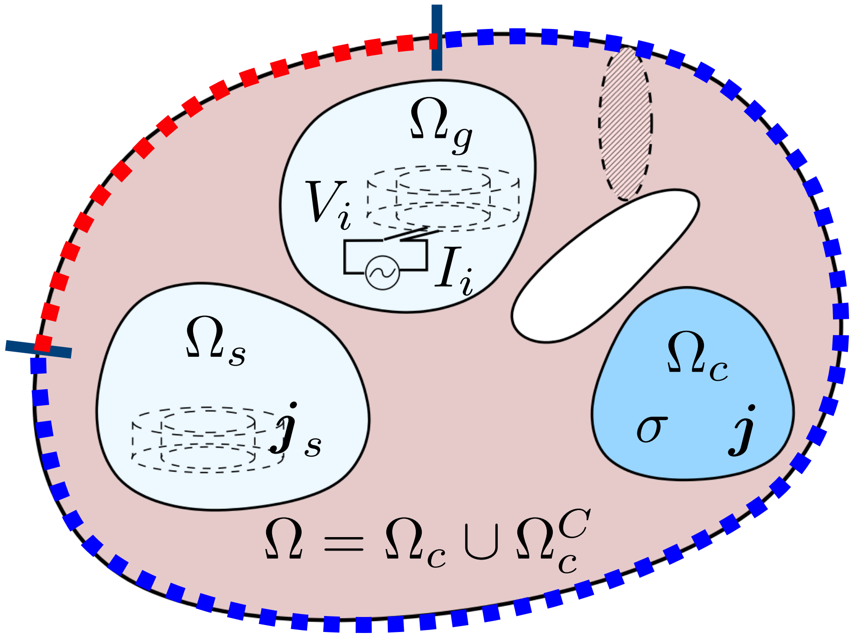
Magnetodynamic formulations and discretization

Magnetodynamics
or magneto-quasi-statics (MQS)

Distribution of magnetic field and eddy currents due to moving magnets and time variable sources



"$\boldsymbol{h}$ side"

Ampère's law verified in a **strong** sense

$\boldsymbol{h} - \varphi$ formulation

$\boldsymbol{t} - \omega$ formulation

"$\boldsymbol{b}$ side"

Faraday's law verified in a **strong** sense

$\boldsymbol{a}^*$ formulation

$\boldsymbol{a} - v$ formulation

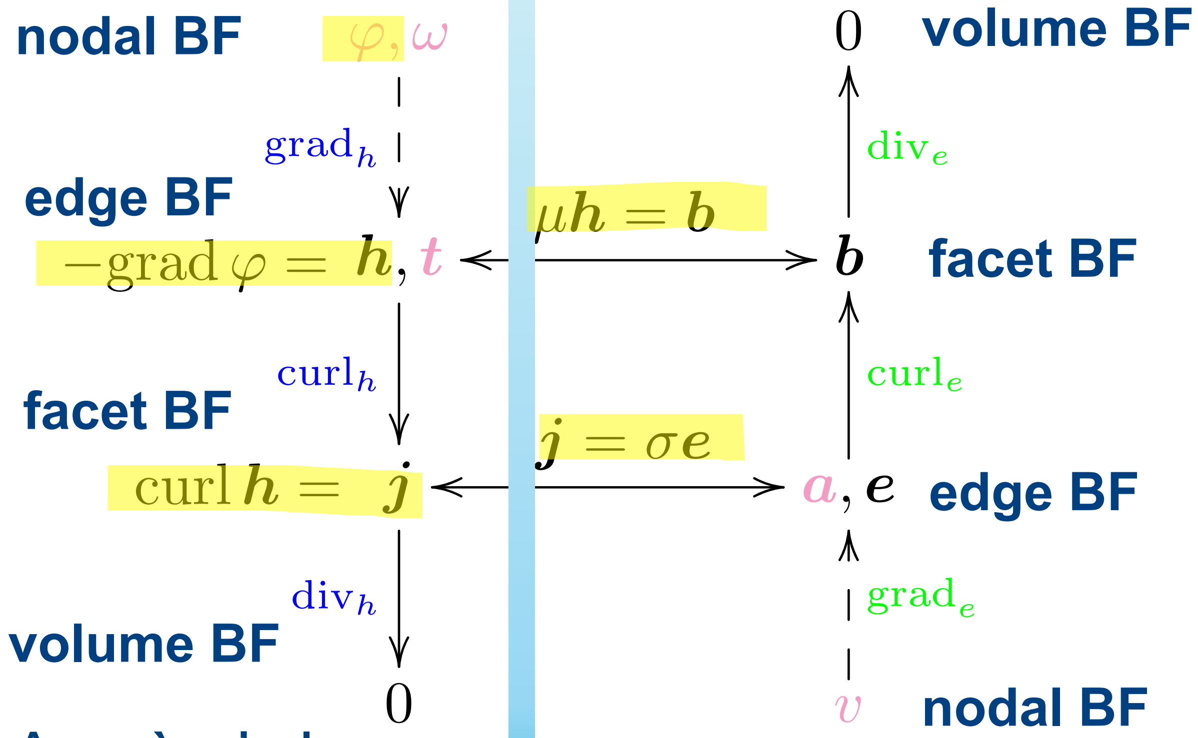
$$\operatorname{curl} \boldsymbol{h} = \boldsymbol{j} \qquad \boldsymbol{b} = \mu\, \boldsymbol{h} (+\boldsymbol{b}_s)$$
$$\operatorname{curl} \boldsymbol{e} = -\partial_t \boldsymbol{b} \qquad \boldsymbol{j} = \sigma\, \boldsymbol{e} (+\boldsymbol{j}_s)$$
$$\operatorname{div} \boldsymbol{b} = 0$$

Displacement currents neglected with regard to eddy currents

KU LEUVEN

Magnetodynamics

$$\operatorname{curl} \boldsymbol{e} = -\partial_t \boldsymbol{b}, \qquad \operatorname{curl} \boldsymbol{h} = \boldsymbol{j}, \qquad \operatorname{div} \boldsymbol{b} = 0$$



Ampère's law verified in a **strong** sense

$\boldsymbol{h} - \varphi$ formulation

$\boldsymbol{h}$ magnetic field

φ magnetic scalar potential

resistivity $\rho = \dfrac{1}{\sigma}$

In non conduction domain:

$$\boldsymbol{h} = \boldsymbol{h}_s + \boldsymbol{h}_r \text{ with } \begin{cases} \operatorname{curl} \boldsymbol{h}_s = \boldsymbol{j}_s \\ \operatorname{curl} \boldsymbol{h}_r = 0 \end{cases}$$

Faraday's law verified in a **weak** sense

$$\operatorname{curl} \rho \operatorname{curl} \boldsymbol{h} + \partial_t (\mu \boldsymbol{h}) = 0 \quad \text{in } \Omega_c$$

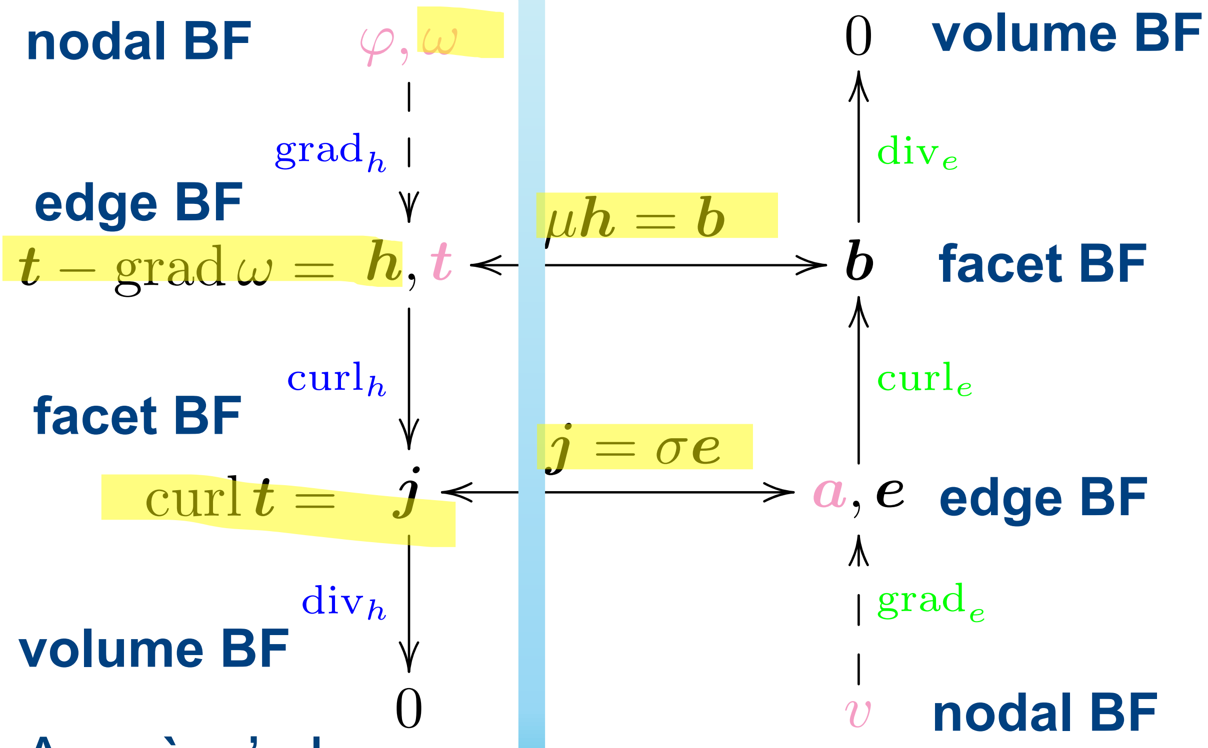
$$\operatorname{div} \Big(\mu (\boldsymbol{h}_s - \operatorname{grad} \varphi) \Big) = 0 \quad \text{in } \Omega_c^C$$

Magnetodynamics

$\boldsymbol{t} - \omega$ formulation
$\boldsymbol{t}$ electric vector potential
ω magnetic scalar potential

$$\operatorname{curl} \boldsymbol{e} = -\partial_t \boldsymbol{b}, \qquad \operatorname{curl} \boldsymbol{h} = \boldsymbol{j}, \qquad \operatorname{div} \boldsymbol{b} = 0$$

resistivity $\rho = \dfrac{1}{\sigma}$



Ampère's law verified in a **strong** sense

Faraday's law verified in a **weak** sense

$+$ Gauge in Ω

$$\operatorname{curl}\left(\rho \operatorname{curl} \boldsymbol{t}\right) + \partial_t\Big(\mu(\boldsymbol{t} - \operatorname{grad} \omega)\Big) = 0 \quad \text{in } \Omega_c$$

