Time-harmonic and transient simulation

Higher harmonics in electro/magneto-quasi-static models due to
- excitations, switches, nonlinear/hysteretic materials, moving parts

simulated by
- frequency domain (time-harmonic, multi-harmonic)
- time integration
- combinations (e.g. enveloped method)

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Type of problems/models

- Full wave models
  (wave propagation problems)
  - (non)**linear** materials
  - **frequency** dependence
  - (un)structured grids
  - time- and frequency-domain
  - **explicit** time integration

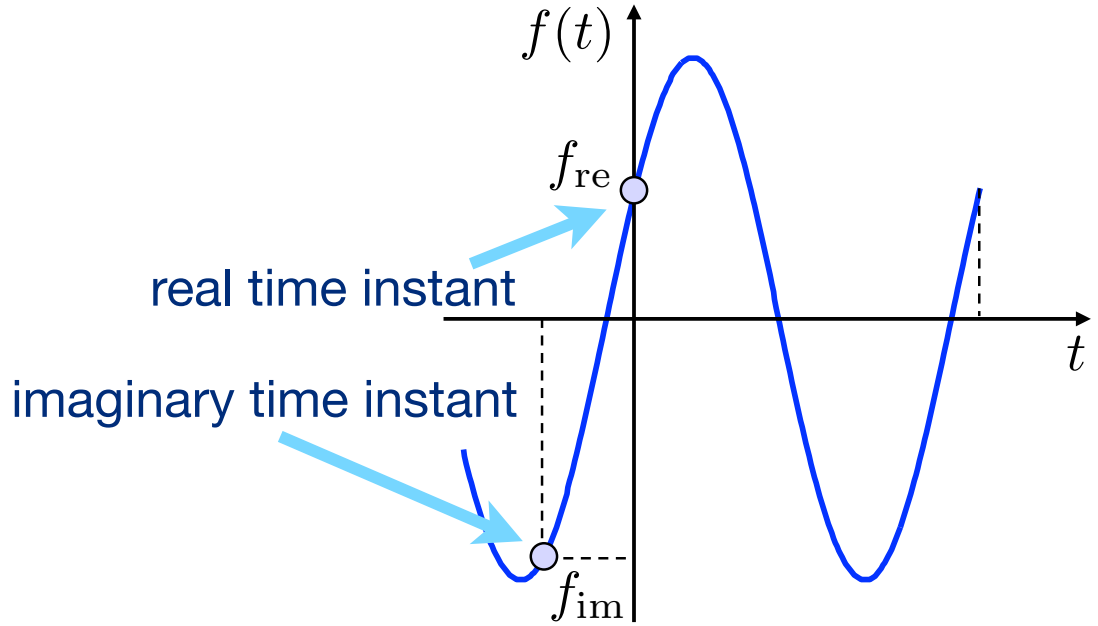
- Magneto-quasi-statics (MQS), electro-quasi-statics (EQS), Darwin models/formulations (EMQS)
  - **(non)linear** materials
  - **magnitude** dependence
  - unstructured grids
  - time- and frequency domain
  - **implicit** time integration

Phasors — Energy and power (reminder!)

$$\boldsymbol{f} = f_{\text{re}} + \imath f_{\text{im}}$$

$$f_{\text{peak}} = \sqrt{f_{\text{re}}^2 + f_{\text{im}}^2}$$

$$\varphi = \arctan\left(\frac{f_{\text{im}}}{f_{\text{re}}}\right)$$



$$\begin{aligned} f(t) &= \text{Re}\left\{\boldsymbol{f}\, e^{\imath\omega t}\right\} \\ &= f_{\text{re}}\cos(\omega t) - f_{\text{im}}\sin(\omega t) \\ &= f_{\text{peak}}\ \cos(\omega t - \varphi) \\ &= f_{\text{eff}}\sqrt{2}\ \cos(\omega t - \varphi) \\ &= \text{Re}\left\{\boldsymbol{f}_{\text{eff}}\sqrt{2}\, e^{\imath\omega t}\right\} \end{aligned}$$

