Maxwell's equations

$$\text{grad}\, f_0 \equiv \nabla f_0 = (\partial_x, \partial_y, \partial_z) f_0$$
$$\text{curl}\, \boldsymbol{f}_1 \equiv \nabla \times \boldsymbol{f}_1 \equiv (\partial_x, \partial_y, \partial_z) \times \boldsymbol{f}_1$$
$$\text{div}\, \boldsymbol{f}_2 \equiv \nabla \cdot \boldsymbol{f}_2 \equiv (\partial_x, \partial_y, \partial_z) \cdot \boldsymbol{f}_2$$

$$\text{curl}\, \boldsymbol{h} - \partial_t \boldsymbol{d} = \boldsymbol{j}$$
$$\text{curl}\, \boldsymbol{e} + \partial_t \boldsymbol{b} = 0$$
$$\text{div}\, \boldsymbol{b} = 0$$
$$\text{div}\, \boldsymbol{d} = q$$

$$\boldsymbol{b} = \mathcal{B}(\boldsymbol{e}, \boldsymbol{h}) = \mu \boldsymbol{h} \ (+\boldsymbol{b}_r)$$
$$\boldsymbol{d} = \mathcal{D}(\boldsymbol{e}, \boldsymbol{h}) = \epsilon \boldsymbol{e} \ (+\boldsymbol{d}_{src})$$
$$\boldsymbol{j} = \mathcal{J}(\boldsymbol{e}, \boldsymbol{h}) = \sigma \boldsymbol{e} \ (+\boldsymbol{j}_{src})$$

v — 0-form = scalar, continuous field

grad ↓

$\boldsymbol{h}, \boldsymbol{e}$ — 1-form = vector field of continuous tangential component

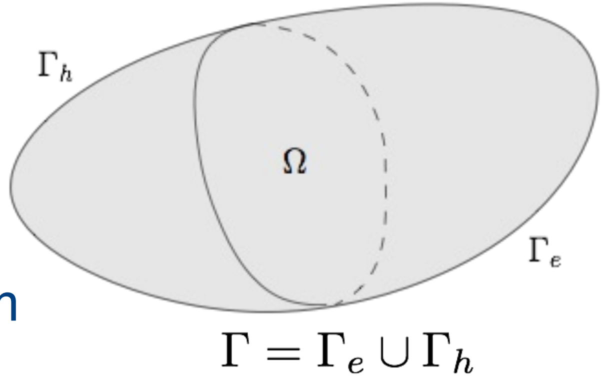
curl ↓

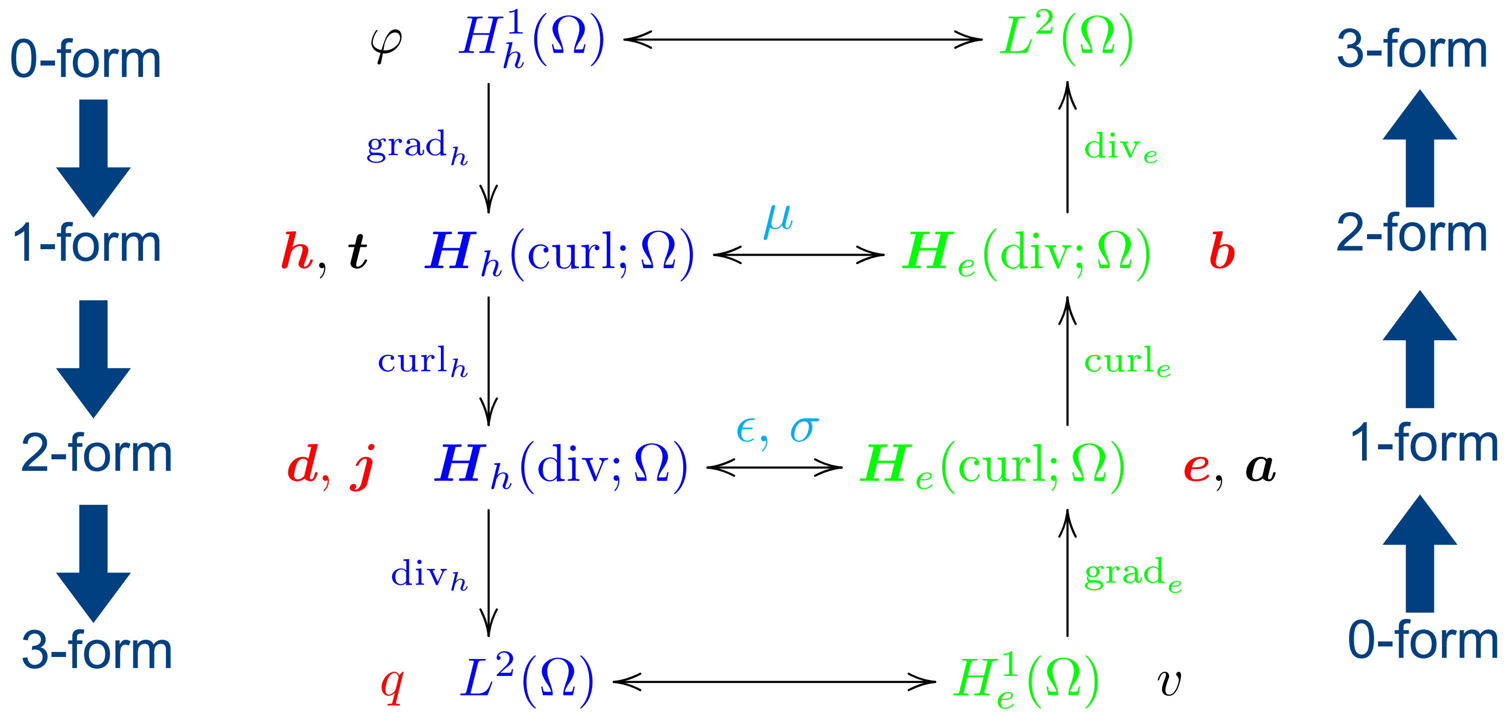
$\boldsymbol{b}, \boldsymbol{j}, \boldsymbol{d}$ — 2-form = vector field of continuous normal component

div ↓

q — 3-form = scalar field

Maxwell's house —Tonti diagram





boundary conditions accounted for in subspaces

boundary split in two parts

square integrable scalar & vector fields: field + field with differential operator

$$H_u^{10}(\Omega) = \{u \in L^2(\Omega) : \mathrm{grad}\, u \in \boldsymbol{L}^2(\Omega), u|_{\Gamma_u} = 0\}$$

$$\boldsymbol{H}_{\boldsymbol{u}}^0(\mathrm{curl};\Omega) = \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega) : \mathrm{curl}\, \boldsymbol{u} \in \boldsymbol{L}^2(\Omega), \boldsymbol{n} \times \boldsymbol{u}|_{\Gamma_u} = 0\}$$

$$\boldsymbol{H}_{\boldsymbol{u}}^0(\mathrm{div};\Omega) = \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega) : \mathrm{div}\, \boldsymbol{u} \in L^2(\Omega), \boldsymbol{n} \cdot \boldsymbol{u}|_{\Gamma_u} = 0\}$$

$u = e \text{ or } h$

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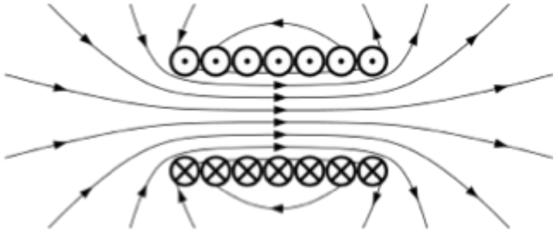
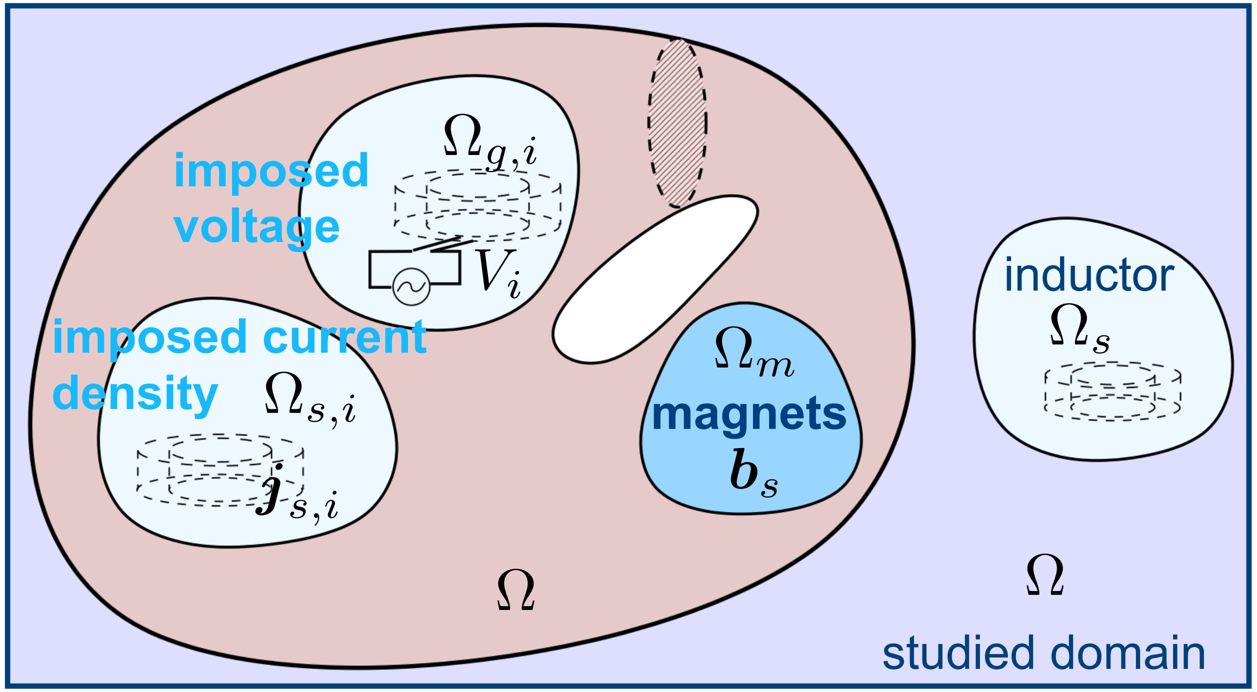
Magnetostatics



solenoid $L = \frac{\Phi}{m.m.f} = n^2 \frac{\mu_0 S}{l}$



$$\begin{aligned} \operatorname{curl} \boldsymbol{h} &= \boldsymbol{j}_s \\ \operatorname{div} \boldsymbol{b} &= 0 \\ \boldsymbol{b} &= \mu\, \boldsymbol{h} (+\boldsymbol{b}_r) \\ \boldsymbol{h} &= \nu\, \boldsymbol{b} (+\boldsymbol{h}_c) \end{aligned}$$

possible sources:

$\boldsymbol{j}_s$ imposed current density in inductor

$\boldsymbol{b}_r$ remanent induction if magnets

$\boldsymbol{h}_c$ coercive magnetic field if magnets

magnetic vector potential formulation

$$\operatorname{curl} \nu \operatorname{curl} \boldsymbol{a} = \boldsymbol{j}_s \,, \quad \boldsymbol{b} = \operatorname{curl} \boldsymbol{a}$$

$$\operatorname{div} \Big(\mu (\boldsymbol{h}_s - \operatorname{grad} \varphi) \Big) = 0 \quad \text{in } \Omega \,, \quad \operatorname{curl} \boldsymbol{h}_s = \boldsymbol{j}_s, \quad \boldsymbol{h} = -\operatorname{grad} \varphi$$