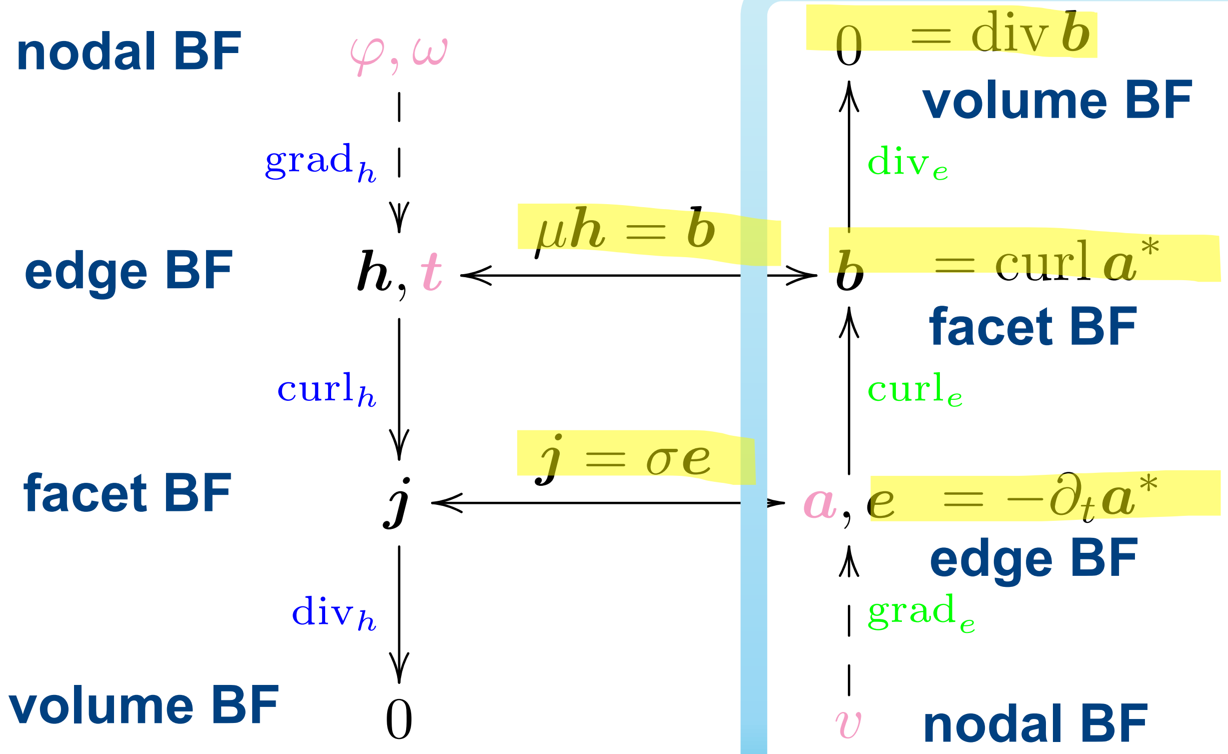
$$\operatorname{div}\Big(\mu(\boldsymbol{t} - \operatorname{grad} \omega)\Big) = 0 \quad \text{in } \Omega_c^C$$

Magnetodynamics

$\boldsymbol{a}^*$ formulation
$\boldsymbol{a}^*$ magnetic vector potential

$$\operatorname{curl} \boldsymbol{e} = -\partial_t \boldsymbol{b}, \qquad \operatorname{curl} \boldsymbol{h} = \boldsymbol{j}, \qquad \operatorname{div} \boldsymbol{b} = 0$$

reluctivity $\nu = \dfrac{1}{\mu}$



Faraday's law verified in a **strong** sense

Ampère's law verified in a **weak** sense

$$\operatorname{curl} \nu \operatorname{curl} \boldsymbol{a}^* + \sigma \partial_t \boldsymbol{a}^* = \boldsymbol{j}_s$$

$+$ Gauge in Ω_c^C

Magnetodynamics

$\boldsymbol{a} - v$ formulation
$\boldsymbol{a}$ magnetic vector potential
v electric scalar potential potential

$$\operatorname{curl} \boldsymbol{e} = -\partial_t \boldsymbol{b}, \qquad \operatorname{curl} \boldsymbol{h} = \boldsymbol{j}, \qquad \operatorname{div} \boldsymbol{b} = 0$$

reluctivity $\nu = \dfrac{1}{\mu}$

$H^1(\Omega)$ — nodal BF — φ, ω
$\downarrow$ grad, grad_h
$\boldsymbol{H}(\operatorname{curl}; \Omega)$ — edge BF — $\boldsymbol{h}, \boldsymbol{t}$
$\downarrow$ curl, curl_h
$\boldsymbol{H}(\operatorname{div}; \Omega)$ — facet BF — $\boldsymbol{j}$
$\downarrow$ div, div_h
$L^2(\Omega)$ — volume BF — 0

$\mu \boldsymbol{h} = \boldsymbol{b}$

$\boldsymbol{j} = \sigma \boldsymbol{e}$

$0 = \operatorname{div} \boldsymbol{b}$ — volume BF
$\uparrow \operatorname{div}_e$
$\boldsymbol{b} = \operatorname{curl} \boldsymbol{a}$ — facet BF
$\uparrow \operatorname{curl}_e$
$\boldsymbol{a}, \boldsymbol{e} = -\partial_t \boldsymbol{a} - \operatorname{grad} v$ — edge BF
$\uparrow \operatorname{grad}_e$
v — nodal BF

Faraday's law verified in a **strong** sense

Ampère's law verified in a **weak** sense

$$\operatorname{curl} \nu \operatorname{curl} \boldsymbol{a} + \sigma(\partial_t \boldsymbol{a} + \operatorname{grad} v) = \boldsymbol{j}_s$$

$$\operatorname{div}\Big(-\sigma(\partial_t \boldsymbol{a} + \operatorname{grad} v)\Big) = 0 \quad \text{in } \Omega^C$$

$+$ Gauge in Ω

Spatial discretization — magnetodynamics

We want to find the modified magnetic vector potential $\boldsymbol{a}^*(\boldsymbol{x})$ in Ω

$$\operatorname{curl}(\nu \operatorname{curl} \boldsymbol{a}^*) + \sigma \partial_t \boldsymbol{a}^* = \boldsymbol{j}_s$$

with given

$\boldsymbol{j}_s(\boldsymbol{x})$ imposed electric current density
$\nu(\boldsymbol{x})$ reluctivity > 0 in part of the domain
$\sigma(\boldsymbol{x})$ sigma > 0 in part of the domain

$$\operatorname{curl}(\nu \operatorname{curl} \boldsymbol{a}^*) + \sigma \partial_t \boldsymbol{a}^* = \boldsymbol{j}_s$$

weighted residual approach

We integrate the equation weighted by (vectorial) weighting or test functions $\boldsymbol{w}_i(\boldsymbol{x})$ over the whole domain Ω:

find $\boldsymbol{a}^*$ such that

$$\int_\Omega \Big(\operatorname{curl}(\nu \operatorname{curl} \boldsymbol{a}^*) + \sigma \partial_t \boldsymbol{a}^*\Big) \cdot \boldsymbol{w}_i \,\mathrm{d}\Omega = \int_\Omega \boldsymbol{j}_s \cdot \boldsymbol{w}_i \,\mathrm{d}\Omega$$

holds $\forall \boldsymbol{w}_i$

Spatial discretization — magnetodynamics (II)

$$\int_\Omega \left(\mathrm{curl}\,(\nu \mathrm{curl}\,\boldsymbol{a}^*) + \sigma\partial_t \boldsymbol{a}^*\right)\cdot \boldsymbol{w}_i \,\mathrm{d}\Omega = \int_\Omega \boldsymbol{j}_s \cdot \boldsymbol{w}_i \,\mathrm{d}\Omega$$

$\boldsymbol{v} = \boldsymbol{w}_i$

$\boldsymbol{u} = \nu\,\mathrm{curl}\,\boldsymbol{a}^*$

$$\boldsymbol{v}\cdot \mathrm{curl}\,\boldsymbol{u} - \boldsymbol{u}\cdot \mathrm{curl}\,\boldsymbol{v} = \mathrm{div}\,(\boldsymbol{u}\times\boldsymbol{v})$$

integration by parts
Green formula

$$\int_\Omega \left(\mathrm{div}\,(\nu\,\mathrm{curl}\,\boldsymbol{a}^*\times \boldsymbol{w}_i) + \nu\,\mathrm{curl}\,\boldsymbol{a}^*\cdot \mathrm{curl}\,\boldsymbol{w}_i + \sigma\partial_t \boldsymbol{a}^*\cdot \boldsymbol{w}_i\right)\mathrm{d}\Omega = \int_\Omega \boldsymbol{j}_s\cdot \boldsymbol{w}_i\,\mathrm{d}\Omega$$

$\boldsymbol{u} = \nu\,\mathrm{curl}\,\boldsymbol{a}^* \times \boldsymbol{w}_i$

$$\int_\Omega \mathrm{div}\,\boldsymbol{u}\,\mathrm{d}\Omega = \oint_\Gamma \boldsymbol{u}\,\mathrm{d}\boldsymbol{\Gamma}\,, \quad \mathrm{d}\boldsymbol{\Gamma} = \boldsymbol{n}\mathrm{d}\Gamma$$

divergence theorem

find $\boldsymbol{a}^*$ such that

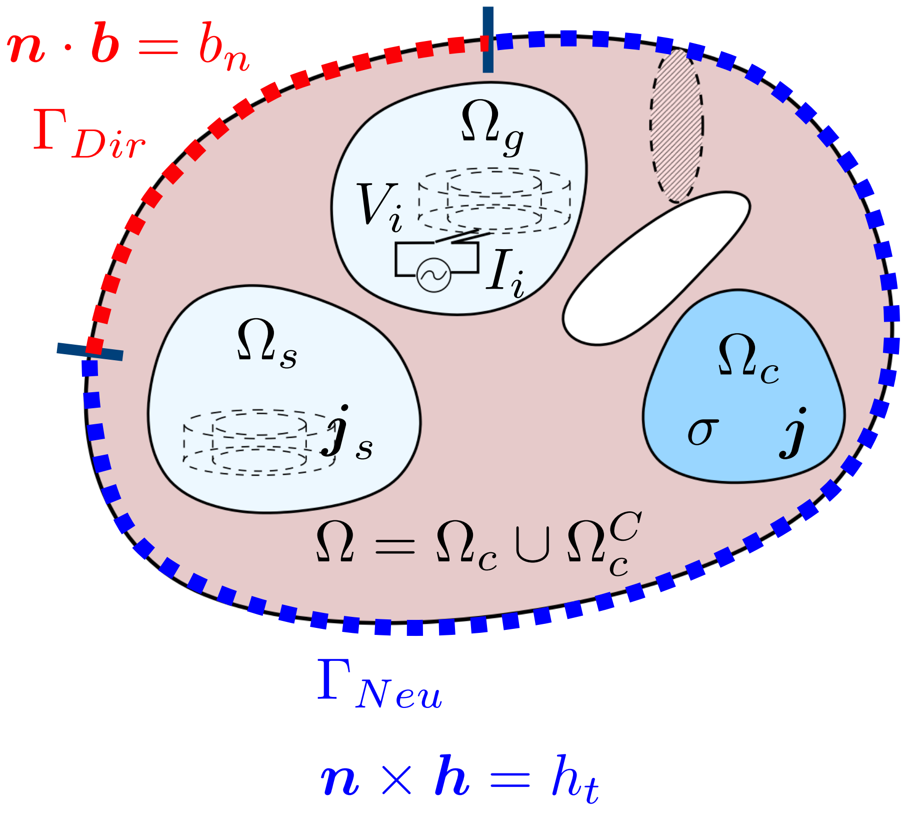
Weak formulation

$$\int_\Gamma (\nu\,\mathrm{curl}\,\boldsymbol{a}^*\times \boldsymbol{w}_i)\,\boldsymbol{n}\mathrm{d}\Gamma + \int_\Omega \left(\nu\,\mathrm{curl}\,\boldsymbol{a}^*\cdot \mathrm{curl}\,\boldsymbol{w}_i + \sigma\partial_t \boldsymbol{a}^*\cdot \boldsymbol{w}_i\right)\mathrm{d}\Omega = \int_\Omega \boldsymbol{j}_s\cdot \boldsymbol{w}_i\,\mathrm{d}\Omega$$

holds $\forall \boldsymbol{w}_i(\boldsymbol{x})$

only the first derivative of the MVP is now required

Spatial discretization — magnetodynamics (III)