$$u(t) = u_{\text{eff}}\sqrt{2}\ \cos(\omega t - \varphi_u)$$

$$i(t) = i_{\text{eff}}\sqrt{2}\ \cos(\omega t - \varphi_i)$$

$$\begin{aligned} p(t) &= u_{\text{eff}} i_{\text{eff}} \cos(\varphi_i - \varphi_u) \\ &\quad + u_{\text{eff}} i_{\text{eff}} \cos(2\omega t - \varphi_i - \varphi_u) \end{aligned}$$

$$u(t) = \text{Re}\left\{\boldsymbol{u}_{\text{eff}}\sqrt{2}\, e^{\imath\omega t}\right\}$$

$$i(t) = \text{Re}\left\{\boldsymbol{i}_{\text{eff}}\sqrt{2}\, e^{\imath\omega t}\right\}$$

$$P_{av} = \boldsymbol{u}_{\text{eff}}\, \boldsymbol{i}_{\text{eff}}^*$$

Phasor fields

Magnetic vector potential

$$\begin{aligned}\underline{a}(x,y,z,t) &= (a_x(x,y,z,t), a_y(x,y,z,t), a_z(x,y,z,t)) \\ &= \mathrm{Re}\left\{\boldsymbol{a}(x,y,z)\, e^{\imath\omega t}\right\} \\ &= \underline{a}_{re}(x,y,z)\ \cos(\omega t) - \underline{a}_{im}(x,y,z)\ \sin(\omega t) \\ &= |\boldsymbol{a}(x,y,z)|\ \cos(\omega t - \varphi(x,y,z))\end{aligned}$$

Magnetic vector potential formulation

$$\mathrm{curl}\,\nu\mathrm{curl}\,\underline{a} + \sigma\partial_t\underline{a} = \underline{j}_s$$

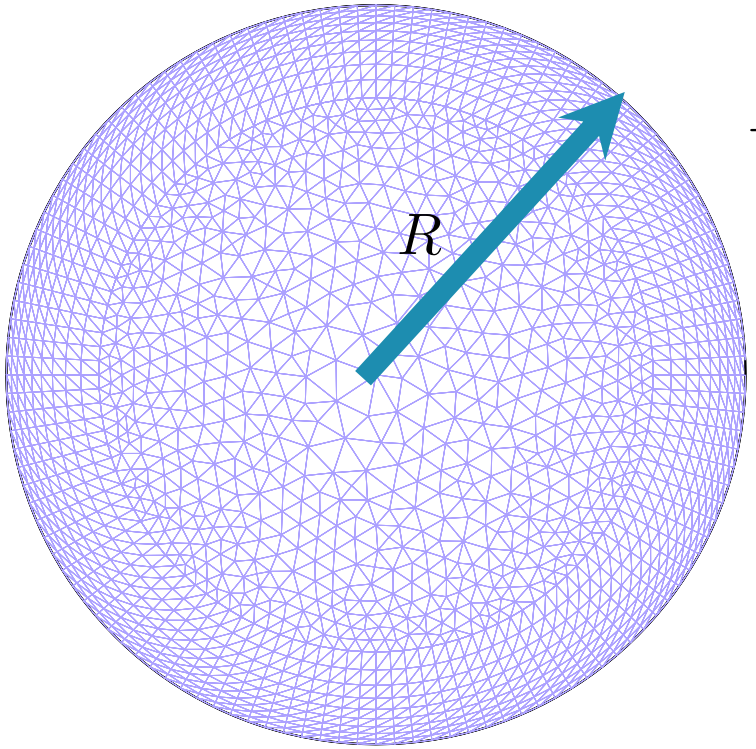
time domain (MVP is a real vector)