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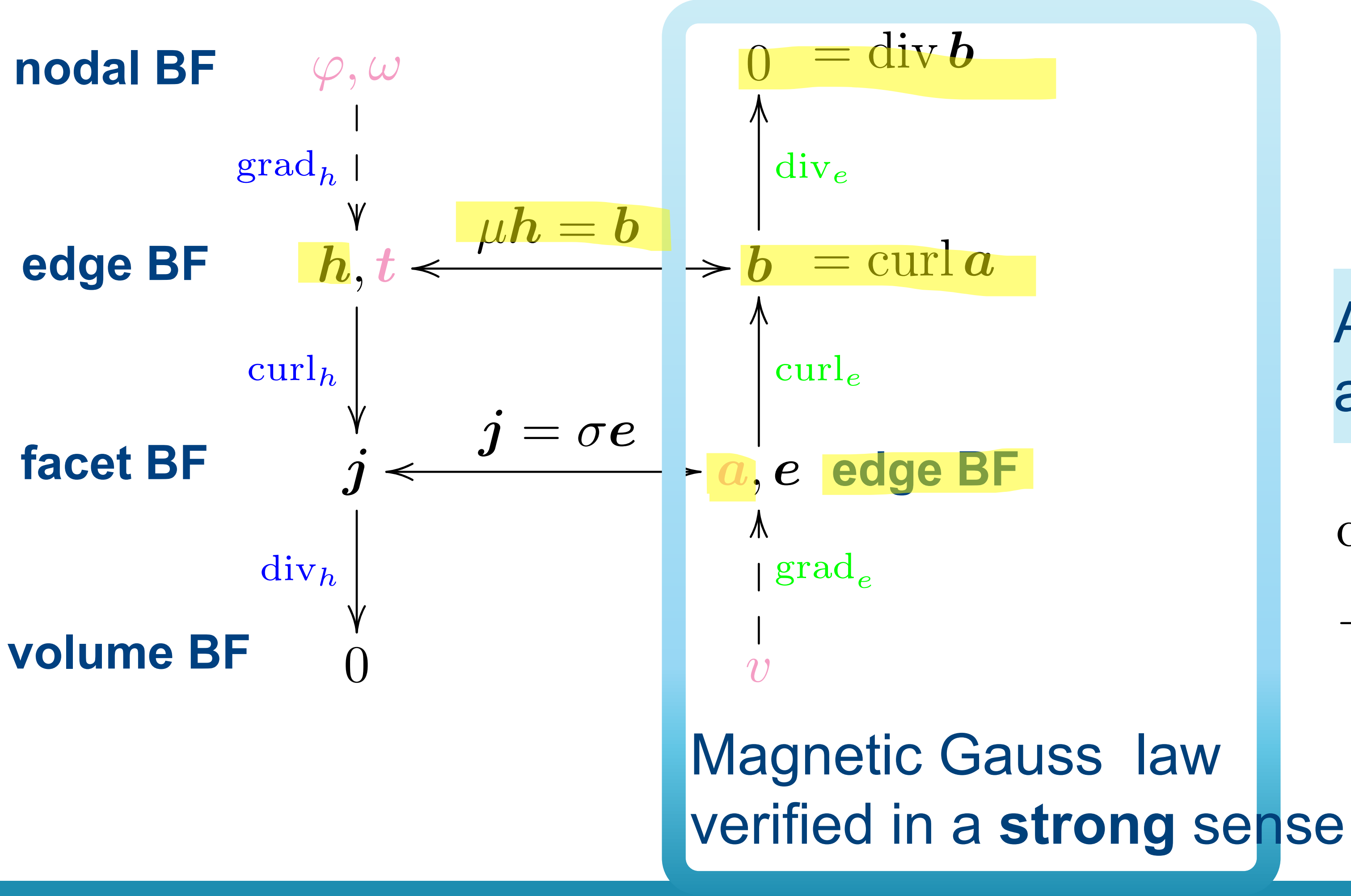
Magnetostatics

$\text{curl}\, \boldsymbol{h} = \boldsymbol{j}_s, \qquad \text{div}\, \boldsymbol{b} = 0$

$\boldsymbol{a}$ formulation

$\boldsymbol{a}$ magnetic vector potential

reluctivity $\nu = \dfrac{1}{\mu}$



Ampère's law verified in a **weak** sense

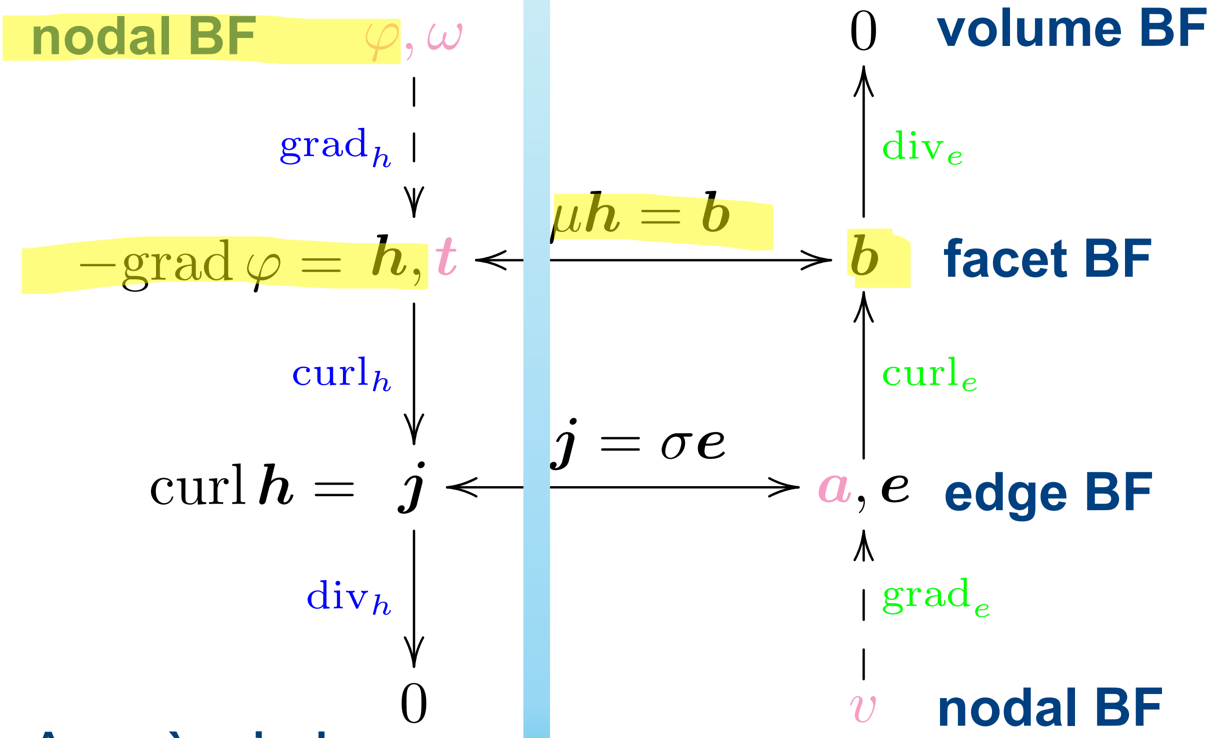
$\text{curl}\, \nu \text{curl}\, \boldsymbol{a} = \boldsymbol{j}_s$

$+$ Gauge in Ω

Magnetostatics

φ formulation
$\boldsymbol{h} = -\operatorname{grad} \varphi$ magnetic field
φ magnetic scalar potential

$$\operatorname{curl} \boldsymbol{h} = \boldsymbol{j}_s, \qquad \operatorname{div} \boldsymbol{b} = 0$$



Ampère's law verified in a **strong** sense

$$\operatorname{curl} \boldsymbol{h}_s = \boldsymbol{j}_s$$

Magnetic Gauss law verified in a **weak** sense

$$\operatorname{div} \Big(\mu(\boldsymbol{h}_s - \operatorname{grad} \varphi) \Big) = 0 \quad \text{in } \Omega$$

Spatial discretization — magnetostatics

We want to find the magnetic vector potential $\boldsymbol{a}(\boldsymbol{x})$ in Ω

$$\mathrm{curl}\,(\nu \mathrm{curl}\,\boldsymbol{a}) = \boldsymbol{j}_s$$

with given

$\boldsymbol{j}_s(\boldsymbol{x})$ imposed electric current density
$\nu(\boldsymbol{x})$ reluctivity > 0 in part of the domain

weighted residual approach

We integrate the equation weighted by (vectorial) weighting or test functions $\boldsymbol{w}_i(\boldsymbol{x})$ over the whole domain Ω:

find $\boldsymbol{a}$ such that

$$\int_\Omega \mathrm{curl}\,(\nu \mathrm{curl}\,\boldsymbol{a}) \cdot \boldsymbol{w}_i \,\mathrm{d}\Omega = \int_\Omega \boldsymbol{j}_s \cdot \boldsymbol{w}_i \,\mathrm{d}\Omega$$

holds $\forall \boldsymbol{w}_i$

Spatial discretization — magnetostatics (II)

$$\int_\Omega \mathrm{curl}\,(\nu \mathrm{curl}\,\boldsymbol{a}) \cdot \boldsymbol{w}_i \,\mathrm{d}\Omega = \int_\Omega \boldsymbol{j}_s \cdot \boldsymbol{w}_i \,\mathrm{d}\Omega$$

$\boldsymbol{v} = \boldsymbol{w}_i$

$\boldsymbol{u} = \nu\,\mathrm{curl}\,\boldsymbol{a}$

$$\boldsymbol{v} \cdot \mathrm{curl}\,\boldsymbol{u} - \boldsymbol{u} \cdot \mathrm{curl}\,\boldsymbol{v} = \mathrm{div}\,(\boldsymbol{u} \times \boldsymbol{v})$$

integration by parts
Green formula

$$\int_\Omega \Big(\mathrm{div}\,(\nu\,\mathrm{curl}\,\boldsymbol{a} \times \boldsymbol{w}_i) + \nu\,\mathrm{curl}\,\boldsymbol{a} \cdot \mathrm{curl}\,\boldsymbol{w}_i\Big)\,\mathrm{d}\Omega = \int_\Omega \boldsymbol{j}_s \cdot \boldsymbol{w}_i \,\mathrm{d}\Omega$$

$\boldsymbol{u} = \nu\,\mathrm{curl}\,\boldsymbol{a} \times \boldsymbol{w}_i$

$$\int_\Omega \mathrm{div}\,\boldsymbol{u}\,\mathrm{d}\Omega = \oint_\Gamma \boldsymbol{u}\,\mathrm{d}\boldsymbol{\Gamma}\,, \quad \mathrm{d}\boldsymbol{\Gamma} = \boldsymbol{n}\mathrm{d}\Gamma$$

divergence theorem

find $\boldsymbol{a}$ such that

Weak formulation

$$\int_\Gamma (\boldsymbol{n} \times \nu\,\mathrm{curl}\,\boldsymbol{a} \cdot \boldsymbol{w}_i)\,\mathrm{d}\Gamma + \int_\Omega \nu\,\mathrm{curl}\,\boldsymbol{a} \cdot \mathrm{curl}\,\boldsymbol{w}_i \,\mathrm{d}\Omega = \int_\Omega \boldsymbol{j}_s \cdot \boldsymbol{w}_i \,\mathrm{d}\Omega$$

holds $\forall \boldsymbol{w}_i(\boldsymbol{x})$

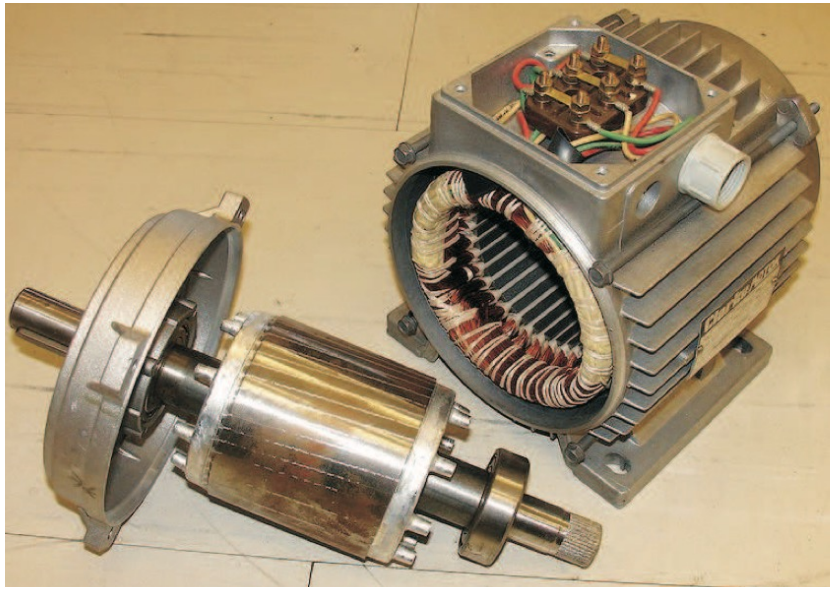
only the first derivative of the MVP is now required