Dirichlet BC at Γ_{Dir}

$$\boldsymbol{a}^*\times\boldsymbol{n}=\boldsymbol{a}^*_{Dir}\times\boldsymbol{n}\iff\boldsymbol{b}\cdot\boldsymbol{n}=b_n$$

Homogeneus Neumann BC at Γ_{Neu}

$$h_t=\nu\,\mathrm{curl}\,\boldsymbol{a}^*\times\boldsymbol{n}=0$$

$$\underbrace{\int_{\Gamma_{Dir}}(\nu\,\mathrm{curl}\,\boldsymbol{a}^*\times\boldsymbol{w}_i)\,\boldsymbol{n}\,\mathrm{d}\Gamma}_{=0}+\underbrace{\int_{\Gamma_{Neu}}(\nu\,\mathrm{curl}\,\boldsymbol{a}^*\times\boldsymbol{w}_i)\,\boldsymbol{n}\,\mathrm{d}\Gamma}_{=0}$$

$$\forall\boldsymbol{w}_i(\boldsymbol{x}):\boldsymbol{w}_i\times\boldsymbol{n}=0$$

essential BC **natural BC**

Spatial discretization — magnetodynamics (IV)

$$\boldsymbol{a}^*(\boldsymbol{x}) \approx \boldsymbol{a}_h^*(\boldsymbol{x}) = \sum_{j=1}^{N} u_j \boldsymbol{s}_j$$

$\boldsymbol{s}_j(\boldsymbol{x}) \times \boldsymbol{n} = 0$ at Γ_{Dir}

$$\begin{cases} \boldsymbol{s}_j(\mathbf{x}) & \text{shape functions} \\ u_j & \text{scalar unknowns, Dofs} \end{cases}$$

Ritz-Galerkin method $\boldsymbol{s}_j(\boldsymbol{x}) = \boldsymbol{w}_j(\boldsymbol{x})$

Petrov-Galerkin method $\boldsymbol{s}_j(\boldsymbol{x}) \neq \boldsymbol{w}_j(\boldsymbol{x})$

$$\int_\Omega \left(\nu \operatorname{curl} \boldsymbol{a}^* \cdot \operatorname{curl} \boldsymbol{w}_i + \sigma \partial_t \boldsymbol{a}^* \cdot \boldsymbol{w}_i\right) \mathrm{d}\Omega = \int_\Omega \boldsymbol{j}_s \cdot \boldsymbol{w}_i \, \mathrm{d}\Omega$$

↓

$$\sum_j \left(u_j \underbrace{\int_\Omega \nu \operatorname{curl} \boldsymbol{w}_j \cdot \operatorname{curl} \boldsymbol{w}_i \, \mathrm{d}\Omega}_{= k_{ij}} + \partial_t u_j \underbrace{\int_\Omega \sigma \boldsymbol{w}_j \cdot \boldsymbol{w}_i \, \mathrm{d}\Omega}_{= m_{ij}}\right) = \underbrace{\int_\Omega \boldsymbol{j}_s \cdot \boldsymbol{w}_i \, \mathrm{d}\Omega}_{= f_i}$$

→

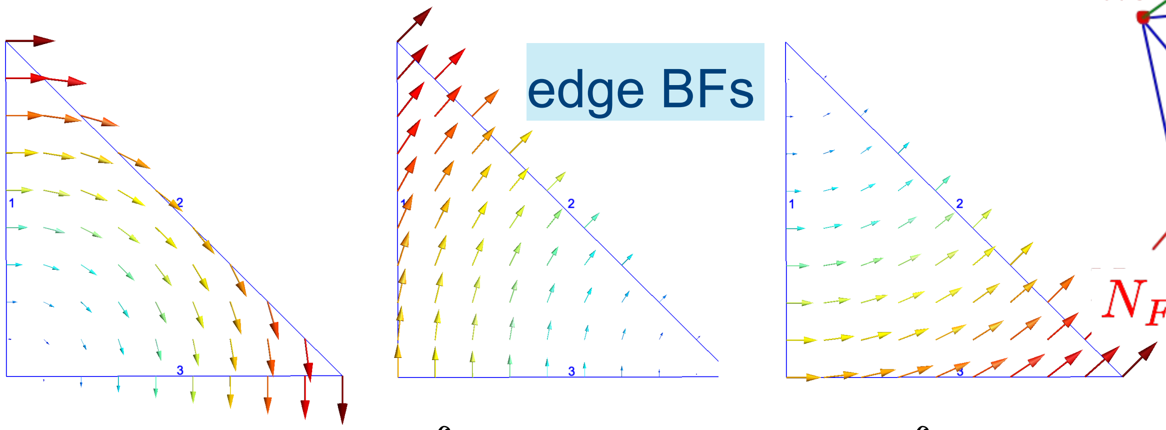
$$[k_{ij}][u_j] + [m_{ij}][\partial_t u_j] = [f_i]$$

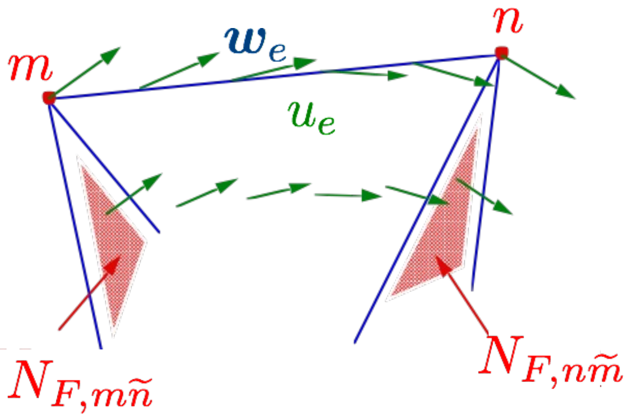
K and M symmetric, semi-positive-definite

Spatial discretization — magnetodynamics (V)

$$\sum_j \left(u_j \int_\Omega \nu \,\mathrm{curl}\, \boldsymbol{w}_j \cdot \mathrm{curl}\, \boldsymbol{w}_i \,\mathrm{d}\Omega + \partial_t u_j \int_\Omega \sigma \boldsymbol{w}_j \cdot \boldsymbol{w}_i \,\mathrm{d}\Omega\right) = \int_\Omega \boldsymbol{j}_s \cdot \boldsymbol{w}_i \,\mathrm{d}\Omega$$

$$\mathrm{curl}\, \boldsymbol{w}_j = \sum_q c_{jq} \boldsymbol{z}_q$$





$$\sum_j \left(u_j \underbrace{\sum_p \sum_q c_{ip} c_{jq}}_{\mathcal{A}_j} \underbrace{\int_\Omega \nu \, \boldsymbol{z}_q \cdot \boldsymbol{z}_p \mathrm{d}\Omega}_{\mathcal{M}^\nu_{pq}} + \partial_t u_j \underbrace{\int_\Omega \sigma \boldsymbol{w}_j \cdot \boldsymbol{w}_i \,\mathrm{d}\Omega}_{\mathcal{M}^\sigma_{ij}}\right) = \underbrace{\int_\Omega \boldsymbol{j}_s \cdot \boldsymbol{w}_i \,\mathrm{d}\Omega}_{\mathcal{J}^s_i}$$

FE matrix system $\tilde{\mathcal{C}} \mathcal{M}^\nu \mathcal{C} \mathcal{A} + \mathcal{M}^\sigma \partial_t \mathcal{A} = \mathcal{J}^s$

Inserting boundary conditions

unconstraint system

$$\begin{bmatrix} \boldsymbol{K}_{bb} & \boldsymbol{K}_{bc} \\ \boldsymbol{K}_{cb} & \boldsymbol{K}_{cc} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_b \\ \boldsymbol{u}_c \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_b \\ \boldsymbol{f}_c \end{bmatrix}$$

potentials living at Dirichlet boundaries

adding constraints leads to

$$\left[\begin{array}{cc|c} \boldsymbol{K}_{bb} & \boldsymbol{K}_{bc} & 0 \\ \boldsymbol{K}_{cb} & \boldsymbol{K}_{cc} & \boldsymbol{B}_{cq} \\ \hline 0 & \boldsymbol{B}_{qc} & 0 \end{array}\right] \begin{bmatrix} \boldsymbol{u}_b \\ \boldsymbol{u}_c \\ \boldsymbol{y}_q \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_b \\ \boldsymbol{f}_c \\ 0 \end{bmatrix}$$

eliminate known potentials

$$\boldsymbol{K}_{bb}\boldsymbol{u}_b = \boldsymbol{f}_b - \boldsymbol{K}_{bc}\boldsymbol{u}_c$$

$$\begin{bmatrix} \boldsymbol{K}_{bb} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_b \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_b - \boldsymbol{K}_{bc}\boldsymbol{u}_c \end{bmatrix}$$

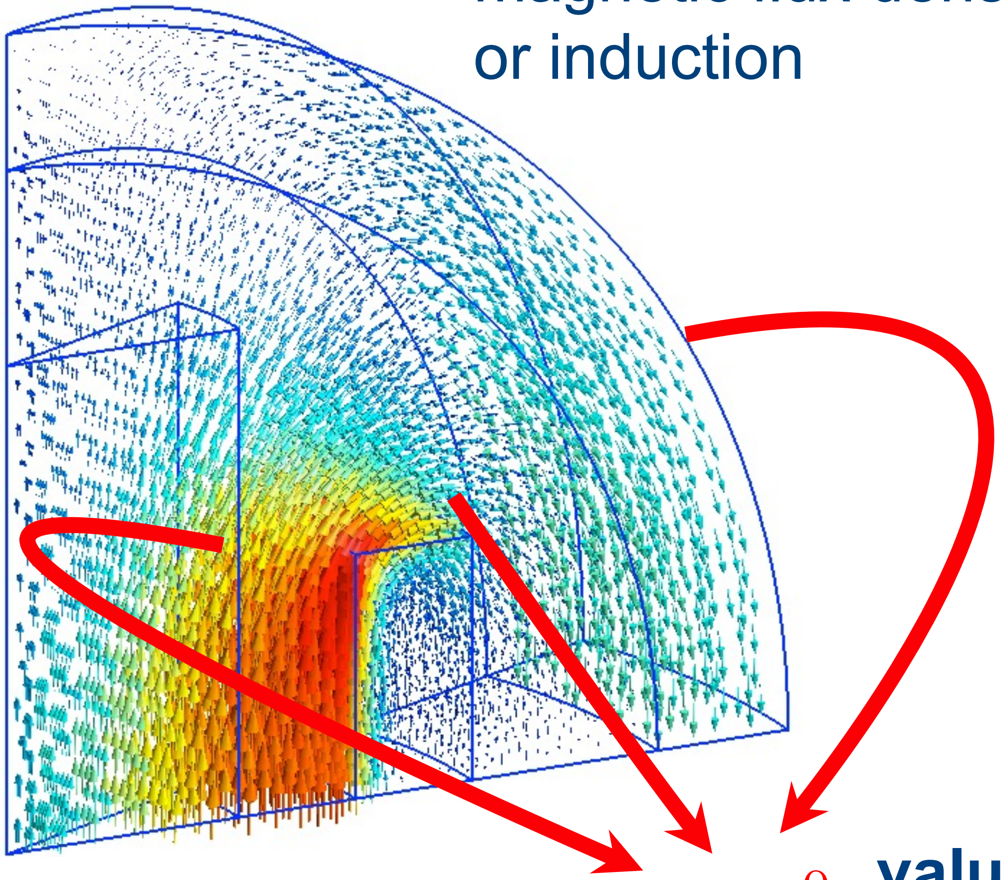
matrix system is shrank

Boundary conditions
Duality between formulations

	electric BC "flux wall" "current gate"	magnetic BC "flux gate" "current wall"
definition	$\boldsymbol{e}_t$	$\boldsymbol{h}_t$
electric current	$\boldsymbol{j}_n \neq 0$	$\boldsymbol{j}_n = 0$
magnetic flux	$\boldsymbol{b}_n = 0$	$\boldsymbol{b}_n \neq 0$
magnetic vector potential formulation — $\boldsymbol{b}$-conform	Dirichlet BC	Neumann BC
magnetic scalar potential formulation — $\boldsymbol{h}$-conform	Neumann BC	Dirichlet BC

