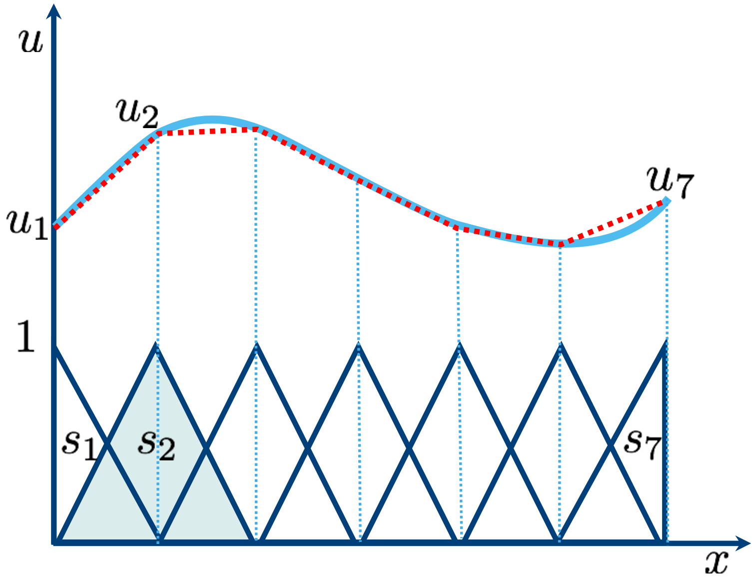
Finite elements

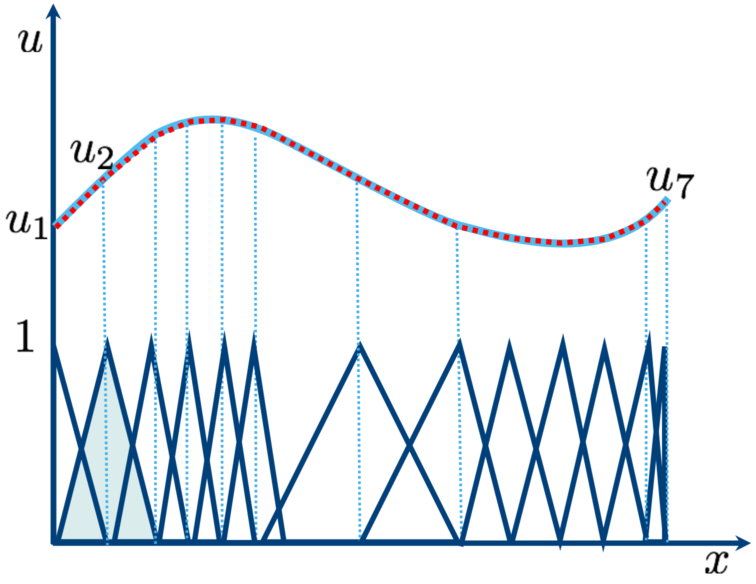
- set of linearly independent basis/shape functions and weighting functions (also called test and trial functions)
- commonly piecewise polynomial
- defined at a structured or unstructured grid/mesh
- compact support
- scalar or vectorial functions



Finite elements
Lowest order basis functions in 1D

$$u(\boldsymbol{x}) \approx u_h(\boldsymbol{x}) = \sum_{i=1}^{N} u_i s_i(\boldsymbol{x})$$