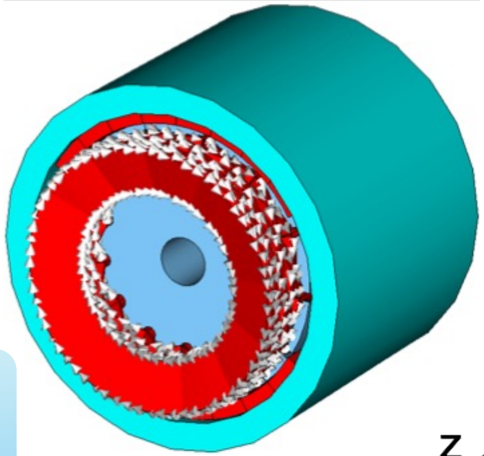
From 3D to 2D models

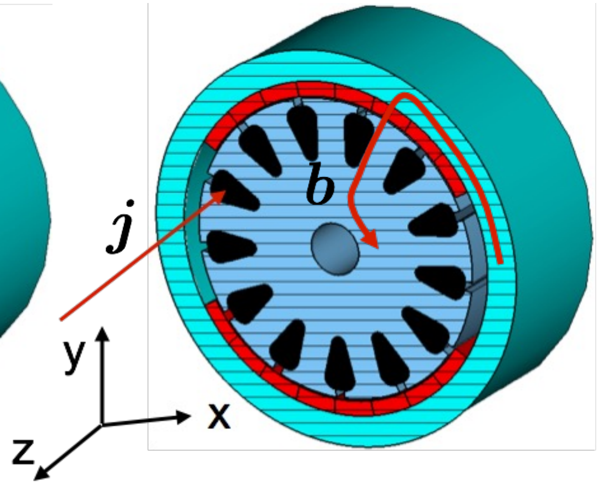
$$\boldsymbol{j}_s = (0, 0, j_s(\boldsymbol{x}))$$
$$\boldsymbol{b} = (b_x(\boldsymbol{x}), b_y(\boldsymbol{x}), 0)$$
$$\boldsymbol{h} = (h_x(\boldsymbol{x}), h_y(\boldsymbol{x}), 0)$$

$$\boldsymbol{a} = (0, 0, a_z(\boldsymbol{x}))$$
$$\boldsymbol{b} = \operatorname{curl} \boldsymbol{a} = (\partial_y a_z, -\partial_x a_z, 0)$$
$$\boldsymbol{h} = \nu \operatorname{curl} \boldsymbol{a} = \nu\,(\partial_y a_z, -\partial_x a_z, 0)$$

$$\operatorname{div} \boldsymbol{b} = \partial_x b_x + \partial_y b_y = \partial^2_{xy} a_z - \partial^2_{xy} a_z = 0$$







$$\operatorname{curl}(\nu \operatorname{curl} \boldsymbol{a}) = \boldsymbol{j}_s$$
$$-\partial_x(\nu \partial_x a_z) - \partial_y(\nu \partial_y a_z) = j_{s,z}$$

Open boundary problems — Low frequency

Truncation of outer boundaries
Asymptotic boundary conditions
Kelvin transformation
Shell transformation

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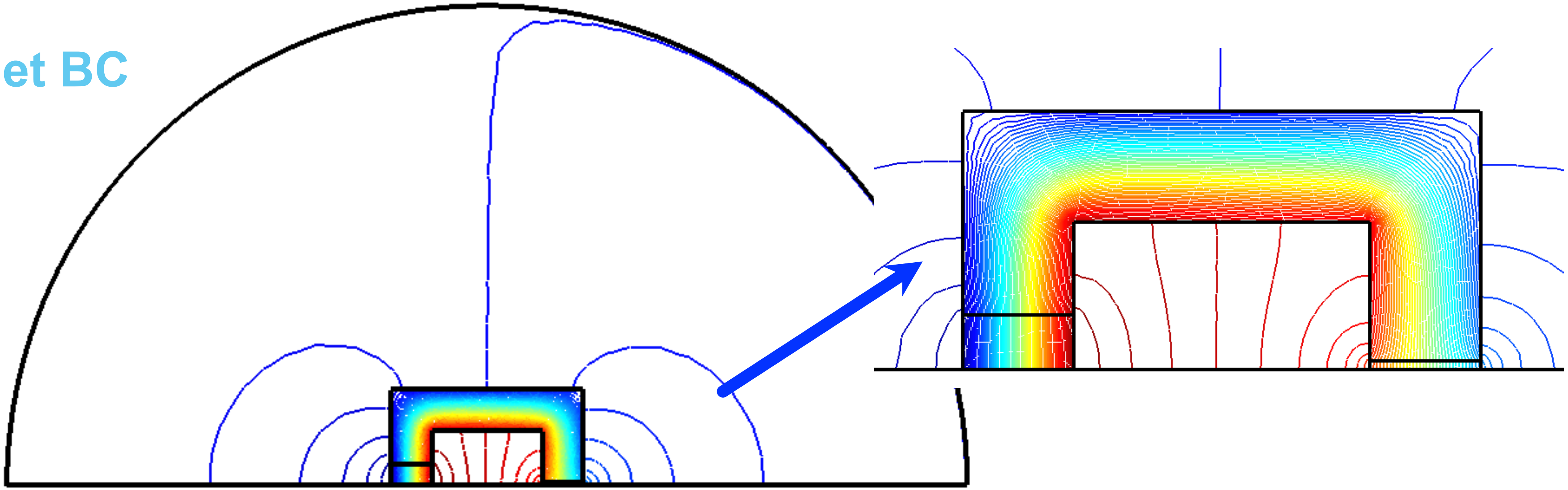
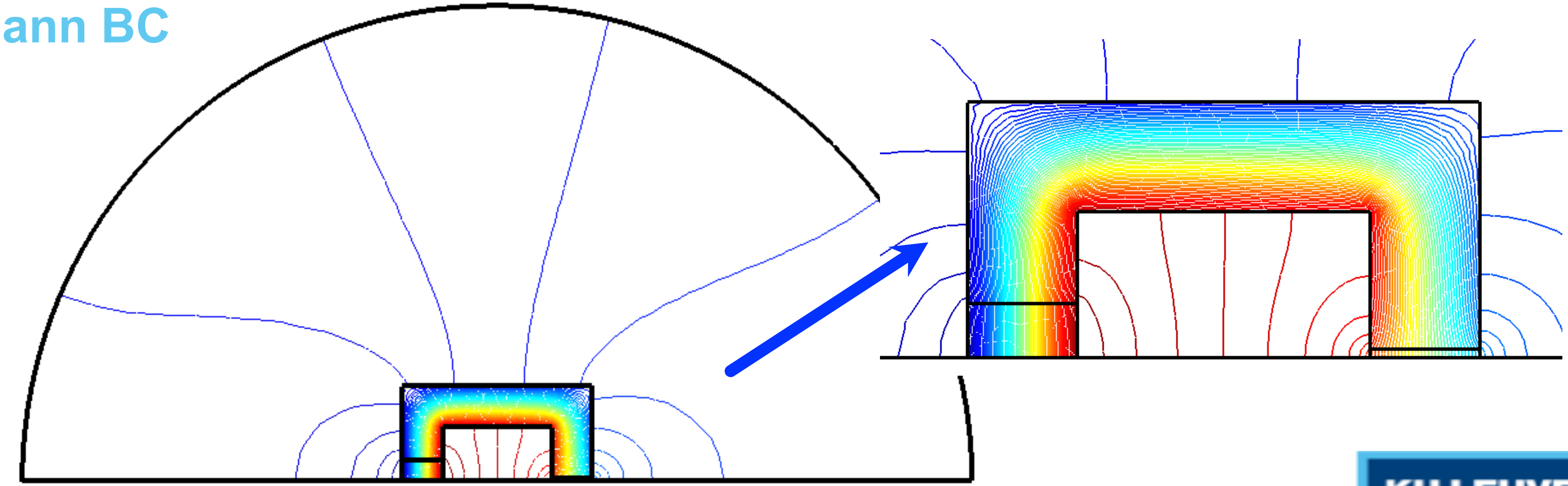
Truncation of outer boundaries

- Pick an arbitrary boundary far enough from the region of interest and impose a homogeneous Dirichlet or Neumann boundary condition
- Rule of thumb:
  distance from centre of problem to outer boundary == 5 times
  distance from centre to outside of region of interest
- Used by most FE electromagnetic software, as it requires no additional effort to implement
- To get an accurate solution a large volume of air around the area of interest must be modelled
- This large area can be modelled with a relatively coarse mesh to limit the extra computational time

Dirichlet BC



Neumann BC



Asymptotic boundary conditions (BCs) mixed BC to impose on a circular outer boundary

- solution inside: finite elements
- solution outside: asymptotic solution of the problem at hand on a circular shell, e.g. for a magnetic vector potential formulation

$$\boldsymbol{a}(r,\theta) = \sum_{m=1}^{\infty} \frac{a_m}{r^m} \cos(m\theta + \alpha_m)$$

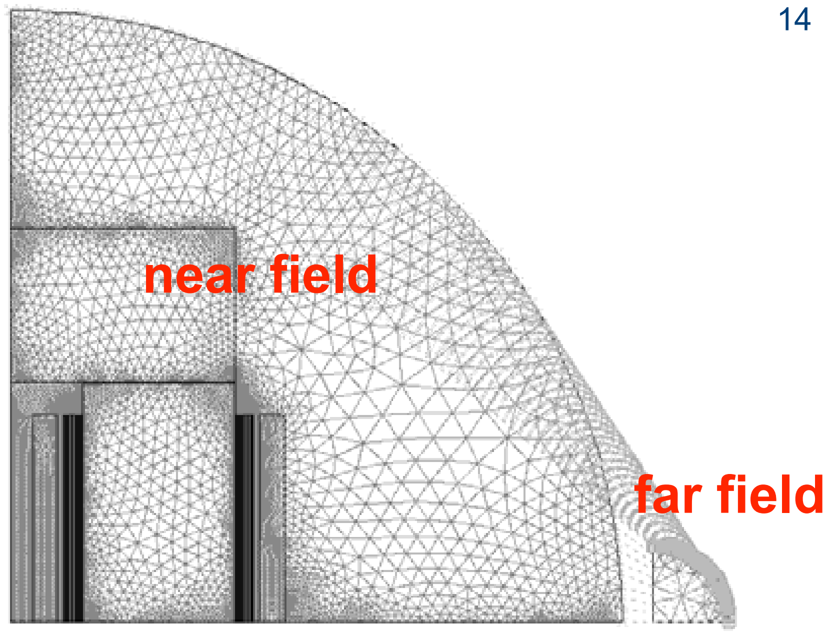
- magnitude of harmonic decreases quickly with distance, only the leading harmonic is kept for describing the open field solution

$$\boldsymbol{a}(r,\theta) \approx \frac{a_m}{r^m} \cos(m\theta + \alpha_m)$$

- substituting $\frac{\partial \boldsymbol{a}}{\partial r} = -m \frac{a_m}{r^{m+1}} \cos(m\theta + \alpha_m)$ into the complete solution, we have

$$\frac{\partial \boldsymbol{a}}{\partial r} + \frac{m}{r} \boldsymbol{a} = 0$$ **mixed BC**

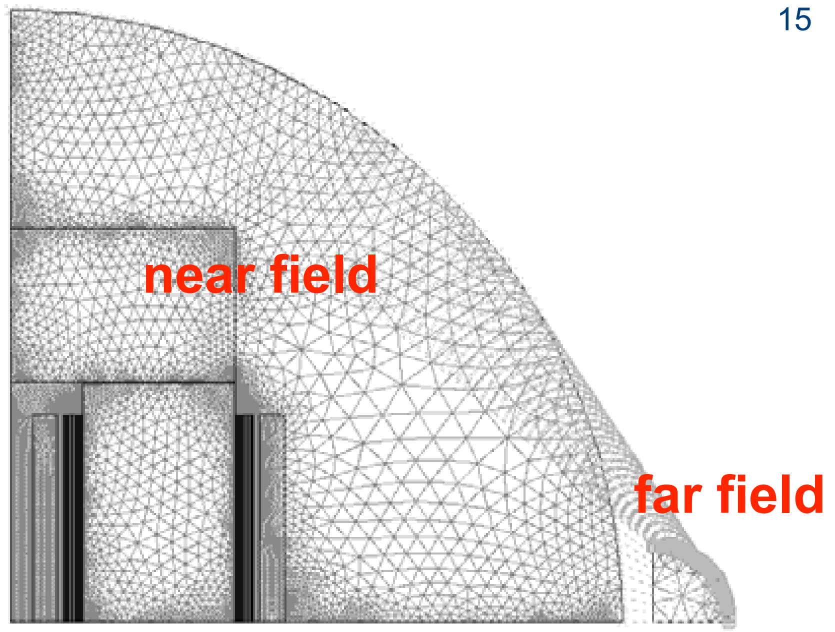
Kelvin transformation



- Strengths
  - effects of the exterior region model exactly
  - sparse matrix representation of the problem kept
  - no special features in FE solver required