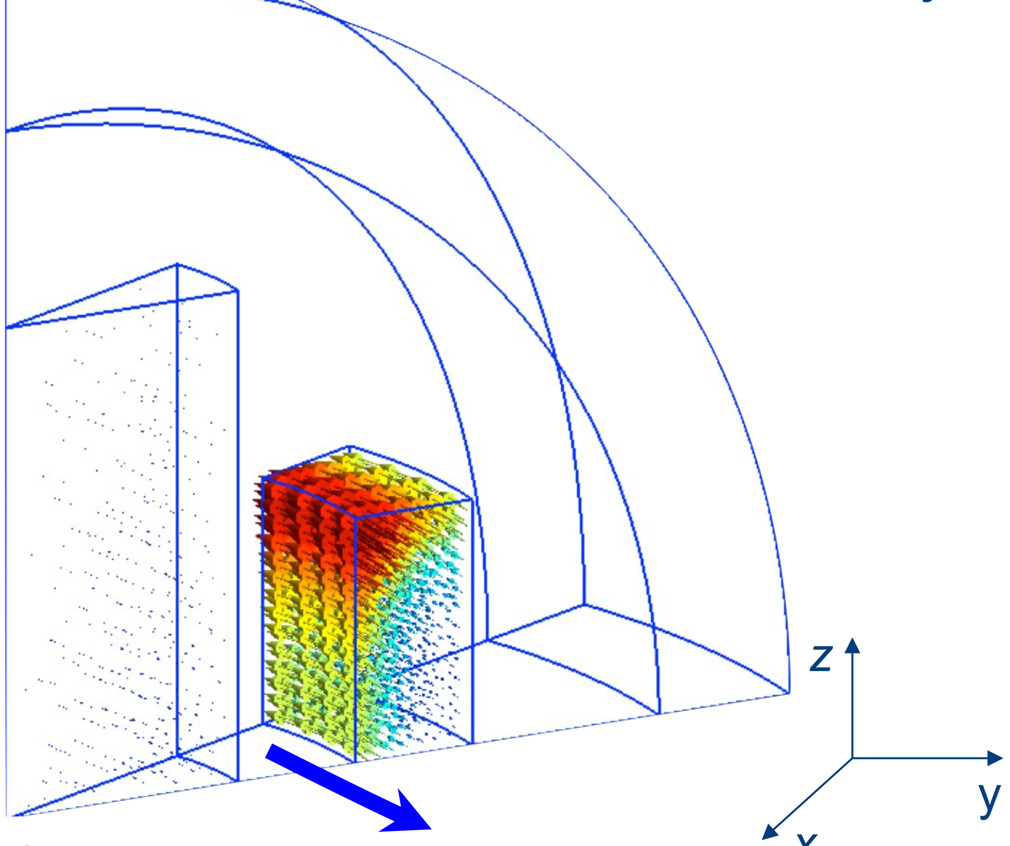
Boundary conditions (II)

magnetic flux density
or induction



$e_t = 0$

electric current density



$b_n = 0$

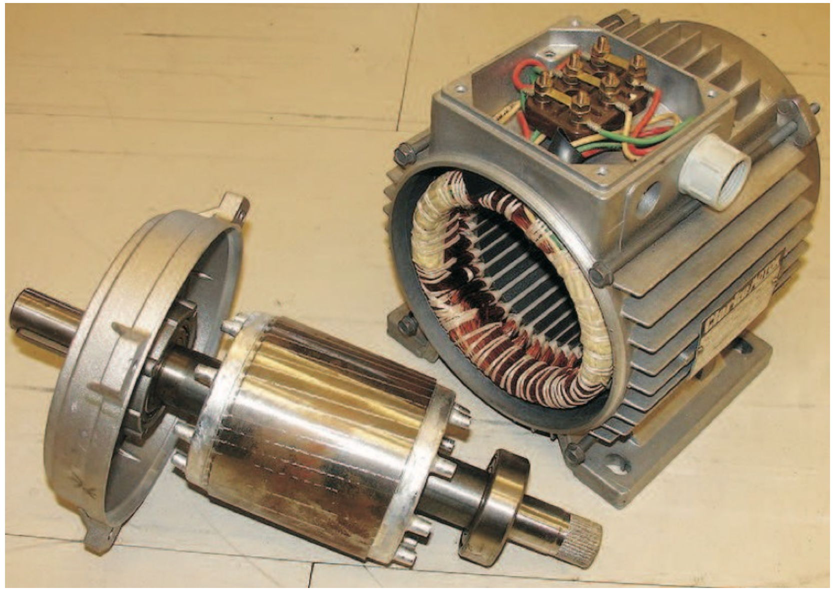
**values to impose
at symmetry planes**

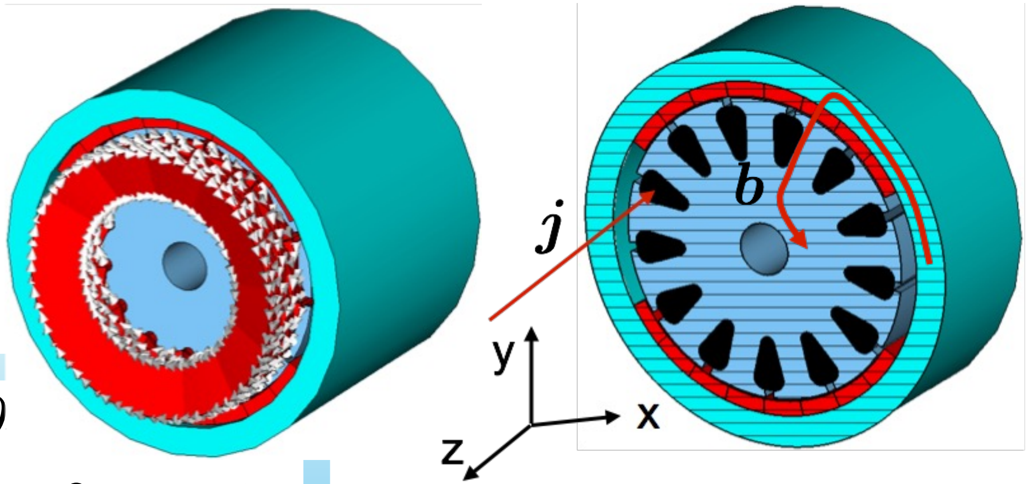
From 3D to 2D models

$$\boldsymbol{j} = (0, 0, j_z(\boldsymbol{x}))$$
$$\boldsymbol{b} = (b_x(\boldsymbol{x}), b_y(\boldsymbol{x}), 0)$$
$$\boldsymbol{h} = (h_x(\boldsymbol{x}), h_y(\boldsymbol{x}), 0)$$

$$\boldsymbol{a} = (0, 0, a_z(\boldsymbol{x}))$$
$$\boldsymbol{b} = \operatorname{curl}\boldsymbol{a} = (\partial_y a_z, -\partial_x a_z, 0)$$
$$\boldsymbol{h} = \nu\operatorname{curl}\boldsymbol{a} = \nu\,(\partial_y a_z, -\partial_x a_z, 0)$$
$$\boldsymbol{j} = (0, 0, \partial_x h_y - \partial_y h_x) = (0, 0, j_{s,z} - \sigma\partial_t a_z)$$

$$\operatorname{div}\boldsymbol{b} = \partial_x b_x + \partial_y b_y = \partial^2_{xy} a_z - \partial^2_{xy} a_z = 0$$





$$-\partial_x(\nu\partial_x a_z) - \partial_y(\nu\partial_y a_z) + \sigma\partial$$

$$\operatorname{curl}(\nu\operatorname{curl}\boldsymbol{a}) + \sigma\partial_t\boldsymbol{a} = \boldsymbol{j}_s$$

2D spatial discretization

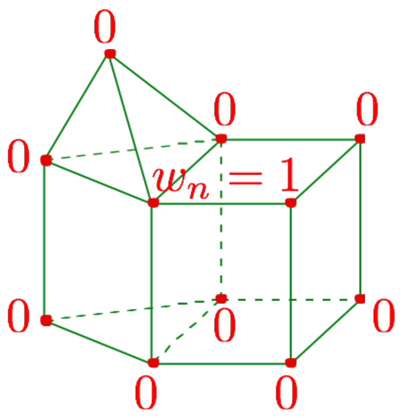
$$\sum_j \left(u_j \int_\Omega \nu \,\mathrm{curl}\, \boldsymbol{w}_j \cdot \mathrm{curl}\, \boldsymbol{w}_i \,\mathrm{d}\Omega + \partial_t u_j \int_\Omega \sigma \boldsymbol{w}_j \cdot \boldsymbol{w}_i \,\mathrm{d}\Omega \right) = \int_\Omega \boldsymbol{j}_s \cdot \boldsymbol{w}_i \,\mathrm{d}\Omega$$

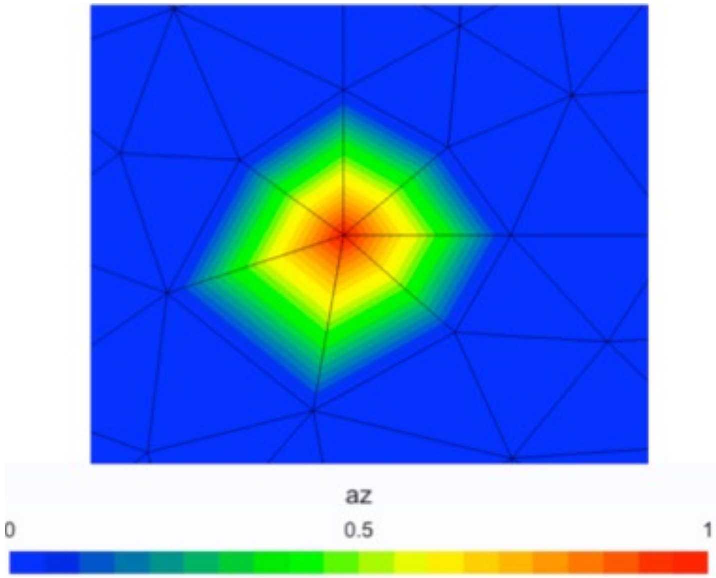
$$\boldsymbol{a} = \sum_j u_j \boldsymbol{w}_{e,j} = \sum_j u_j \frac{w_{n,j}(\boldsymbol{x})}{l_z} \hat{\boldsymbol{z}}$$