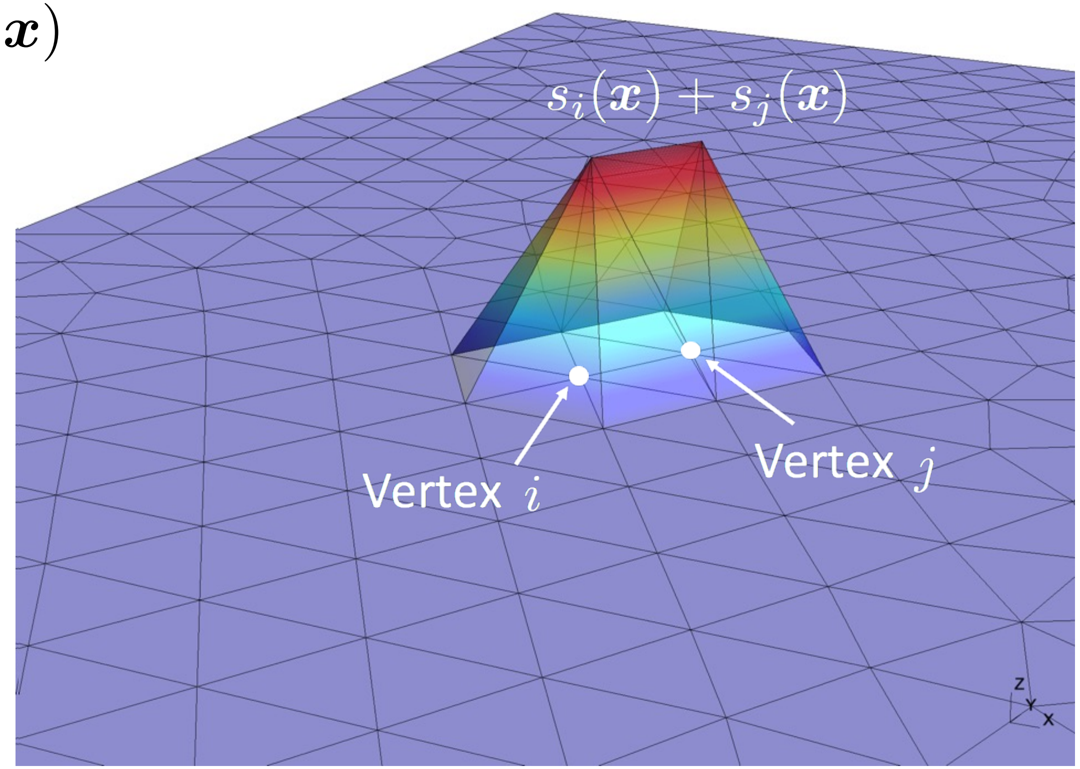




$$F^0(\Omega) = \mathrm{span}\{s_1(\boldsymbol{x}), s_2(\boldsymbol{x}), \ldots s_N(\boldsymbol{x})\}$$

Finite elements
Lowest order basis functions in 2D

$$u(\boldsymbol{x}) \approx u_h(\boldsymbol{x}) = \sum_{i=1}^{N} u_i s_i(\boldsymbol{x})$$

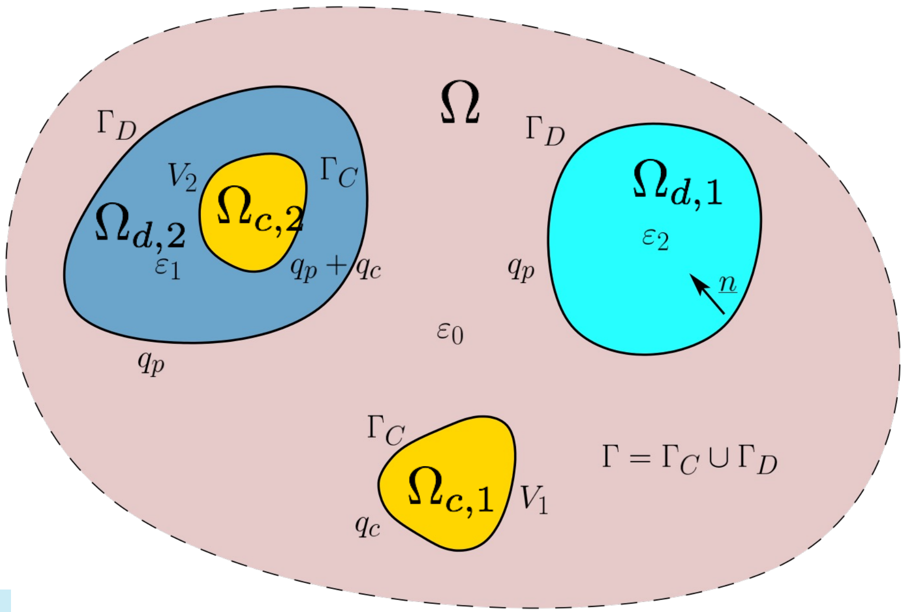


$$F^0(\Omega) = \mathrm{span}\{s_1(\boldsymbol{x}), s_2(\boldsymbol{x}), \ldots s_N(\boldsymbol{x})\}$$