- exterior domain modelled by forcing a link between two circular regions:
  - a circular region with devices of interest and surrounding air, where we actually want to compute the field (‘near field/internal’)
  - an additional circular region representing the ‘far field/external’
- periodic boundary constraints between the two circles to enforce the continuity of the local quantity of interest (e.g MVP)
- the additional circular region models exactly the infinity space solution, but on a bounded domain

Kelvin transformation (cond'd)



- 'far field/exterior' region with homogeneous material govern by

polar coordinates

$$\Delta \boldsymbol{a} = 0 \quad \Longrightarrow \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \boldsymbol{a}}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \boldsymbol{a}}{\partial \phi^2} = 0$$

- 'near field/interior' region is a circle of radius r_0, 'far field' is everything outside
- Map unbounded region onto a bounded region by defining in the mapped space

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \boldsymbol{a}}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 \boldsymbol{a}}{\partial \phi^2} = 0 \qquad R = \frac{r_0^2}{r}$$

- the field at any point can always be recovered by applying an inverse mapping
- Axisymmetry, Dirichlet or Neumann BCs simulated by modifying material parameters (e.g. permeabilty)

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$\{X^I = X, Y, Z\}$
$\{y^j = x, y, z\}$
$C^I = C^j \delta_j^I$

Shell transformation

Map unbounded region into a shell $X^I - C^I = (y^i - C^j)\delta_j^I \, F(R_{int}, R_{ext}, r(y^j))$

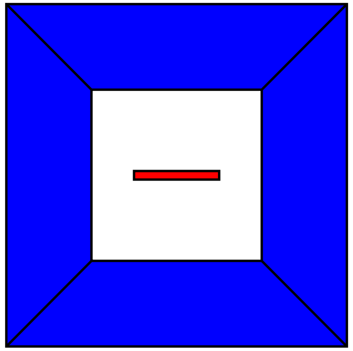
$$F(R_{int}, R_{ext}, r) = \left(\frac{R_{int}(R_{ext} - R_{int})}{r(R_{ext} - r)} \right)^p \qquad \frac{dF}{dr} = -\theta \frac{F}{r}, \qquad \theta = \frac{R_{ext} - 2r}{p(R_{ext} - r)}$$

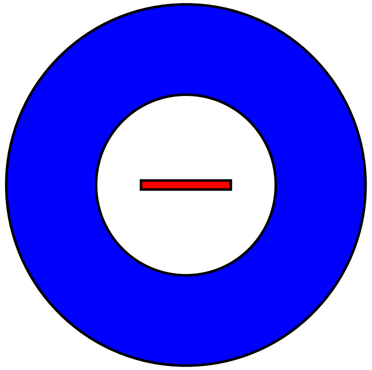
This transformation applies to shells that are:

cylindrical $r(y^i) = \sqrt{(x - C^x)^2 + (y - C^y)^2}$

Parallelepipedic (or trapezoidal) $r(y^i) = (y^k - C^k)$

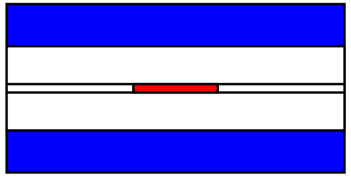
spherical $r(y^i) = \sqrt{(x - C^x)^2 + (y - C^y)^2 + (z - C^z)^2}$





Jacobian matrix of the mapping

$$\Lambda_j^I = \begin{pmatrix} 1 - \theta n^x \partial_x r & - \theta n^x \partial_y r & - \theta n^x \partial_z r \\ - \theta n^y \partial_x r & 1 - \theta n^y \partial_y r & - \theta n^y \partial_z r \\ - \theta n^z \partial_x r & - \theta n^z \partial_y r & 1 - \theta n^z \partial_z r \end{pmatrix} \qquad n^i = \frac{y^i - C^i}{r}$$



Unidirectional

determinant $\Lambda_{shell}(R_{int}, R_{ext}, r) = F^2(1 - \theta)$

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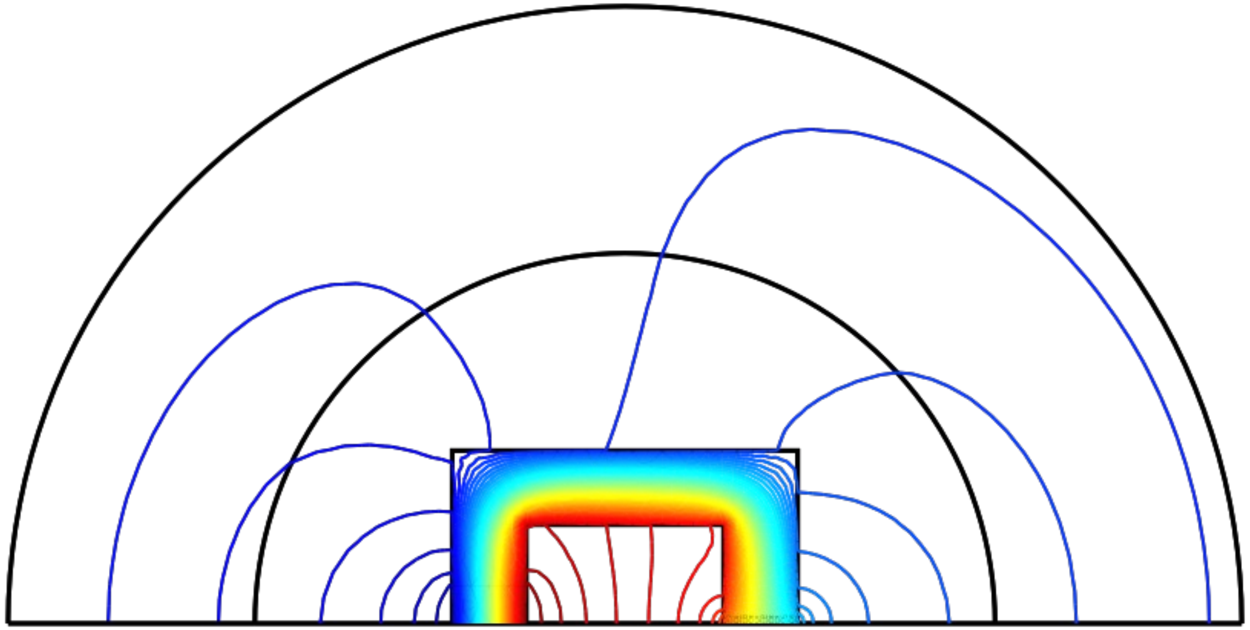
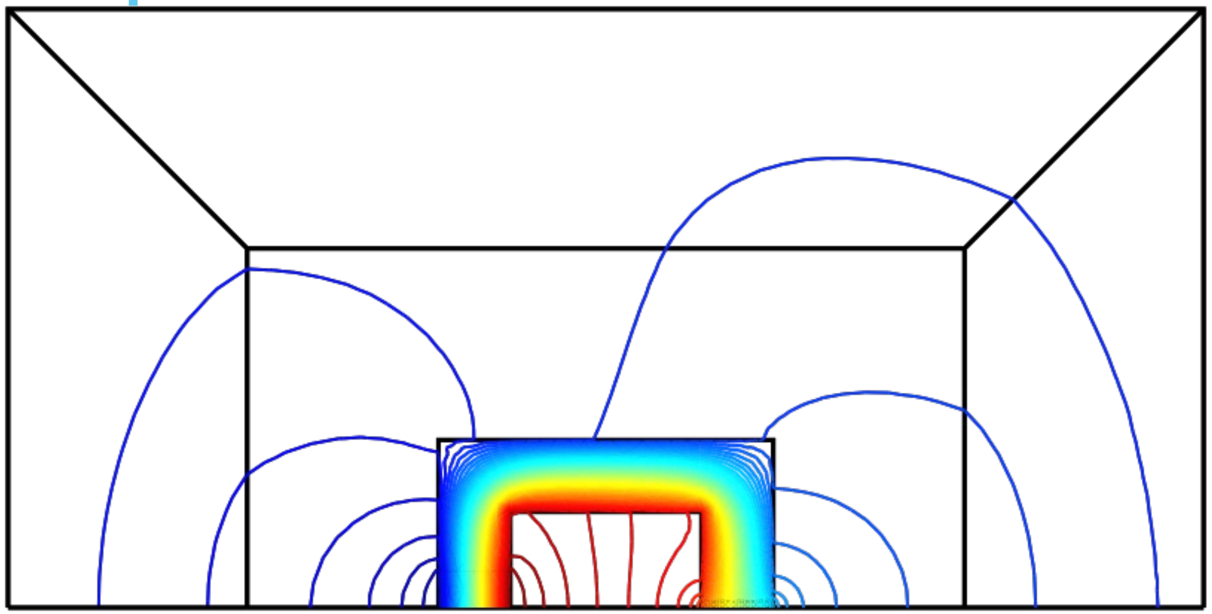
Spherical shell



Parallelepipedic or trapezoidal shell



Discrete mathematical structure
Whitney elements

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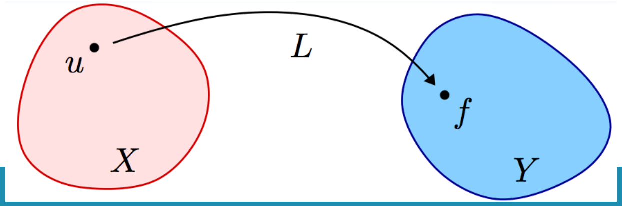
Discrete mathematical structure

Replace the continuous spaces (infinite dimension) by discrete spaces (finite dimension)

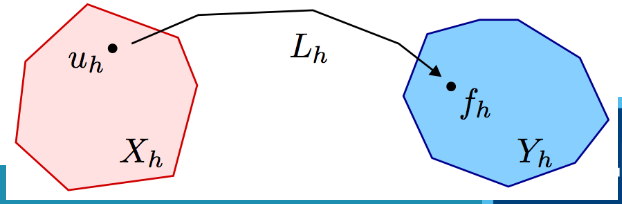
$$
\begin{array}{ccc}
H^1_h(\Omega) & \longleftrightarrow & L^2(\Omega) \\
\big\downarrow \mathrm{grad}_h & & \big\uparrow \mathrm{div}_e \\
\boldsymbol{H}_h(\mathrm{curl};\Omega) & \longleftrightarrow & \boldsymbol{H}_e(\mathrm{div};\Omega) \\
\big\downarrow \mathrm{curl}_h & & \big\uparrow \mathrm{curl}_e \\
\boldsymbol{H}_h(\mathrm{div};\Omega) & \longleftrightarrow & \boldsymbol{H}_e(\mathrm{curl};\Omega) \\
\big\downarrow \mathrm{div}_h & & \big\uparrow \mathrm{grad}_e \\
L^2(\Omega) & \longleftrightarrow & H^1_e(\Omega)
\end{array}
$$