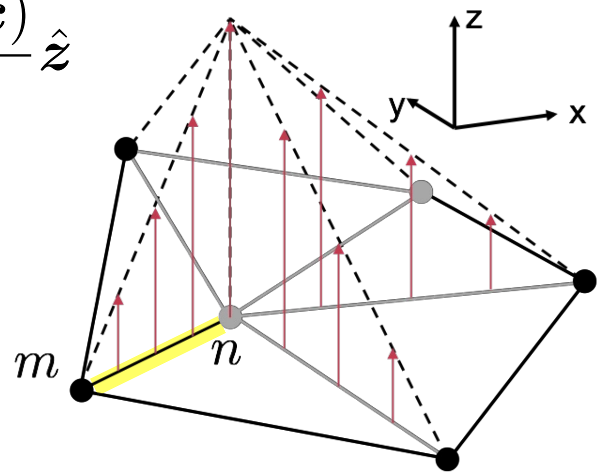
$$\boldsymbol{w}_e = \varsigma_m \,\mathrm{grad}\, \varsigma_n - \varsigma_n \,\mathrm{grad}\, \varsigma_m$$
$$w_n = \varsigma_n$$

$$\boldsymbol{a}(\boldsymbol{x}_n) = \frac{u_n}{l_z} \hat{\boldsymbol{z}}$$

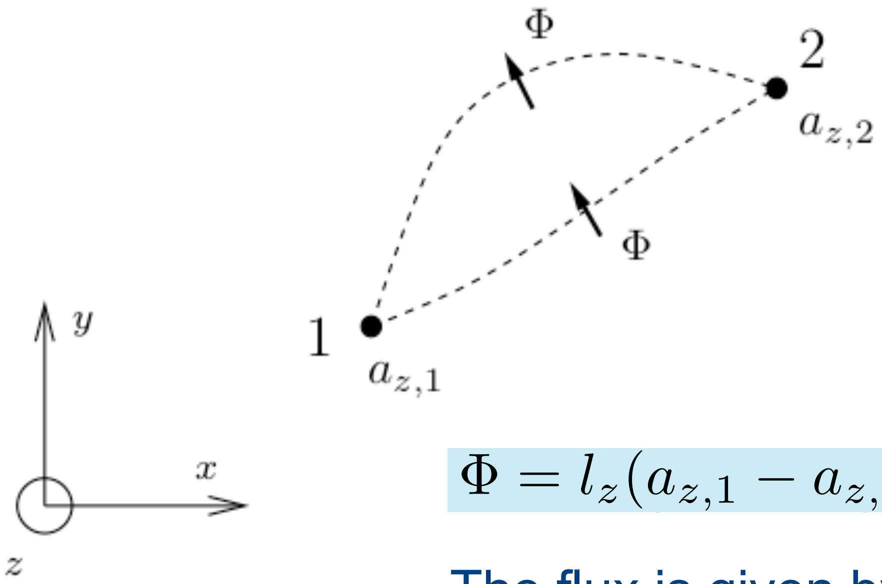
$$\boldsymbol{w}_e = \frac{w_n(\boldsymbol{x})}{l_z} \hat{\boldsymbol{z}}$$





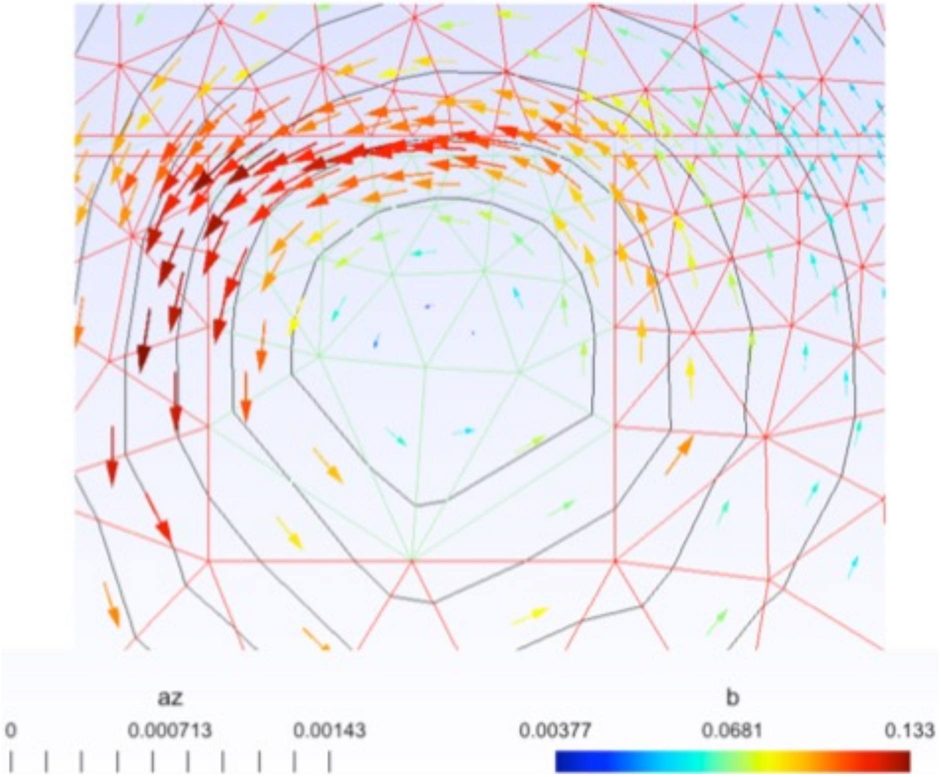


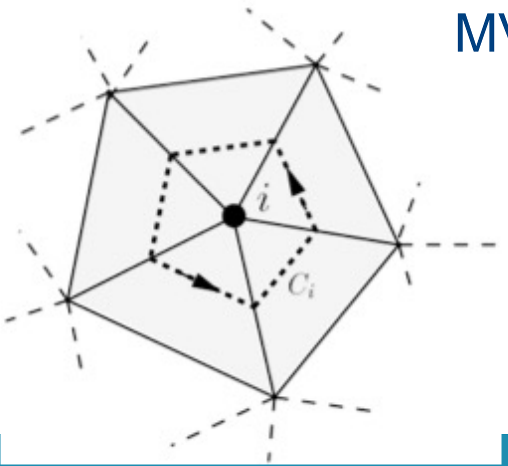
Physical meaning of MVP & FE equations



$$\Phi = l_z(a_{z,1} - a_{z,2})$$

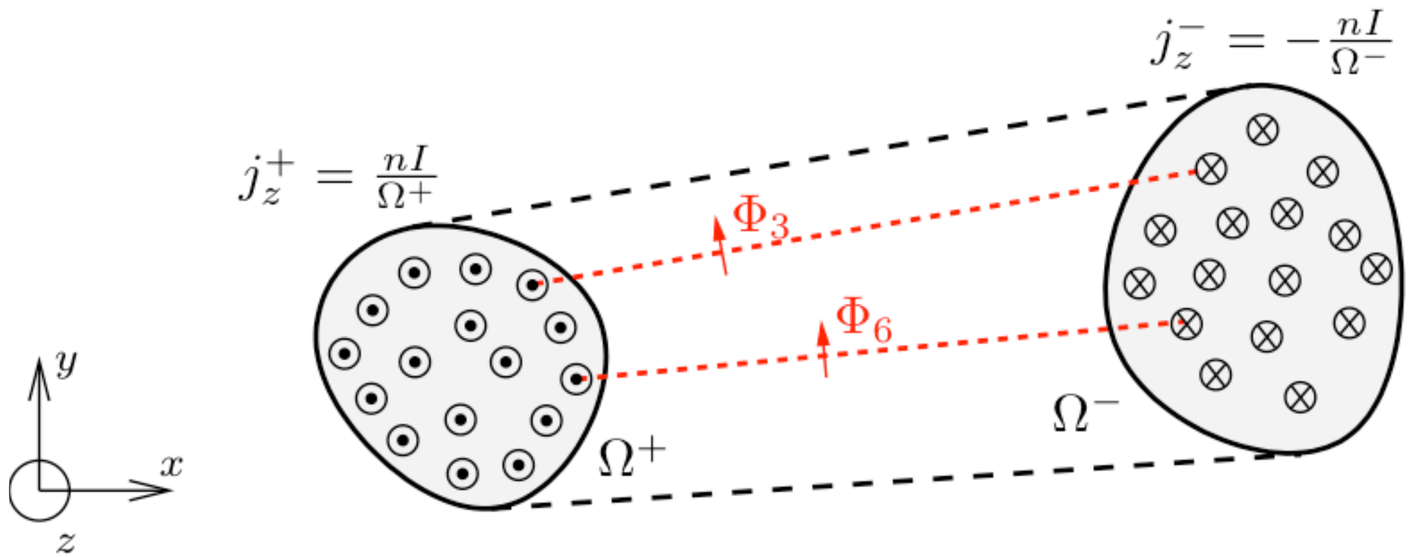
The flux is given by difference of z-MVPs multiplied by the axial length





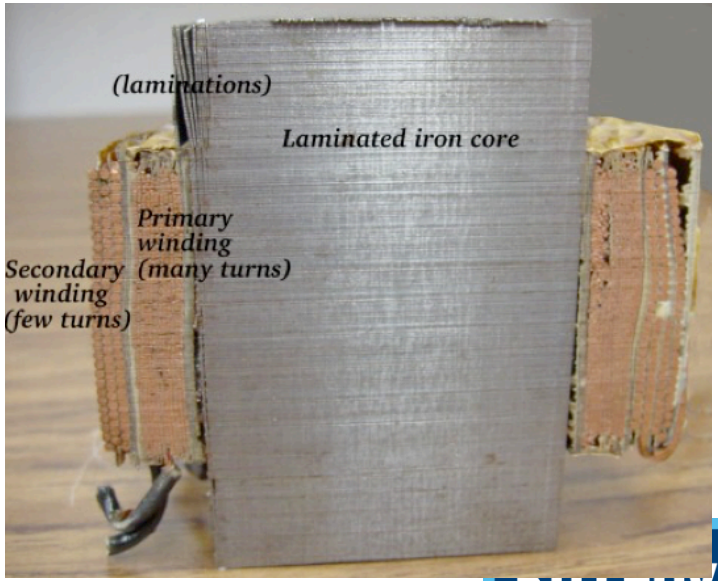
$$\int_\Omega \boldsymbol{h} \cdot \operatorname{curl} \boldsymbol{w}_i \, \mathrm{d}\Omega = \oint_{C_i} \boldsymbol{h} \cdot \mathrm{d}\boldsymbol{l} = 0$$