Electrostatic formulations and discretization

Electrostatics

Phenomena involving time-independent distributions of charges & fields



“$\boldsymbol{d}$ side”

electric vector potential ($\boldsymbol{u}$-) formulation

“$\boldsymbol{e}$ side”

electric scalar potential (v-) formulation

$$\operatorname{curl} \boldsymbol{e} = 0$$
$$\operatorname{div} \boldsymbol{d} = q$$
$$\boldsymbol{d} = \epsilon\, \boldsymbol{e}$$

boundary conditions

$$\boldsymbol{n} \times \boldsymbol{e}|_{\Gamma_e} = 0$$
$$\boldsymbol{n} \cdot \boldsymbol{d}|_{\Gamma_d} = 0$$

Electrostatics

$\operatorname{curl} \boldsymbol{e} = 0, \qquad \operatorname{div} \boldsymbol{d} = q, \qquad \boldsymbol{d} = \epsilon\, \boldsymbol{e}$

Heat conduction problem
ϵ by $\kappa =$ thermal conductivity
v by $\theta =$ temperature
q by $f =$ thermal power

grad_h | div_e

edge BF $\quad u \longleftrightarrow 0 = \operatorname{curl} \boldsymbol{e}$

curl_h | curl_e

facet BF $\quad \boldsymbol{d} \longleftrightarrow \boldsymbol{e} = -\operatorname{grad} v$, with $\boldsymbol{d} = \epsilon \boldsymbol{e}$ (**edge BF**)

div_h | grad_e

volume BF $\quad \operatorname{div} \boldsymbol{d} = q \qquad$ **nodal BF** $\quad v$

electric scalar potential formulation

$$-\operatorname{div}(\epsilon \operatorname{grad} v) = q$$

Poisson equation for inhomogeneous media

$\Downarrow$ no charges & homogeneous medium

$$-\operatorname{div} \operatorname{grad} v = -\Delta v = 0$$

Laplace equation

Ampere's law (div of) verified in a **weak** sense

Spatial discretization — electrostatics

We want to find the electric scalar potential $v(\boldsymbol{x})$ everywhere in Ω

$$-\mathrm{div}\,(\epsilon\,\mathrm{grad}\,v) = q \quad \text{in } \Omega$$

with given

$q(\boldsymbol{x})$ known charge density (possibly $= 0$)
$\epsilon(\boldsymbol{x})$ permittivity > 0 in part of the domain

Weighted residual approach

We integrate the equation weighted by test functions $w_i(\boldsymbol{x})$ over the whole domain:

find v such that

$$\int_\Omega \Big(-\,\mathrm{div}\,(\epsilon\,\mathrm{grad}\,v)\Big) w_i \,\mathrm{d}\Omega = \int_\Omega q\, w_i \,\mathrm{d}\Omega$$

holds $\forall w_i(\boldsymbol{x})$

Spatial discretization — electrostatics (II)

$$\int_\Omega \Big(- \operatorname{div}(\epsilon \operatorname{grad} v)\Big) w_i \,\mathrm{d}\Omega = \int_\Omega q\, w_i \,\mathrm{d}\Omega$$

search wiki vector calculus

$$\boldsymbol{v} \cdot \operatorname{grad} s + s \operatorname{div} \boldsymbol{v} = \operatorname{div}(s\,\boldsymbol{v})$$

integration by parts
Green formula

$$\int_\Omega \Big(- \operatorname{div}(w_i \epsilon \operatorname{grad} v) + \epsilon \operatorname{grad} v \cdot \operatorname{grad} w_i \Big) \,\mathrm{d}\Omega = \int_\Omega q\, w_i \,\mathrm{d}\Omega$$

$$\int_\Omega \operatorname{div} \boldsymbol{v} \,\mathrm{d}\Omega = \oint_\Gamma \boldsymbol{v}\,\mathrm{d}\boldsymbol{\Gamma}\,, \qquad \mathrm{d}\boldsymbol{\Gamma} = \boldsymbol{n}\,\mathrm{d}\Gamma$$

divergence theorem

Weak formulation

find v such that

$$\int_\Gamma w_i(-\epsilon \operatorname{grad} v)\boldsymbol{n}\,\mathrm{d}\Gamma + \int_\Omega \epsilon \operatorname{grad} v \cdot \operatorname{grad} w_i \,\mathrm{d}\Omega = \int_\Omega q\, w_i \,\mathrm{d}\Omega$$

holds $\forall w_i$

only the first derivative of the electric potential is now required

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Spatial discretization — electrostatics (III)