local function spaces on geometrical element $\mathcal{G}$

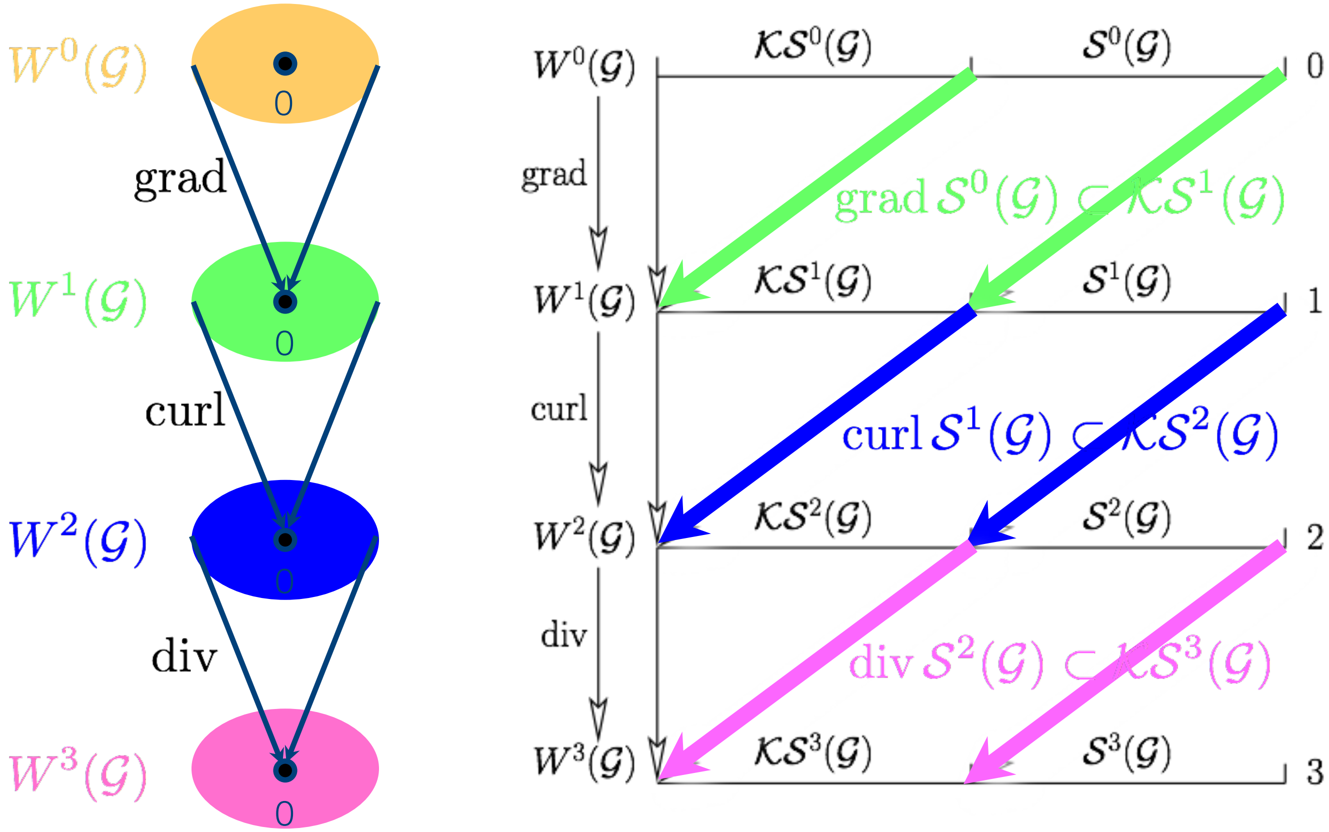
$$W^0(\mathcal{G}) \subset H^1(\mathcal{G})$$
$$W^1(\mathcal{G}) \subset \boldsymbol{H}(\mathrm{curl};\mathcal{G})$$
$$W^2(\mathcal{G}) \subset \boldsymbol{H}(\mathrm{div};\mathcal{G})$$
$$W^3(\mathcal{G}) \subset L^2(\mathcal{G})$$

$$
\begin{array}{ccc}
W^0(\mathcal{G}) & \longleftrightarrow & W^3(\mathcal{G}) \\
\big\downarrow \mathrm{grad} & & \big\uparrow \mathrm{div} \\
W^1(\mathcal{G}) & \longleftrightarrow & W^2(\mathcal{G}) \\
\big\downarrow \mathrm{curl} & & \big\uparrow \mathrm{curl} \\
W^2(\mathcal{G}) & \longleftrightarrow & W^1(\mathcal{G}) \\
\big\downarrow \mathrm{div} & & \big\uparrow \mathrm{grad} \\
W^3(\mathcal{G}) & \longleftrightarrow & W^0(\mathcal{G})
\end{array}
$$



The Whitney complex

$W^p(\mathcal{G})$ is the finite dimensional subspace spanned by the p-Whitney elements on $\mathcal{G}$. They satisfy the property of conformity:

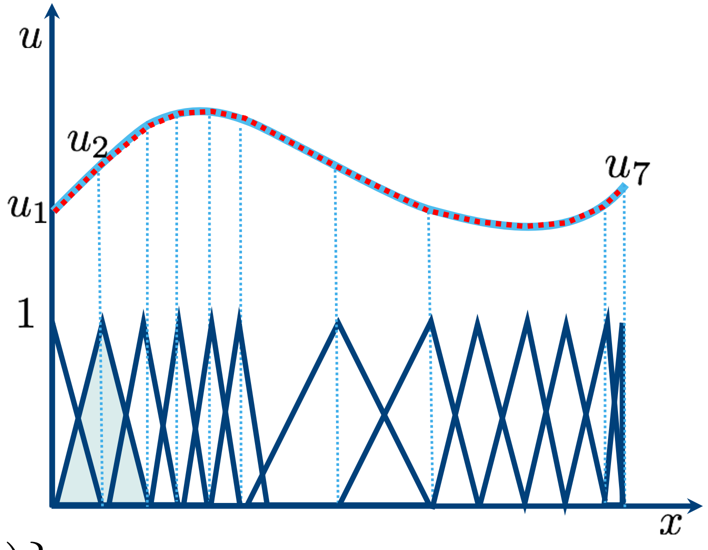


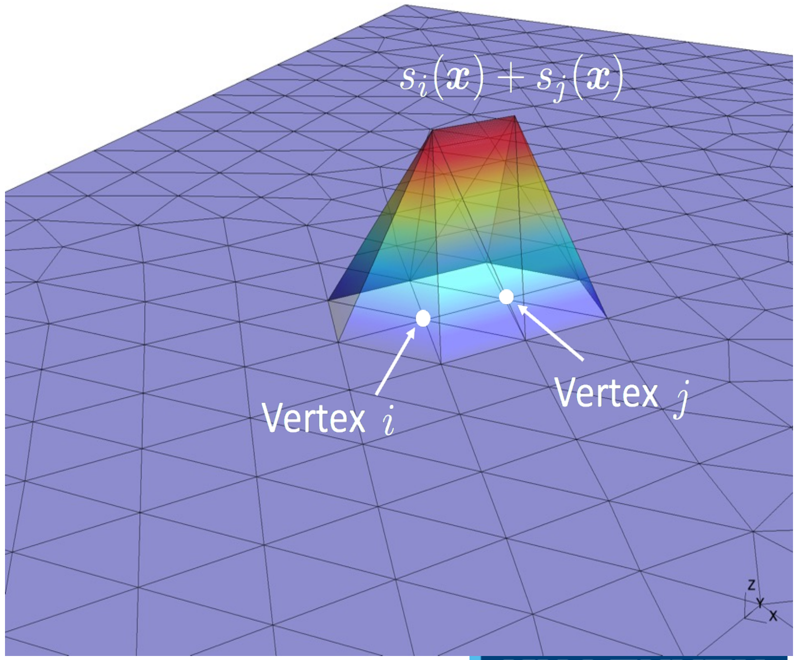
Finite elements

- set of linearly independent basis/shape functions and weighting functions (also called test and trial functions)
- commonly piecewise polynomial
- defined at a structured or unstructured grid/mesh
- compact support
- scalar or vectorial functions

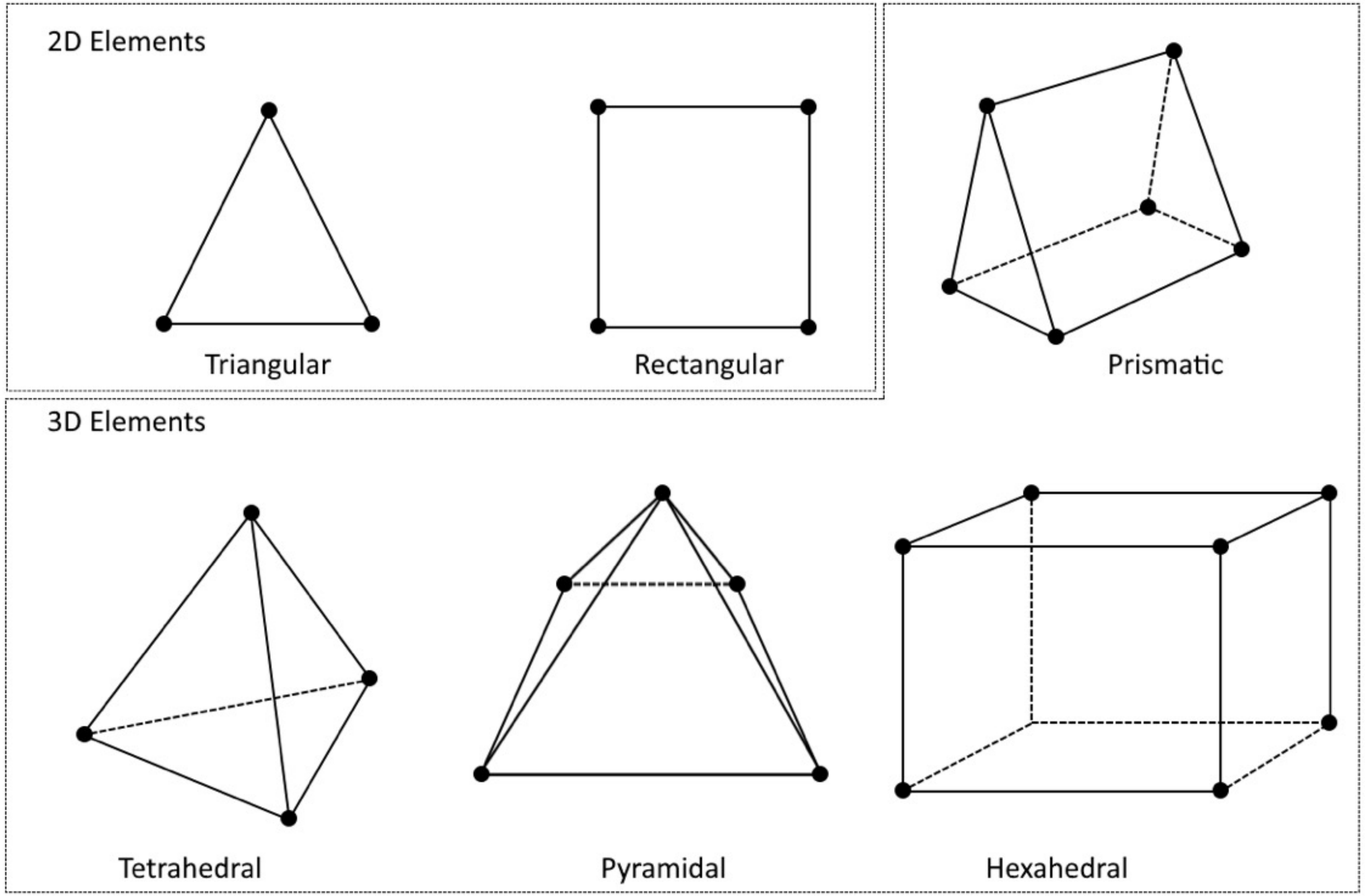
$$u(\boldsymbol{x}) \approx u_h(\boldsymbol{x}) = \sum_{i=1}^{N} u_i s_i(\boldsymbol{x})$$

$$F^0(\Omega) = \mathrm{span}\{s_1(\boldsymbol{x}), s_2(\boldsymbol{x}), \dots s_N(\boldsymbol{x})\}$$





Typical FE elements in 2D and 3D
First order

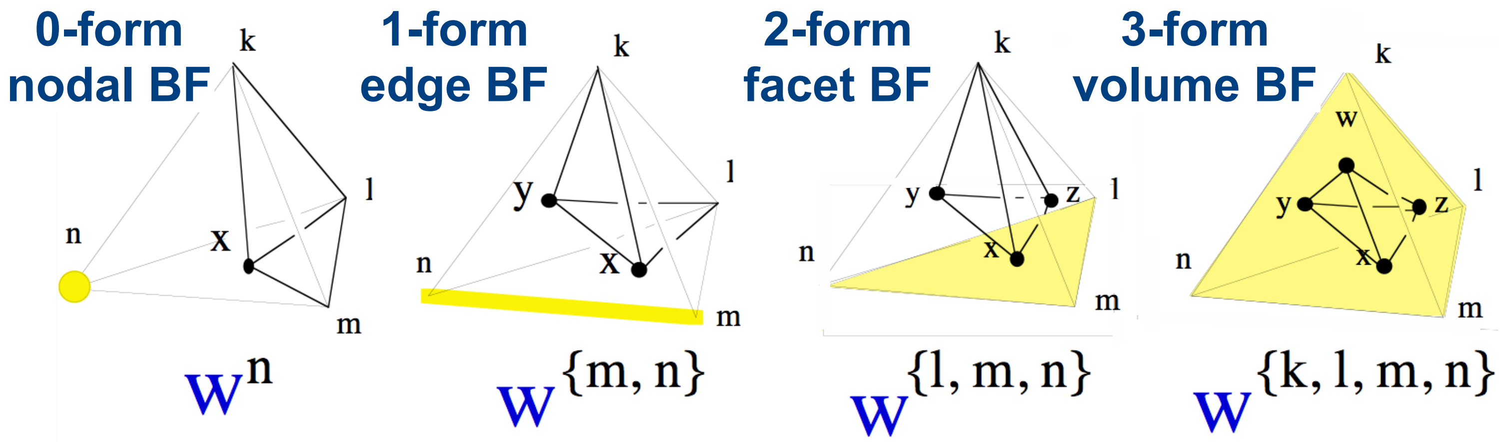


The Whitney elements

Finite element $(\mathcal{G}, \Sigma, S)$:

- ✓ geometrical element $\mathcal{G}$
- ✓ Σ = set of N Dofs
- ✓ S function of finite dimension N

Let us consider a mesh of Ω formed by geometrical elements $\mathcal{G}$ with nodes $\mathcal{N}$, edges $\mathcal{E}$, faces $\mathcal{F}$, volumes $\mathcal{V}$,



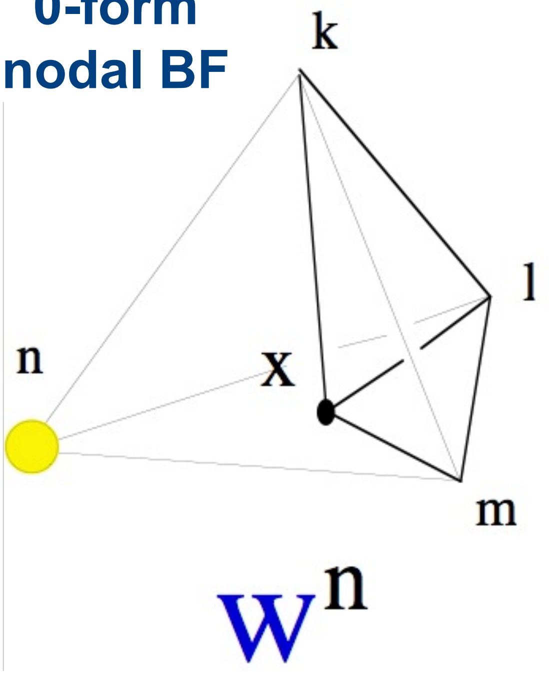
The Whitney elements of order p are expressed as

$$\boldsymbol{w}_{n_0,\cdots,n_p} = p! \sum_{j=0}^{p} (-1)^m \varsigma_{n_m} \operatorname{grad} \varsigma_{n_0} \times \cdots \times \operatorname{grad} \varsigma_{n_{m-1}} \times \operatorname{grad} \varsigma_{n_{m+1}} \times \cdots \times \operatorname{grad} \varsigma_{n_p}$$

with $\varsigma_n(x)$ barycentric weight of x with respect to node n in $\mathcal{G}$

The Whitney elements of order 0
Nodal elements

0-form
nodal BF



$$w_n = \varsigma_n$$

with $n \in \mathcal{N}$ (node set)

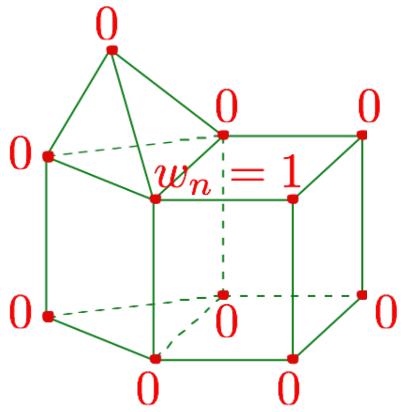
span space $W^0(\mathcal{G})$

The interpolation of a function u is given by

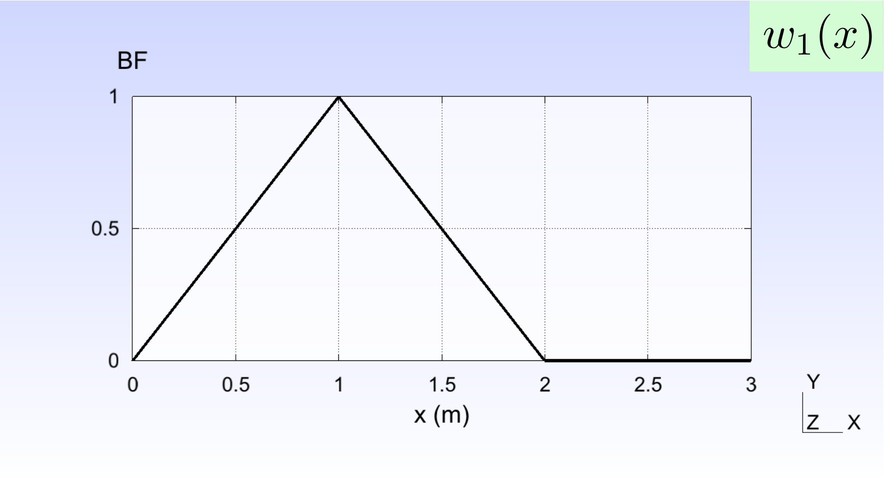
$$u \approx u_h = \sum_{x_i \in \mathcal{N}} u_i w_i$$

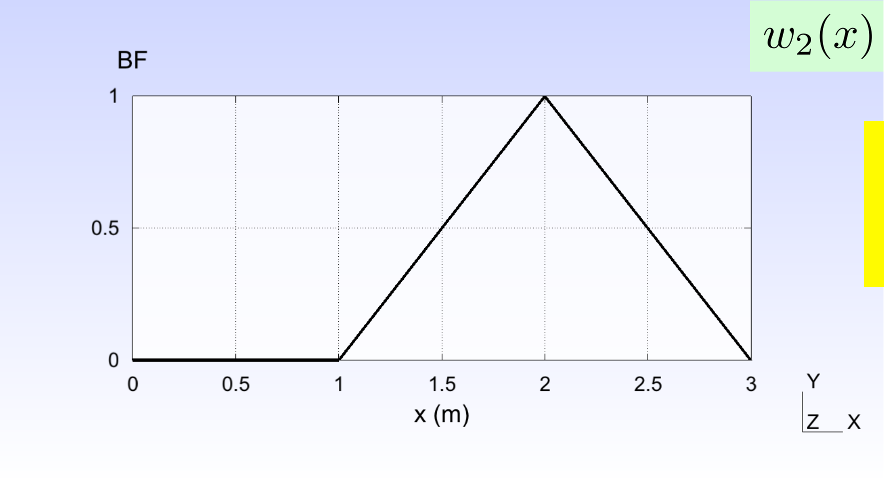
with $u_i = \alpha_i(u) = u(x_i)$

- ✓ piecewise linear continuous:
  first order scalar Lagrange finite elements
- ✓ discretisation of scalar fields
- ✓ $w_n = 1$ at node n, 0 at other nodes
- ✓ $w_n = 1$ is continuous across faces

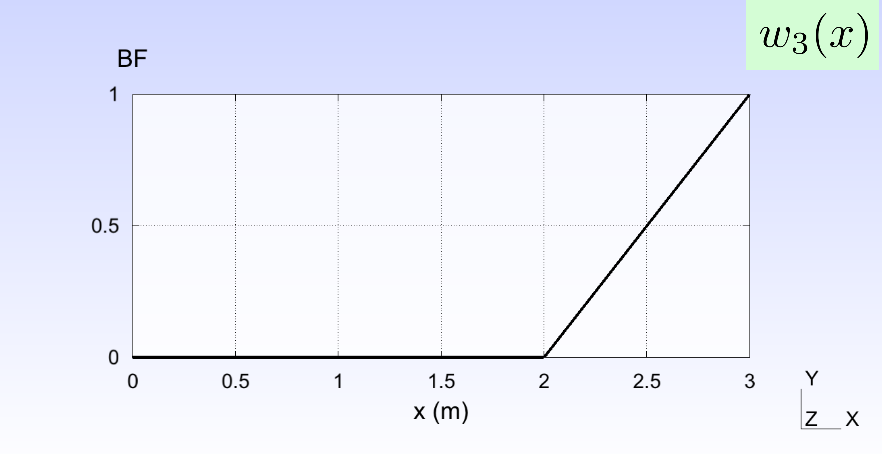


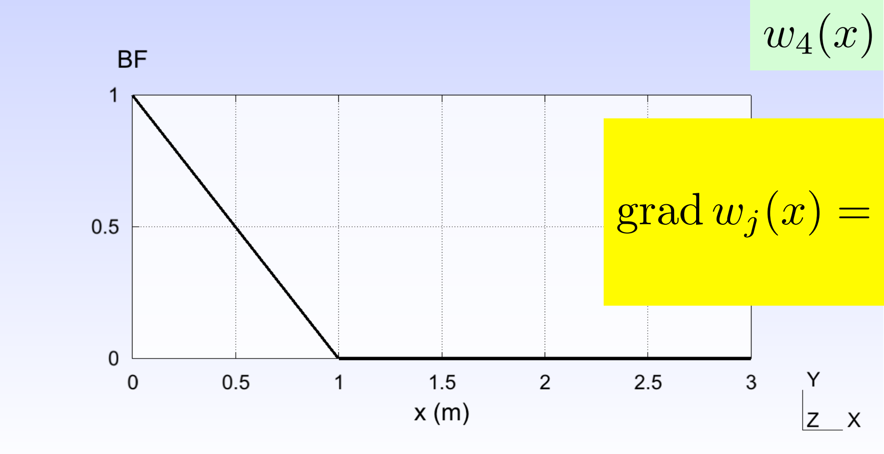
The Whitney elements of order 0
Nodal elements on a line





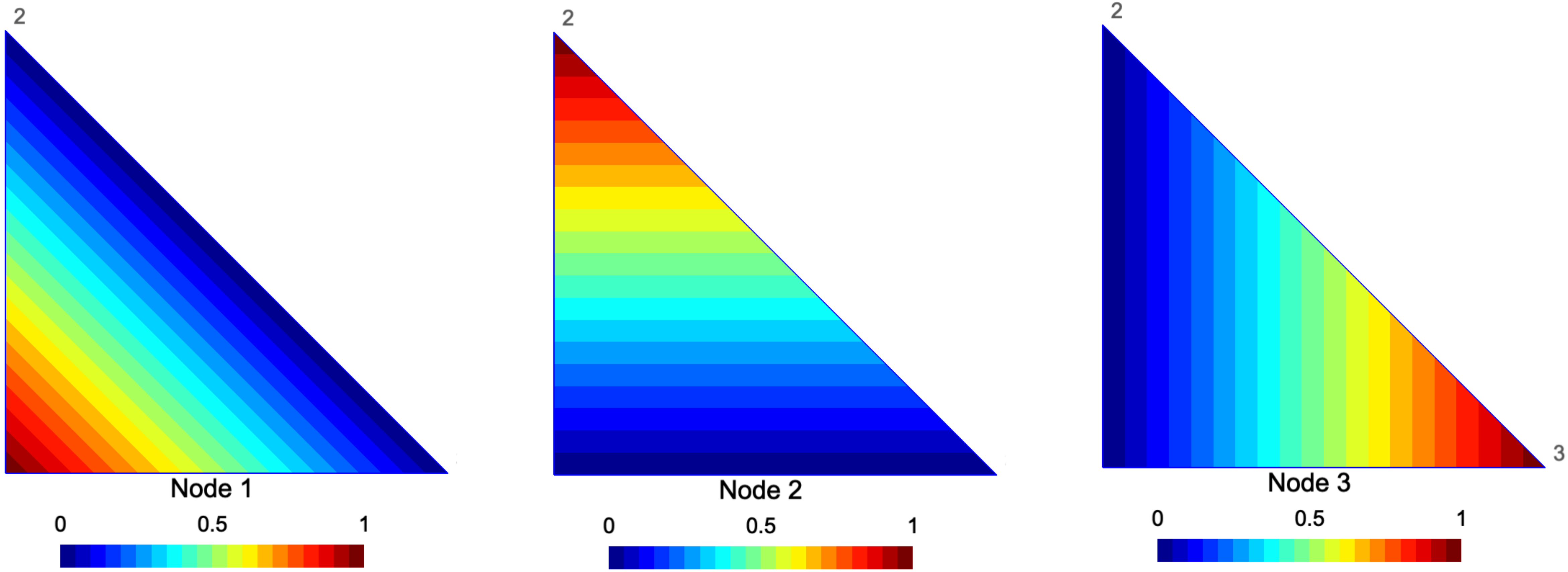
numbering
of DOFs
as in GetDP





$$\operatorname{grad} w_j(x) = \left\{ \begin{array}{c} +1 \\ -1 \\ 0 \end{array} \right. \hat{\boldsymbol{x}}$$