Flux and flux linkage of a coil



Coil with *n*=16 turns with two coil sides (+,-)
Flux of two turns is depicted
Current density, *I* A

$$\Phi = \sum_{k=1}^{n} \Phi_k$$ Total flux-linkage of the coil (Vs or Wb)



$$j_z = \pm \frac{nI}{\Omega^\pm} \text{ in } \Omega_{\text{coil}} = \Omega^+ \cup \Omega^-$$

uniform current region per coil-side region, i.e. **homogenisation**

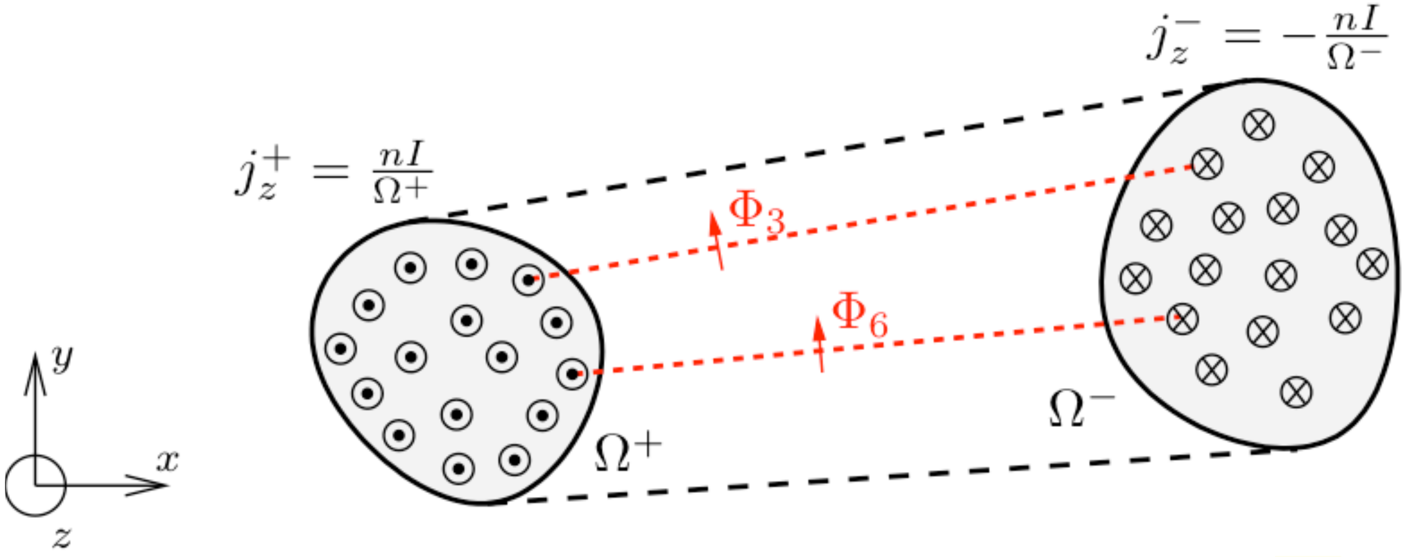
$$j_{z,1\text{A}} = \pm \frac{n}{\Omega^\pm} \text{ in } \Omega_{\text{coil}} = \Omega^+ \cup \Omega^-$$

homogenised 1D current density (1/m^2)

$$\Psi = l_z \int_{\Omega_{\text{coil}}} a_z \, j_{z,1\text{A}} \, \mathrm{d}\Omega$$

flux-linkage (in Vs) from a FE solution in terms of MVP

Flux and flux linkage of a coil



$$j_{z,1\mathrm{A}} = \pm\frac{n}{\Omega^\pm} \text{ in } \Omega_{\mathrm{coil}} = \Omega^+ \cup \Omega^-$$

$$\Psi = l_z \int_{\Omega_{\mathrm{coil}}} a_z\, j_{z,1\mathrm{A}}\, \mathrm{d}\Omega$$

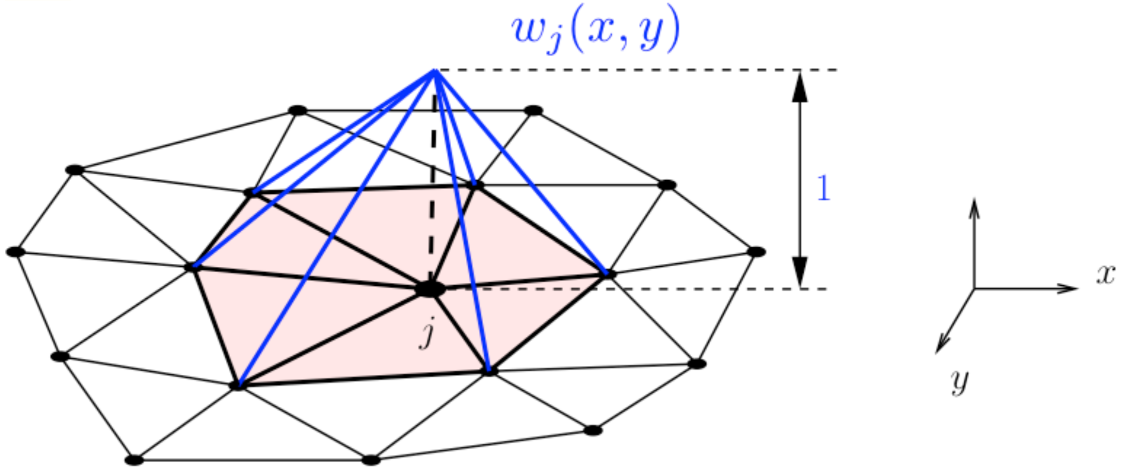
$$\mathbf{S\,A} = \mathbf{J}$$

$$j_z = j_{z,1\mathrm{A}} I$$

$$\mathbf{J_i} = \int_{\boldsymbol{\Omega}} \boldsymbol{j} \cdot \boldsymbol{w}_\mathbf{i}\, \mathrm{d}\Omega$$

$$\mathbf{S\,A} = \mathbf{K} I$$

$$\mathbf{K}_i = \int_\Omega j_{z,1\mathrm{A}}\, w_i\, \mathrm{d}\Omega$$



2D Coil model — example electrical machine

with S the area of the slot
and N_i the number of turns of the winding

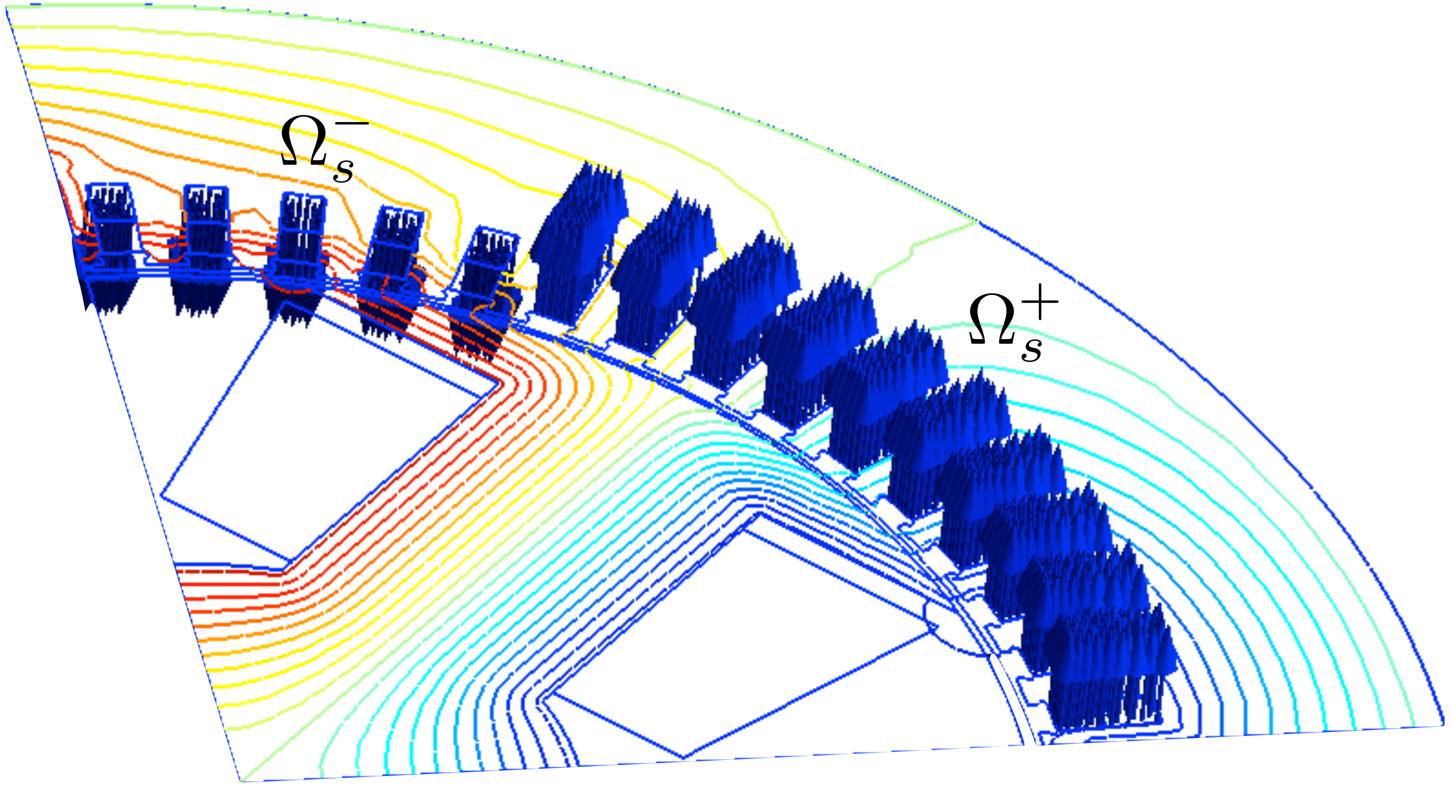
$$\boldsymbol{j} = (0, 0, j_z)$$

$$\boldsymbol{t}_{\Omega_s^+} = \frac{N_i}{S}\hat{\boldsymbol{z}}$$

$$\boldsymbol{t}_{\Omega_s^-} = -\frac{N_i}{S}\hat{\boldsymbol{z}}$$

$$\Psi = \int_{\Omega_s} \boldsymbol{a} \cdot \hat{\boldsymbol{t}}\, \mathrm{d}S$$

$$= \frac{N_i}{S}\int_{\Omega_s^+} a_z\, \mathrm{d}S - \frac{N_i}{S}\int_{\Omega_s^-} a_z\, \mathrm{d}S$$



Coil model (cont'd)

induced voltage $\sim$ flux linkage

what flux is linked?