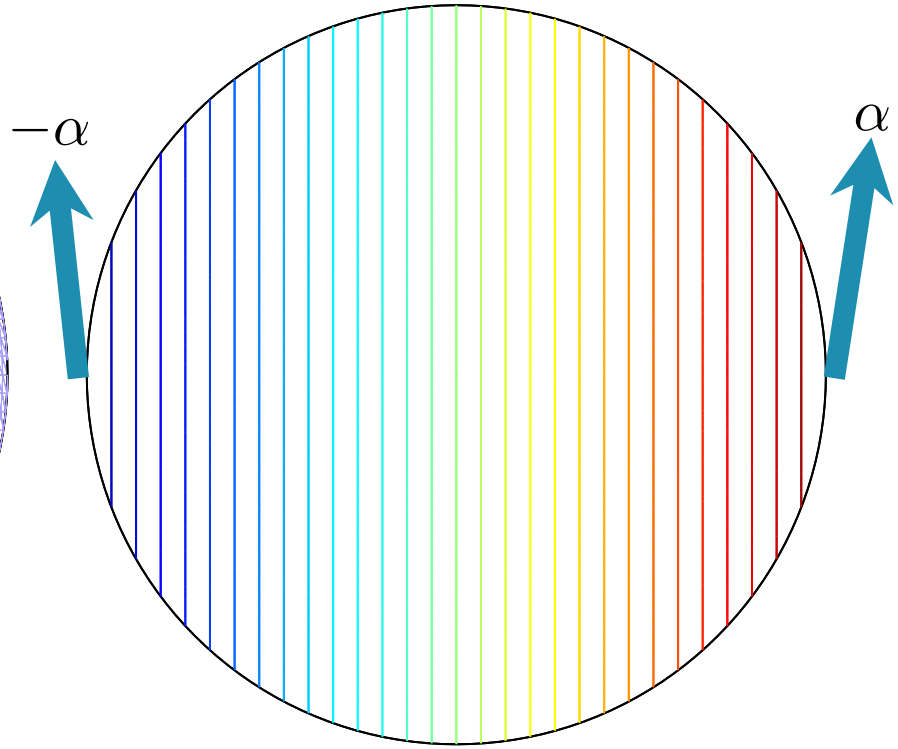
$$\mathrm{curl}\,\nu\mathrm{curl}\,\boldsymbol{a} + \imath\omega\sigma\boldsymbol{a} = \boldsymbol{j}_s$$

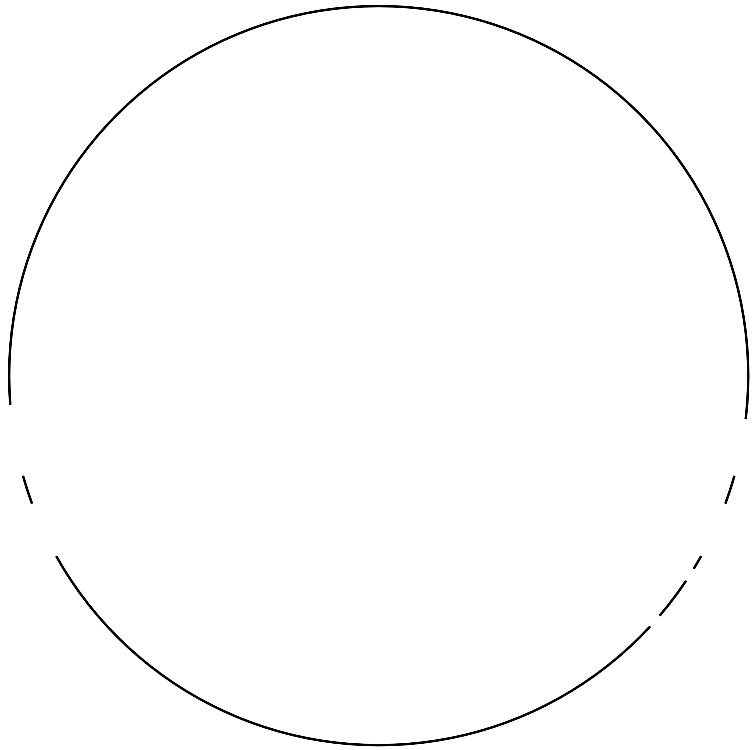
frequency domain (MVP is a complex vector)