The Whitney elements of order 0
Nodal elements on a triangle



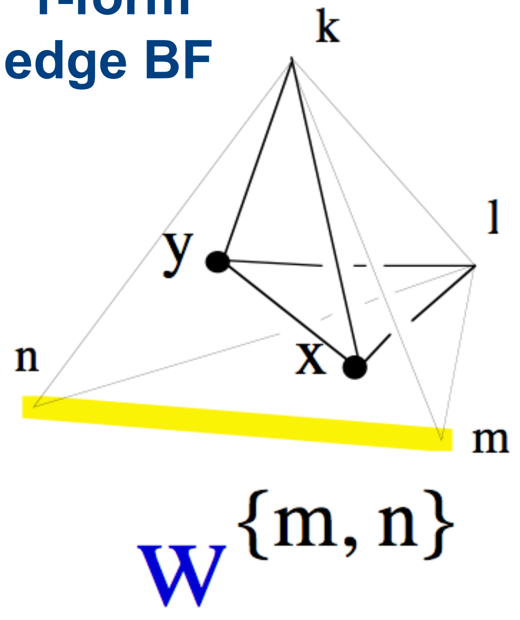
The Whitney elements of order 1
Edge elements

$$\boldsymbol{w}_e = \boldsymbol{w}_{\{m,n\}} = \varsigma_m \operatorname{grad} \varsigma_n - \varsigma_n \operatorname{grad} \varsigma_m,$$

with $e \in \mathcal{E}$ (edge set)

span space $W^1(\mathcal{G})$

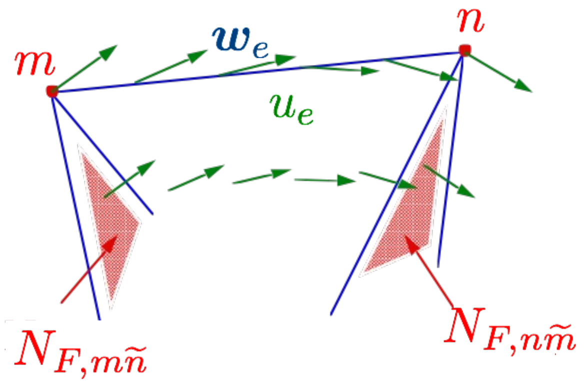
1-form edge BF



The interpolation of a function $\boldsymbol{u}$ is given by

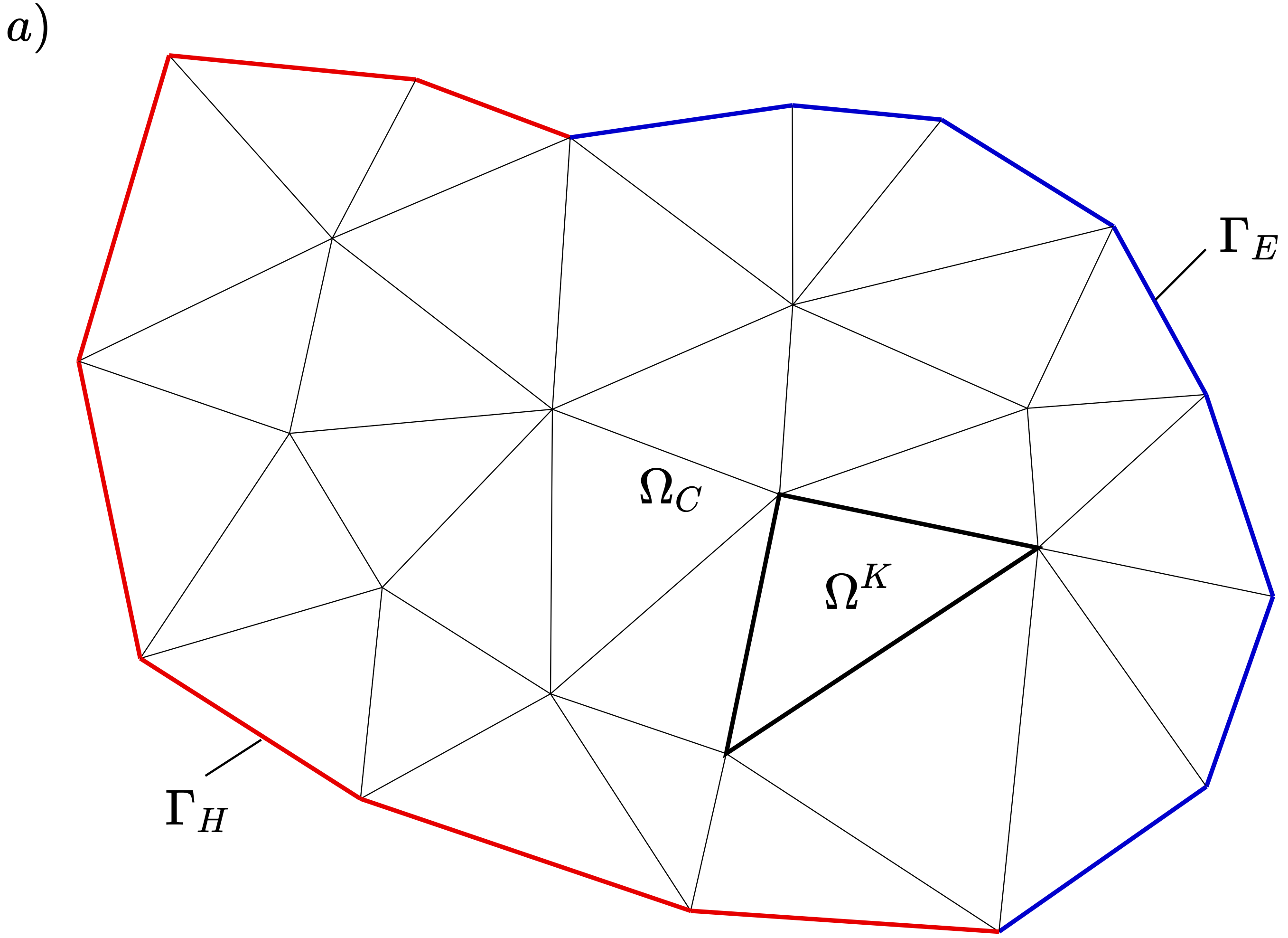
$$\boldsymbol{u} \approx \boldsymbol{u}_h = \sum_{e \in \mathcal{E}} u_e \boldsymbol{w}_e$$

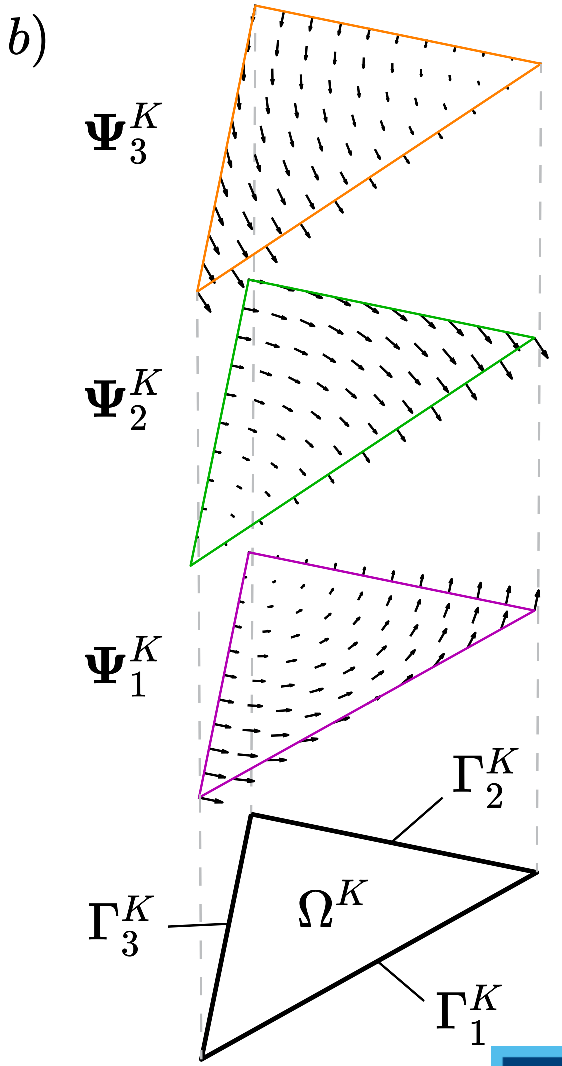
$$\text{with } u_e = \alpha_e(\boldsymbol{u}) = \int_e \boldsymbol{u} \cdot d\boldsymbol{l}\,,\ \forall e \in \mathcal{E}$$



- ✓ Dof = circulations of field along edges of mesh
- ✓ discretisation of 1-forms, e.g. $\boldsymbol{h}$, $\boldsymbol{e}$
- ✓ tangential component continuous across faces
- ✓ circulation of $\boldsymbol{w}_e = 1$ along edge e, 0 across other edges

Edge elements on a triangle

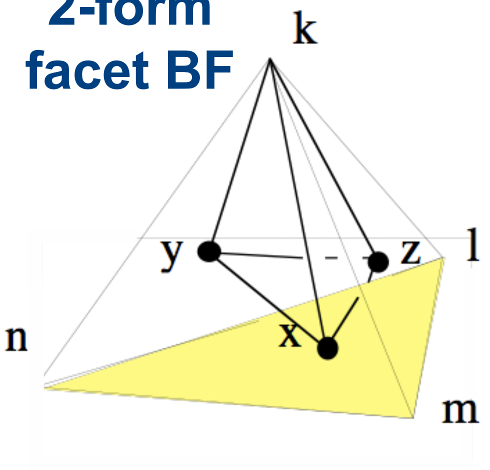




Whitney elements of order 2
Face elements

$$\boldsymbol{w}_f = \boldsymbol{w}_{\{l,m,n\}} =$$
$$2\left(\varsigma_l \operatorname{grad}\varsigma_m \times \operatorname{grad}\varsigma_n - \varsigma_m \operatorname{grad}\varsigma_l \times \operatorname{grad}\varsigma_n + \varsigma_n \operatorname{grad}\varsigma_l \times \operatorname{grad}\varsigma_m\right)$$

2-form facet BF

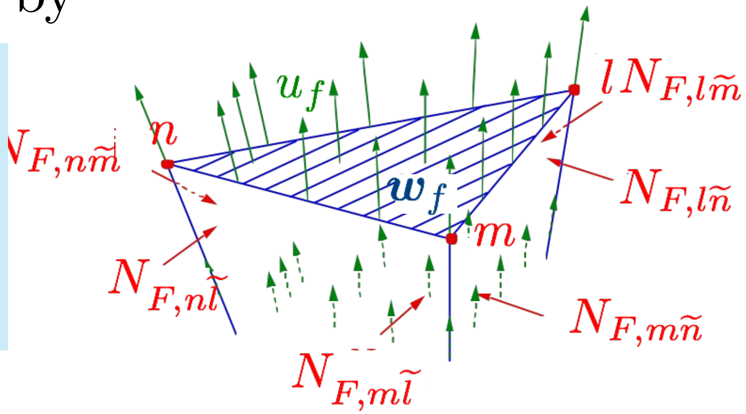


$\mathbf{W}^{\{l,m,n\}}$

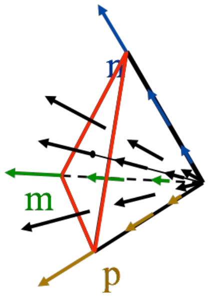
The interpolation of a function $\boldsymbol{u}$ is given by

$$\boldsymbol{u} \approx \boldsymbol{u}_h = \sum_{f\in\mathcal{F}} u_f \boldsymbol{w}_f$$

$$\text{with } u_f = \alpha_f(\boldsymbol{u}) = \int_f \boldsymbol{u}\cdot\boldsymbol{n}\, ds\,,\ \forall f \in \mathcal{F}$$



- ✓ Dof = flux through faces of mesh
- ✓ discretisation of 2-forms, e.g. $\boldsymbol{b}$, $\boldsymbol{j}$
- ✓ normal component continuous across interfaces
- ✓ flux of $\boldsymbol{w}_f = 1$ across face, 0 across other faces of $\mathcal{G}$