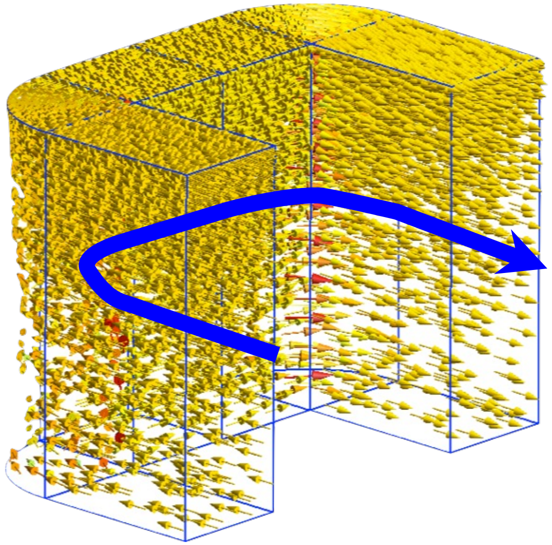
for a single path $\Phi = \oint_{\gamma} \boldsymbol{a}\,\mathrm{d}l$

for a coil

- ✓ integrating along the coil
- ✓ average of the coil cross-section

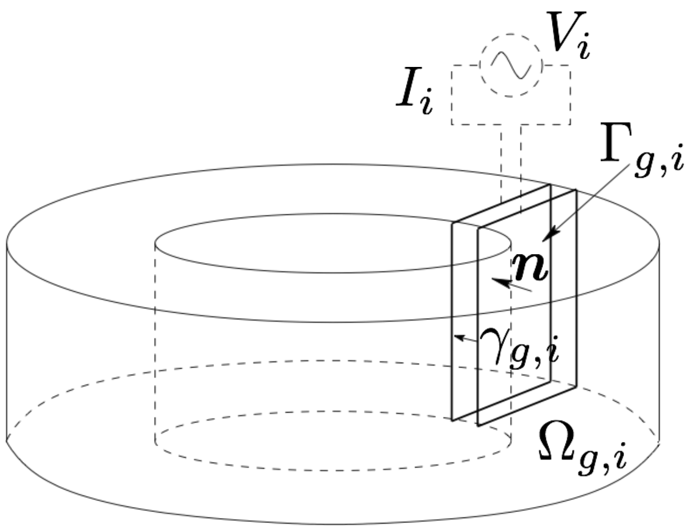
$$\Psi = \frac{n}{S_{coil}} \int_{S_{coil}} \boldsymbol{a} \cdot \hat{\boldsymbol{t}}\,\mathrm{d}S$$

$\hat{\boldsymbol{t}}$ current direction vector

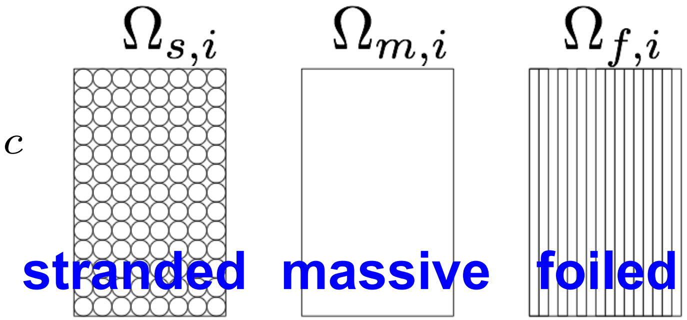


$\hat{\boldsymbol{t}}$ tangent vector from geometry

Coil model



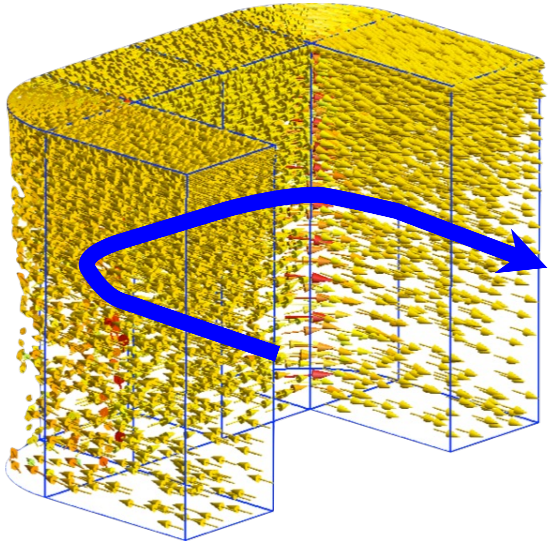
sources in
$\Omega_s \in \Omega_c^C$
$\Omega_m, \Omega_f \in \Omega_c$



cuts inductors

stranded inductors (N_i turns) $\Omega_s \in \Omega_c^C$

✓ imposed current density $\boldsymbol{j}_s = i_s(t)\,\hat{\boldsymbol{t}}$



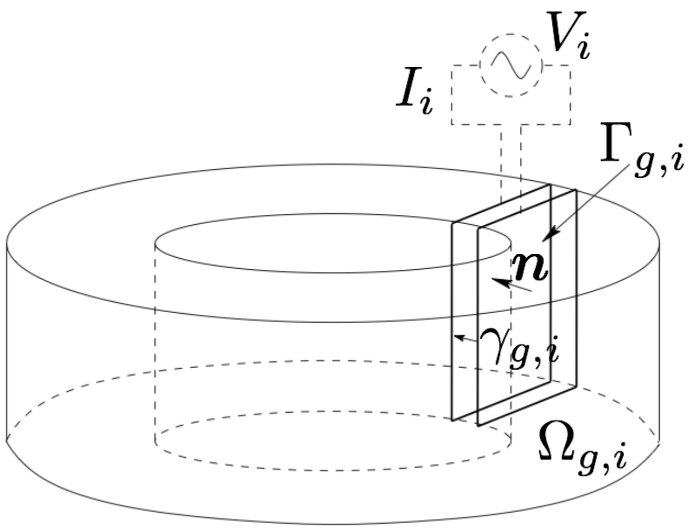
$\hat{\boldsymbol{t}}$ tangent vector from geometry

✓ imposed current or voltage
$\Rightarrow \boldsymbol{j}_s$ unknown

$$\begin{cases} \operatorname{curl} \boldsymbol{h}_s = \boldsymbol{j}_s & \text{in } \Omega_s \\ \operatorname{curl} \boldsymbol{h}_s = 0 & \text{in } \Omega_s^C \end{cases}$$

$\boldsymbol{h}_s$ computed via FEs
$\boldsymbol{h}_s$ not unique

Coil model (cont'd)

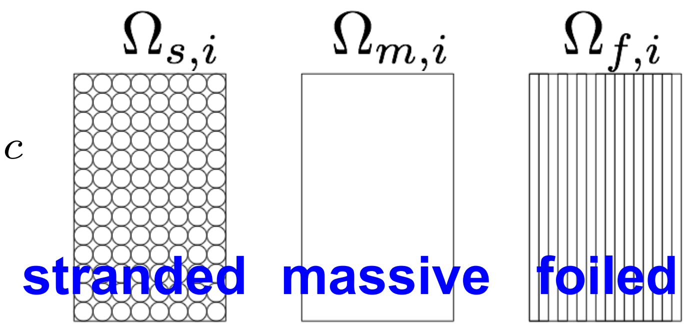


massive inductors $\Omega_m \in \Omega_c$

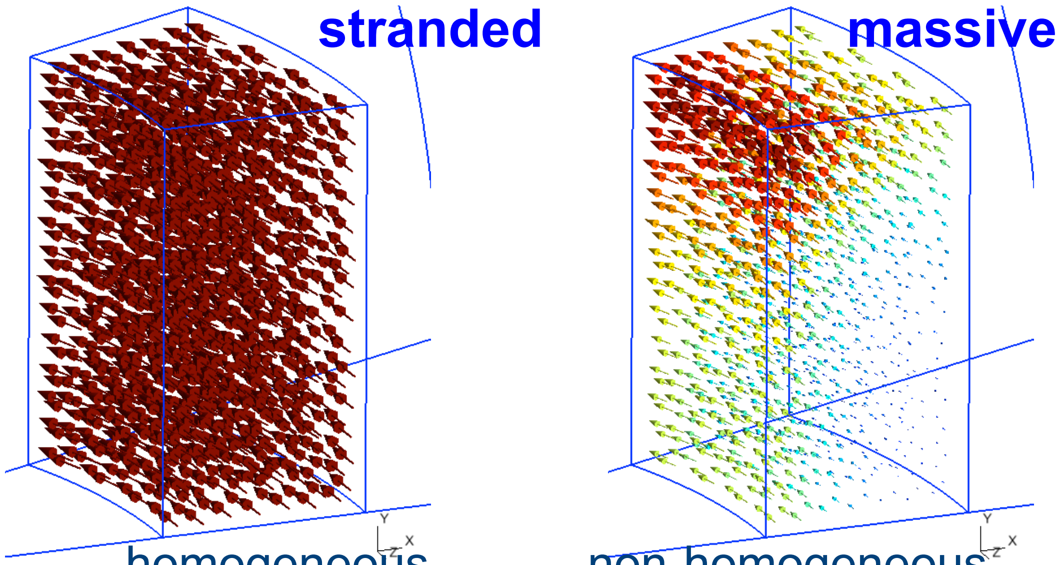
$$\int_{\gamma_{g,i}} \boldsymbol{e} \cdot d\boldsymbol{l} = V_i$$

$$\int_{\Gamma_{g,i}} \boldsymbol{n} \cdot \boldsymbol{j} \, \mathrm{d}s = I_i$$

sources in
$\Omega_s \in \Omega_c^C$
$\Omega_m, \Omega_f \in \Omega_c$



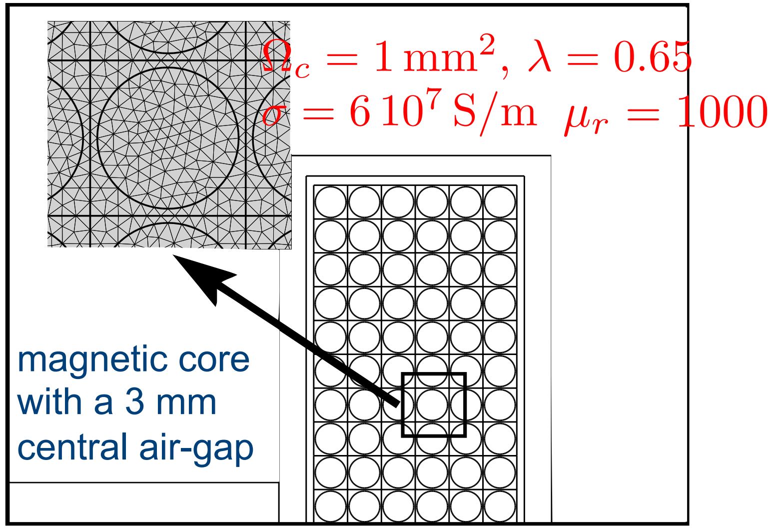
cuts inductors



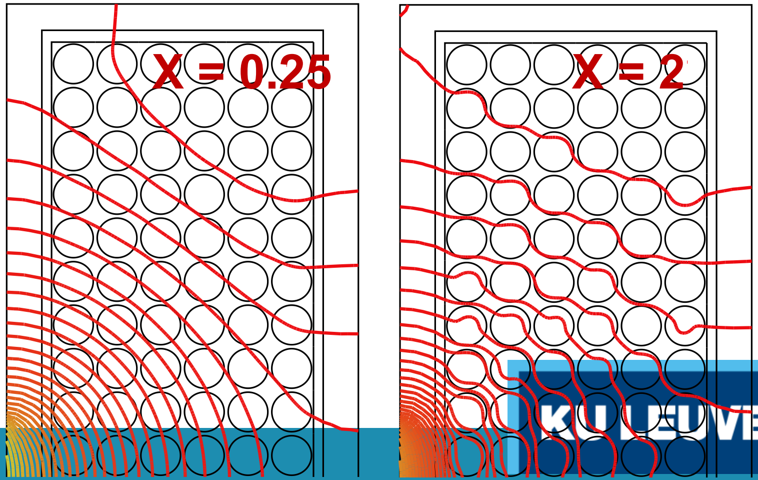
homogeneous distribution

non-homogeneous distribution

Multi-turn windings and eddy current effects



significant EC effect, affecting visibly the flux lines

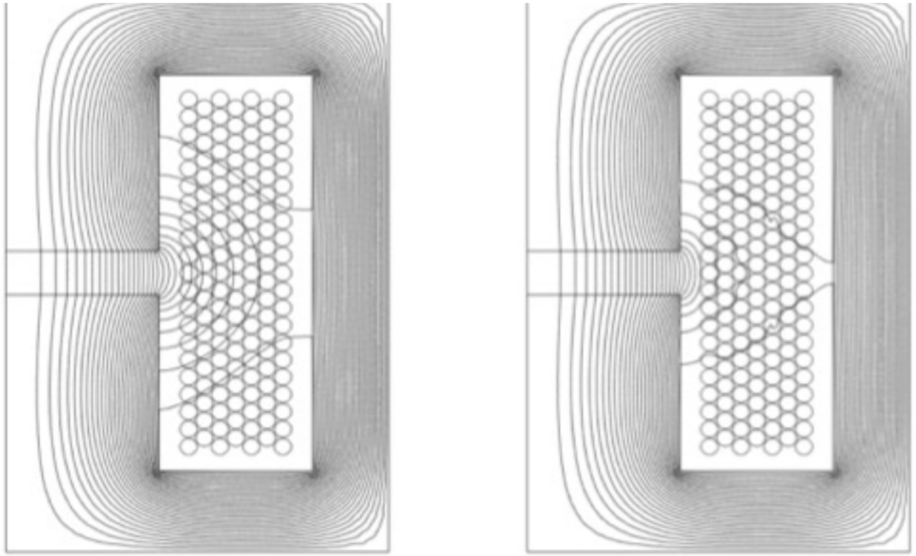


Homogenization methods are indispensable!