Boundary integral

$$\int_\Gamma w_i (-\epsilon \,\mathrm{grad}\, v)\, \mathrm{d}\Gamma$$

$$\boldsymbol{n} \cdot \boldsymbol{d} = d_n$$

$$\forall w_i(\boldsymbol{x})$$

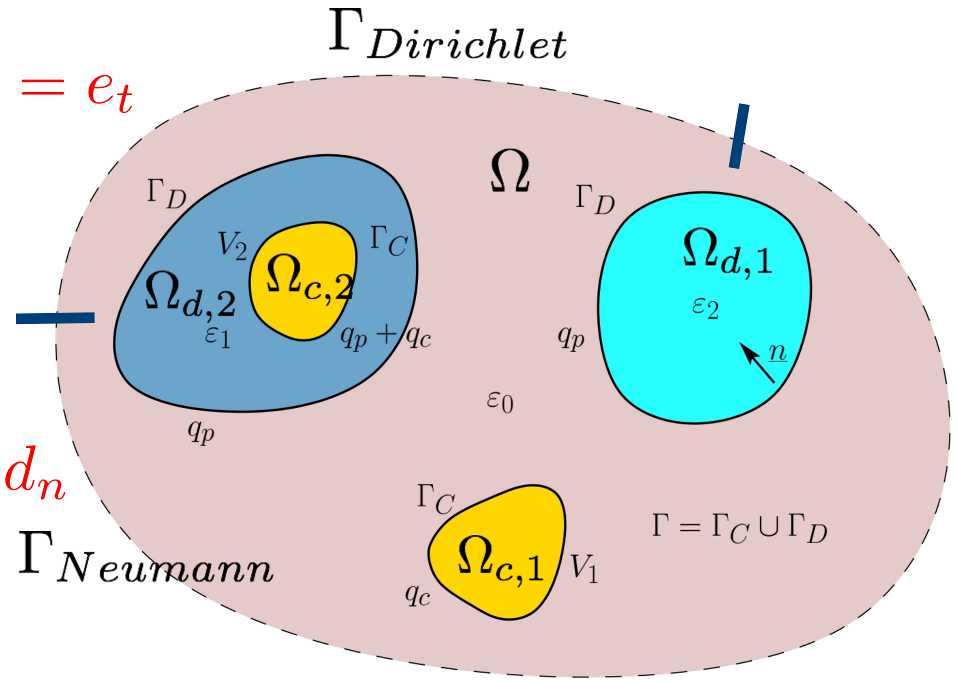
$$\int_{\Gamma_{Dirichlet}} w_i (-\epsilon \,\mathrm{grad}\, v)\, \mathrm{d}\Gamma + \int_{\Gamma_{Neumann}} w_i (-\epsilon \,\mathrm{grad}\, v)\, \mathrm{d}\Gamma$$

$= 0 \quad \forall w_i(\boldsymbol{x})$ if $w_i = 0$ on $\Gamma_{Dirichlet}$ — **essential BC**

$= 0$ — **natural BC**

$\boldsymbol{n} \times \boldsymbol{e} = e_t$ on $\Gamma_{Dirichlet}$

$\boldsymbol{n} \cdot \boldsymbol{d} = d_n$ on $\Gamma_{Neumann}$



Spatial discretization — electrostatics (IV)

$$v \approx v_h = \sum_j u_j s_j \quad \text{with } s_j(\boldsymbol{x}) = 0 \text{ at } \Gamma_{Dirichlet}$$

=> We use Whitney elements, i.e., nodal basis functions for discretising the electric scalar potential v

$$\begin{cases} s_j(\mathbf{x}) & \text{basis/shape functions} \\ u_j & \text{unknowns=degrees of freedom (Dof)} \end{cases}$$