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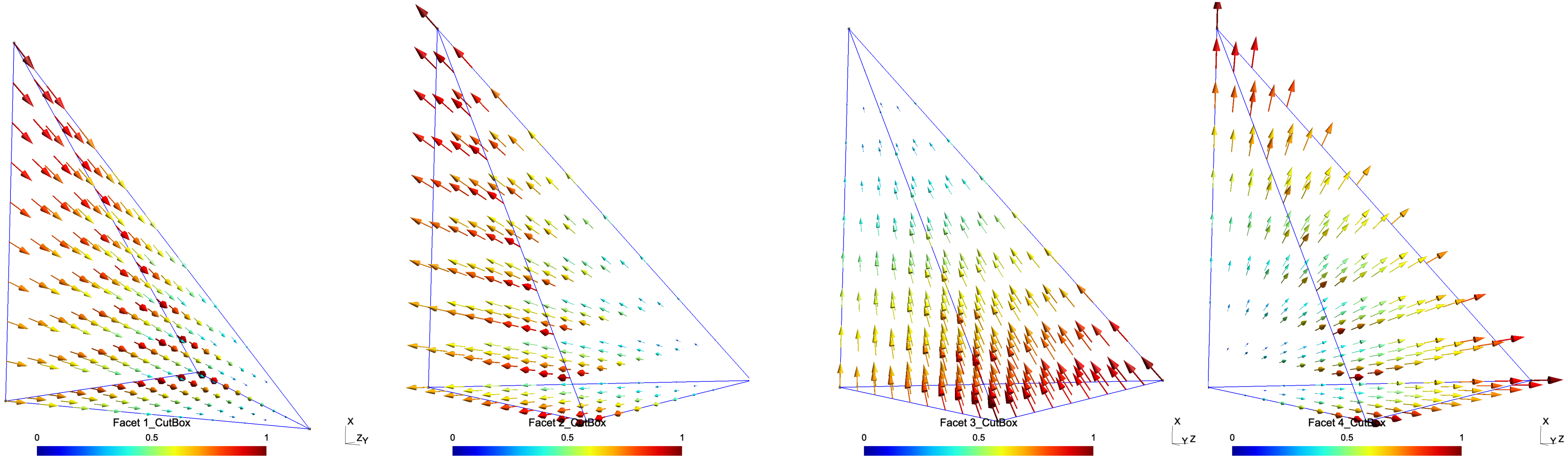
Face elements on a tetrahedron



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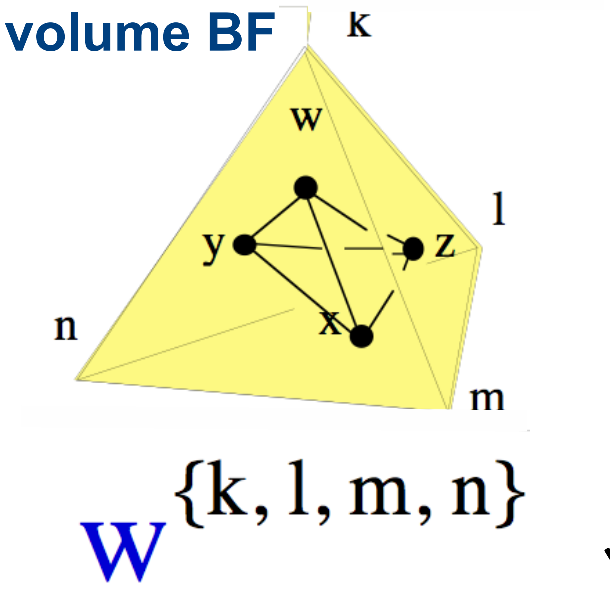
Whitney elements of order 3
Volume elements

$$w_v = w_{\{k,l,m,n\}} = 6\,(\varsigma_k \operatorname{grad}\varsigma_l \times \operatorname{grad}\varsigma_m \times \operatorname{grad}\varsigma_n - \varsigma_l \operatorname{grad}\varsigma_k \times \operatorname{grad}\varsigma_m \times \operatorname{grad}\varsigma_n + \varsigma_n \operatorname{grad}\varsigma_k \times \operatorname{grad}\varsigma_l \times \operatorname{grad}\varsigma_m - \varsigma_n \operatorname{grad}\varsigma_k \times \operatorname{grad}\varsigma_l \times \operatorname{grad}\varsigma_m)$$

with $v = \{k,l,m,n\} \in \mathcal{V}$ (volume set)

span space $W^3(\mathcal{G})$

3-form volume BF



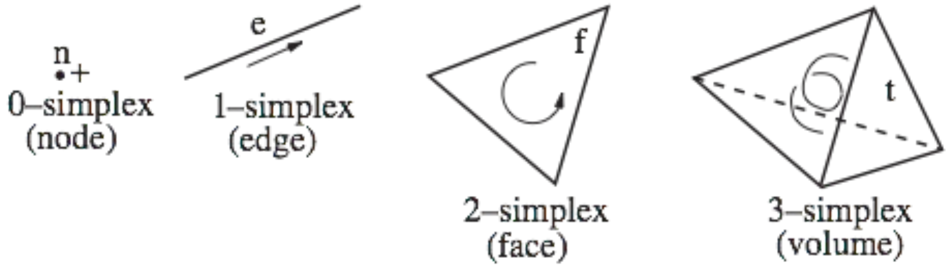
The interpolation of a function u is given by

$$u \approx u_h = \sum_{v\in\mathcal{V}} u_v w_v$$

with $u_v = \alpha_v(u) = \int_v u\,dv$

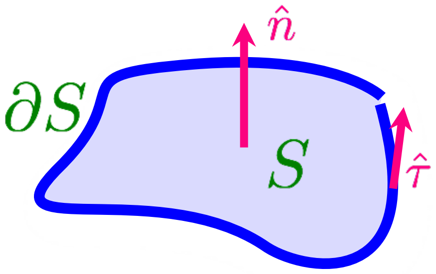
- ✓ piecewise constant functions
- ✓ Dof = integration over its volume
- ✓ discretisation of densities
- ✓ $\sum w_v = 1$ over the volume of $\mathcal{G}$, 0 over other volumes

Conformity



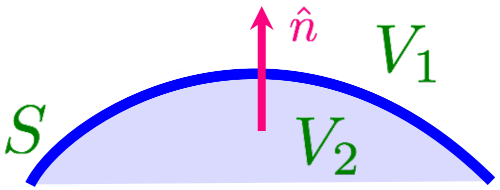
- ✓ **Nodal elements**: Conforming finite elements (in $\mathcal{H}^1(\Omega)$) interpolate scalar fields that are **continuous across any interface**.
  Discretisation of scalar quantities: potentials φ, v, temperature...

- ✓ **Edge elements**: Curl-conforming finite elements (in $\boldsymbol{H}(\mathrm{curl};\Omega)$) ensure the **continuity of the tangential component** of the field.
  Discretisation of the magnetic field $\boldsymbol{h}$, the magnetic vector potential $\boldsymbol{a}$ or the electric field $\boldsymbol{e}$.



$$\int_{\partial S} \boldsymbol{h}\cdot\hat{\tau}\,dl = \int_S (\boldsymbol{j}+\partial_t\boldsymbol{d})\cdot\hat{n}\,ds$$

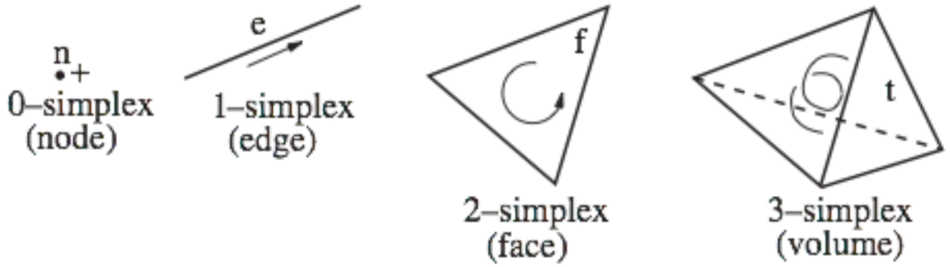
$$\int_{\partial S} \boldsymbol{e}\cdot\hat{\tau}\,dl = -\int_S \partial_t\boldsymbol{b}\cdot\hat{n}\,ds$$



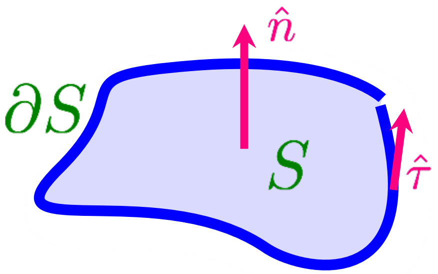
$$\hat{n}\times(\boldsymbol{h}_2-\boldsymbol{h}_1)|_S = \boldsymbol{j}_s$$

$$\hat{n}\times(\boldsymbol{e}_2-\boldsymbol{e}_1)|_S = 0$$

Conformity



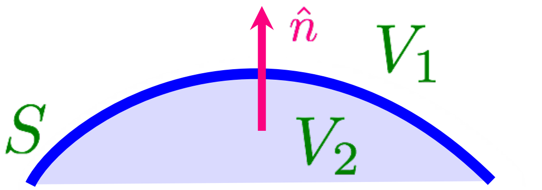
✓ **Face elements:** Div-conforming FEs (in $\boldsymbol{H}(\text{div};\Omega)$) ensure the **continuity of the normal component** of the interpolated field. Discretisation of magnetic flux density $\boldsymbol{b}$, current density $\boldsymbol{j}$ or electric flux density $\boldsymbol{d}$.



$$\int_S \boldsymbol{b}\cdot\hat{n}\,ds = 0$$

$$\int_S \boldsymbol{d}\cdot\hat{n}\,ds = \int_V \rho\,dv$$

$$\int_S \boldsymbol{j}\cdot\hat{n}\,ds = 0$$



$$\hat{n}\cdot(\boldsymbol{b}_2 - \boldsymbol{b}_1)|_S = 0$$

$$\hat{n}\cdot(\boldsymbol{d}_2 - \boldsymbol{d}_1)|_S = \rho_s$$

✓ **Volume elements:** FEs in $L^2(\Omega)$ do not impose any continuity (**discontinuous**) between elements on the interpolated field. Discretisation of quantities that may vary from one element to the other e.g. the electric charge density ρ.

$$\int_S \boldsymbol{d}\cdot\hat{n}\,ds = \int_V \rho\,dv$$

Finite elements spaces of Whitney forms

D	$\mathcal{H}(\mathsf{D},\Omega)$	$V_h(\mathsf{D}) \subset \mathcal{H}(\mathsf{D},\Omega)$	FE space	Reference
grad	$H^1(\Omega)$ $H^1_0(\Omega)$	$V_h(\mathbf{grad})$	linear Lagrangian FE (or node elements)	[13]
curl	$\mathbf{H}(\mathbf{curl},\Omega)$ $\mathbf{H}_0(\mathbf{curl},\Omega)$	$V_h(\mathbf{curl})$	edge elements	[29]
div	$\mathbf{H}(\mathrm{div},\Omega)$ $\mathbf{H}_0(\mathrm{div},\Omega)$	$V_h(\mathrm{div})$	face elements	[29]
0	$L^2(\Omega)$ $L^2_0(\Omega)$	$V_h(0)$	p.w. constants	

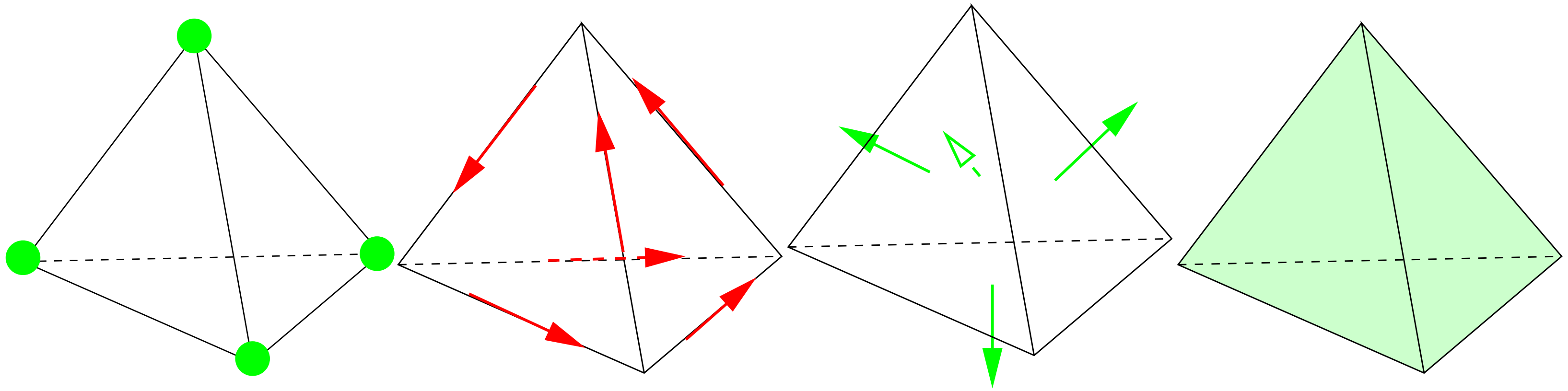


FIG. 4.1. *Symbolic notation for local degrees of freedom for* $V_h(\mathbf{grad}), V_h(\mathbf{curl}), V_h(\mathrm{div})$, *and* $V_h(0)$ *(left to right).*