Multi-turn windings and eddy-current effects: Skin and proximity effects

The *equivalent radius* of the conductors, of arbitrary but symmetrical cross-sectional shape, is given by $r = \sqrt{\Omega_c/\pi}$, where Ω_c is the cross-sectional surface area. The skin depth at frequency f or pulsation $\omega = 2\pi f$ is given by $\delta = \sqrt{2/(\sigma\omega\mu_0)}$, where σ is the conductivity of the conductors and $\mu_0 = 4\pi 10^{-7}$ H/m their permeability. The normalized or *reduced frequency* X is defined as follows:

$$X = \frac{r}{\delta} = \sqrt{f} \cdot r\sqrt{\pi\sigma\mu_0}\,. \qquad (3)$$



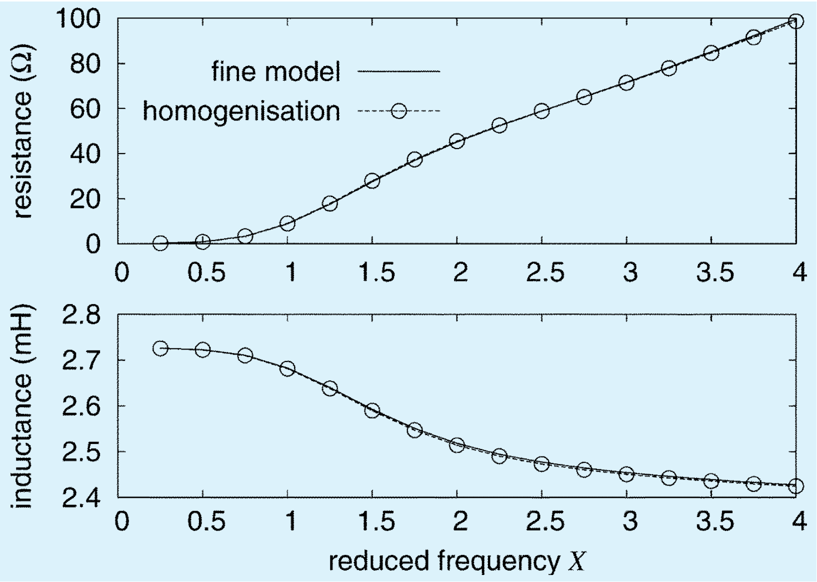
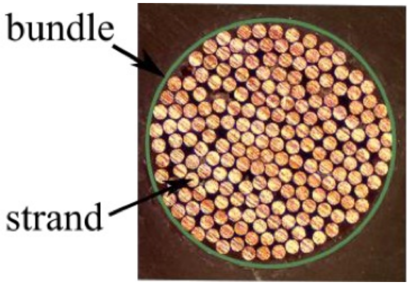
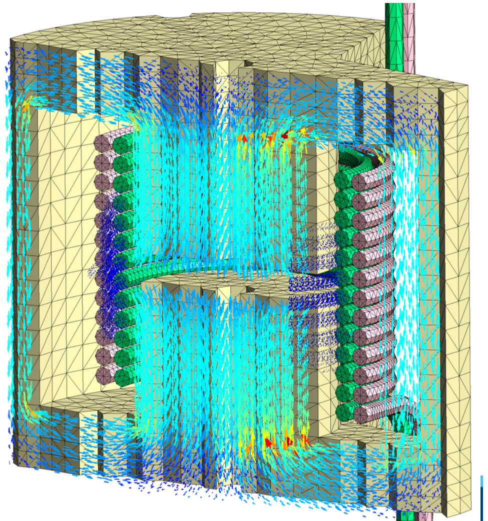
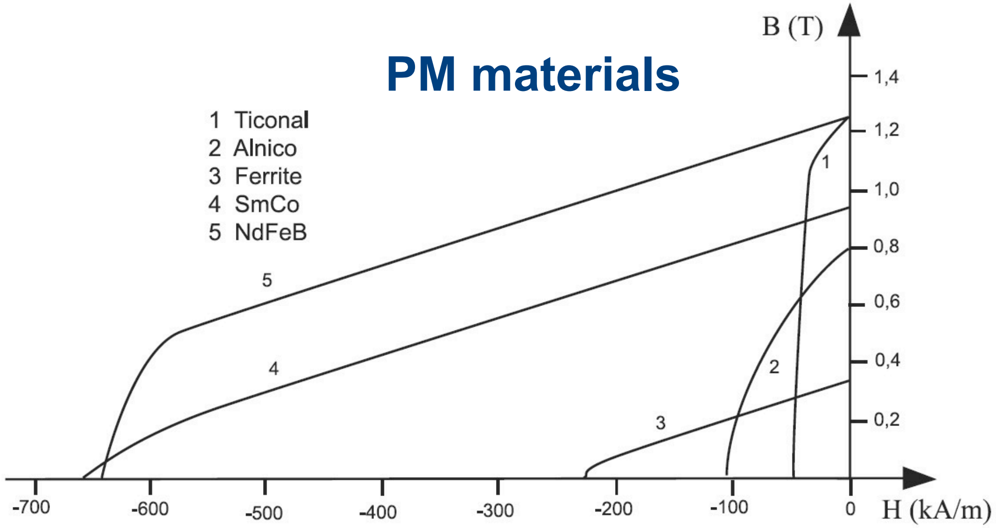


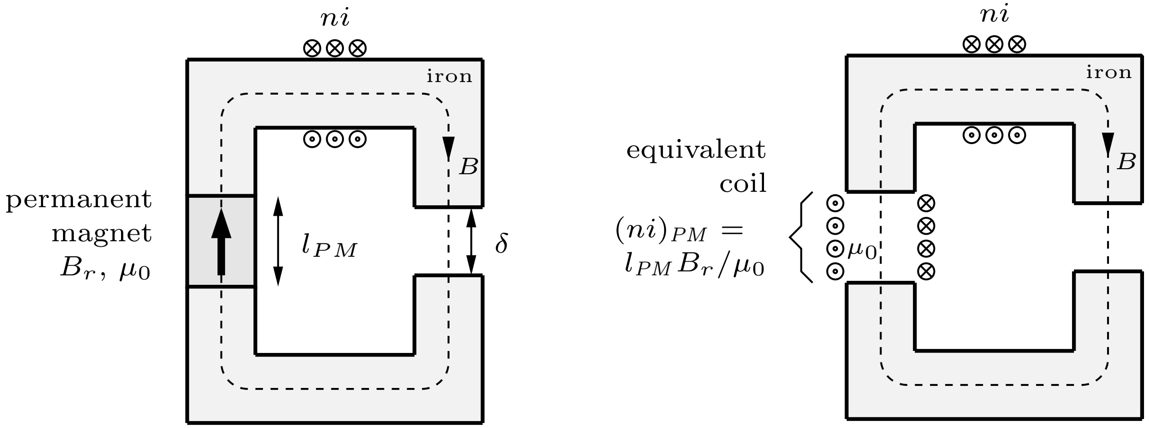
Fig. 5: 2D axisymmetric model of inductor (round conductors and hexagonal packing) – flux lines (real part, i.e. in phase with imposed current), at $X = 0.1$ (left) and $X = 2$ (right), obtained with fine model





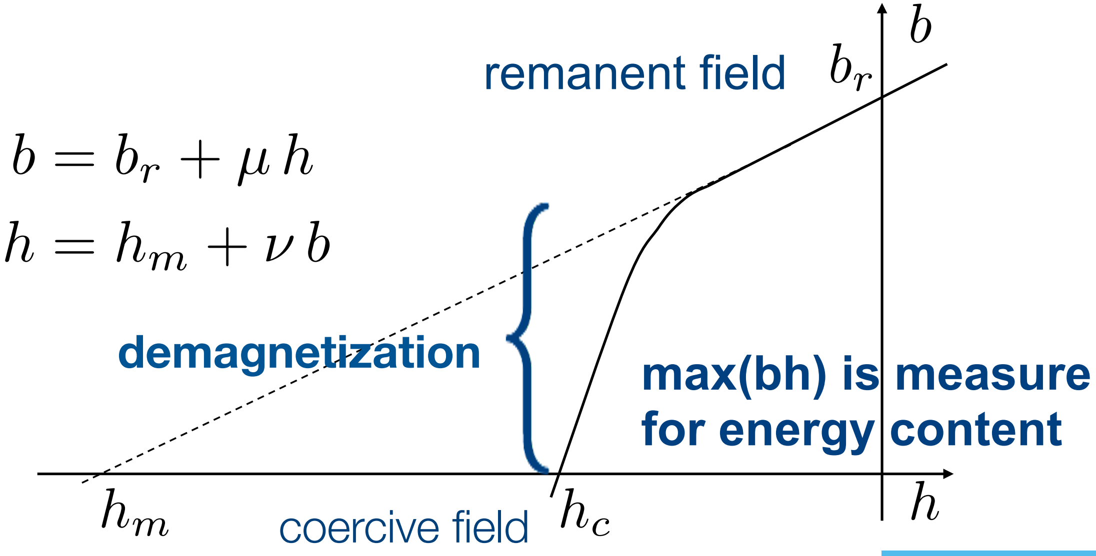
Permanent magnet
Scalar & linear model





Very simple magnetic model
✓ resulting flux approximately aligned
✓ operation point in linear range

$$b = b_r + \mu\, h$$
$$h = h_m + \nu\, b$$



MVP formulation

$$\mathrm{curl}\,(\nu \mathrm{curl}\,\boldsymbol{a}) = \boldsymbol{j} - \underbrace{\mathrm{curl}\,\boldsymbol{h}_m}_{\text{magnetization current}}$$

magnetization current

2D

$$-\partial_x(\nu \partial_x a_z) - \partial_y(\nu \partial_y a_z) =$$
$$j_z - \partial_x h_{m,y} + \partial_y h_{m,x}$$