Ritz-Galerkin method $w_j(\boldsymbol{x}) = s_j(\boldsymbol{x})$

Petrov-Galerkin method $w_j(\boldsymbol{x}) \neq s_j(\boldsymbol{x})$

$$\int_\Omega \epsilon \,\mathrm{grad}\, v \cdot \mathrm{grad}\, w_i \,\mathrm{d}\Omega = \int_\Omega q\, w_i \,\mathrm{d}\Omega$$

$$\sum_j u_j \int_\Omega \epsilon \,\mathrm{grad}\, w_j \cdot \mathrm{grad}\, w_i \,\mathrm{d}\Omega = \int_\Omega q\, w_i \,\mathrm{d}\Omega$$

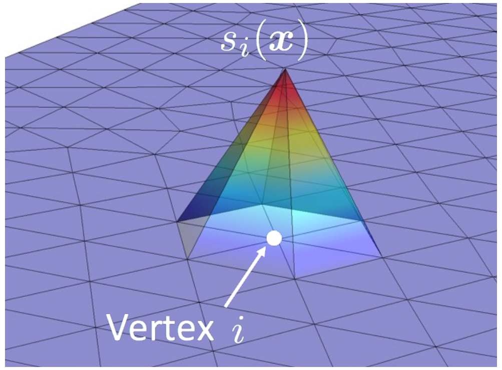
$$\underbrace{\sum_j u_j \epsilon_j \sum_i \mathrm{grad}\, w_j \cdot \mathrm{grad}\, w_i}_{= k_{ij}} = \underbrace{\sum_i q_i\, w_i}_{= f_i}$$

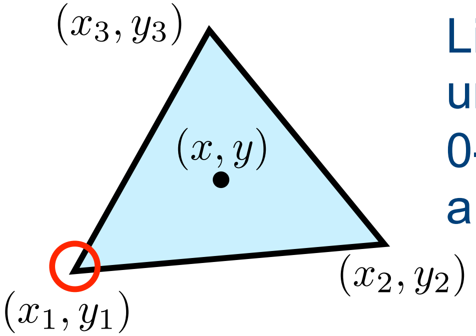
matrix system $[k_{ij}][u_j] = [f_i]$

Function space - electric scalar potential

$$v \approx v_h = \sum_{n \in \mathcal{N}} v_n s_n$$

```
FunctionSpace{
  { Name Hgrad_v; Type Form0; //discrete function space for H1_h
    BasisFunction {
      { Name sn; NameOfCoef vn; Function BF_Node; //‘‘P1 FEs’’
        Support Domain; Entity NodesOf[All]; }
    }
    Constraint {
      { NameOfCoef vn; EntityType NodesOf; NameOfConstraint ElectricScalarPotential; }
    }
  }
}
```





Linear triangular element with unknowns associated to nodes 0-form nodal BF associated to a triangle (2D mesh on slide)

Electrostatic formulation - build equation

```
Formulation {
  { Name Electrostatics_v ; Type FemEquation ;
    Quantity {
      { Name v ; Type Local ; NameOfSpace Hgrad_v ; }
    }
    Equation {
      Galerkin { [ epsilon[] * Dof{d v} , {d v} ] ;
        In Domain_Ele ; Jacobian Vol ; Integration GradGrad ; }
    }
  }
}

Constraint {
  { Name ElectricScalarPotential ;
    Case {
      { Region Dirichlet0 ; Value 0. ; }
      { Region Dirichlet1 ; Value V_imposed ; }
    }
  }
}
```

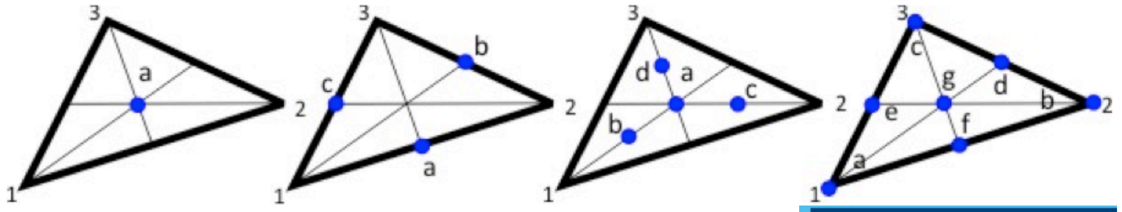
Set of basis functions and associated Dof

Set of weighing functions

$$\int_{\partial\Omega} w_i(-\epsilon\,\mathrm{grad}\,v)\,\mathrm{d}\Gamma + \int_{\Omega} \epsilon\,\mathrm{grad}\,v \cdot \mathrm{grad}\,w_i\,\mathrm{d}\Omega = 0$$

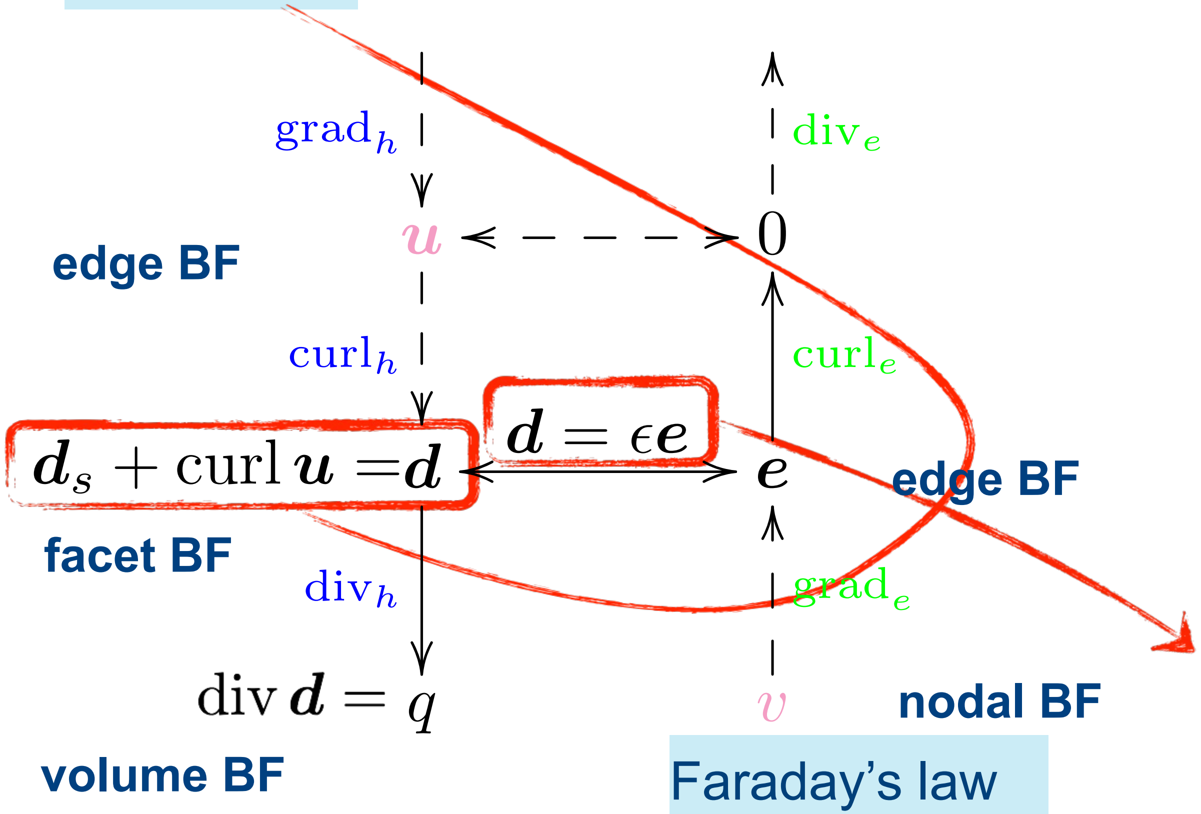
no charges

Dirichlet constraint



Electrostatics

$\operatorname{curl} \boldsymbol{e} = 0, \qquad \operatorname{div} \boldsymbol{d} = q, \qquad \boldsymbol{d} = \epsilon\, \boldsymbol{e}$



Faraday's law verified in a **weak** sense

$\boldsymbol{d}_s$ is any source field verifying
$\operatorname{div} \boldsymbol{d}_s = q$
$+$ gauge condition so that
$\operatorname{div} \operatorname{curl} \boldsymbol{u} = 0$

electric vector potential formulation

$$\operatorname{curl}\left(\frac{1}{\epsilon}\operatorname{curl} \boldsymbol{u} + \frac{1}{\epsilon}\boldsymbol{d}_s\right) = 0$$

$\Downarrow$ no charges

$$\operatorname{curl}\left(\frac{1}{\epsilon}\operatorname{curl} \boldsymbol{u}\right) = 0$$