Static field

non-conducting material







polar coordinates $a_z(R,\theta) = \alpha \cos\theta$

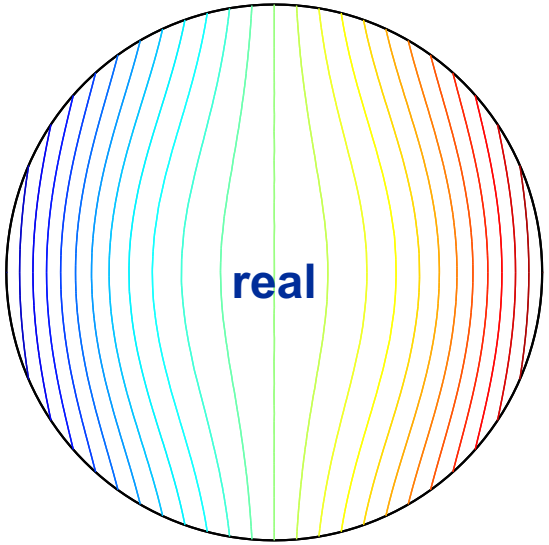
cartesian coordinates $a_z(x,y) = \alpha \dfrac{x}{R}$

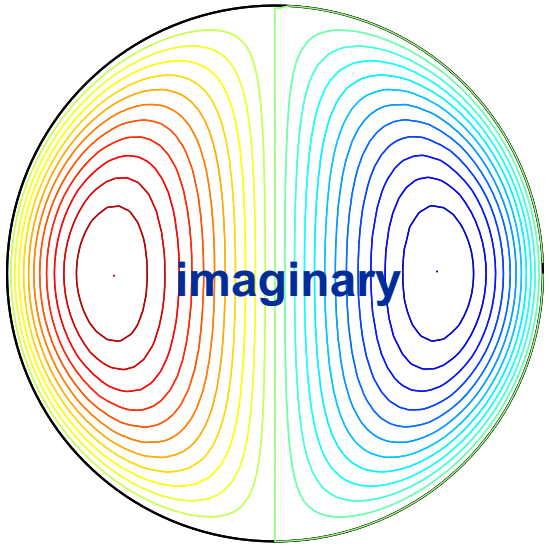
$$\underline{b} = (0, b_y(x,y), 0)$$
$$= (0, -\partial_x a_z, 0) = \left(0, -\frac{\alpha}{R}, 0\right)$$

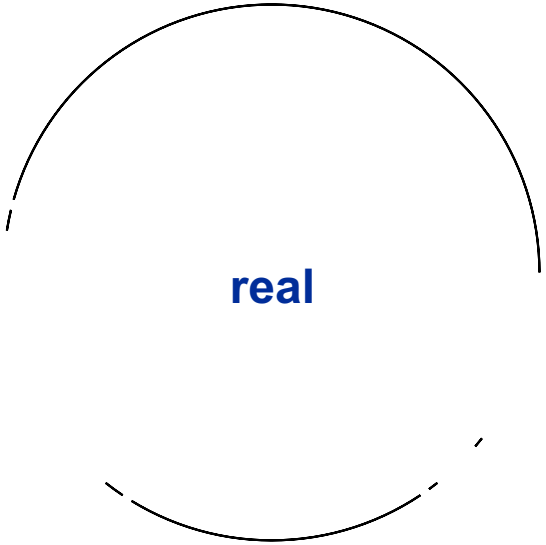
Pulsating field

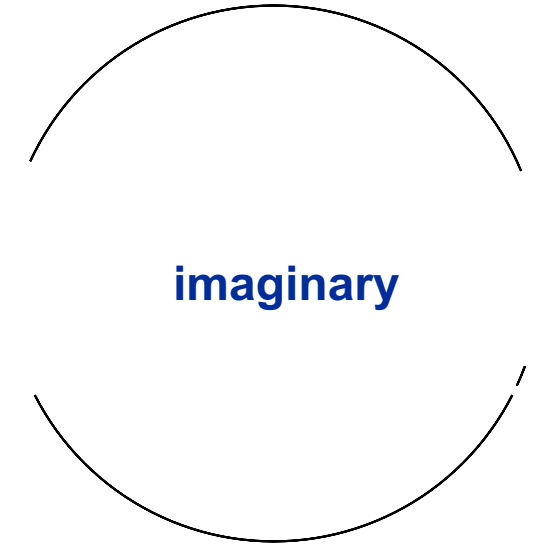
$$a_z(R,\theta,t) = \alpha\ \cos\theta\cos(\omega t)$$

$$b_y(x,y,t) = -\frac{\alpha}{R}\cos(\omega t)$$

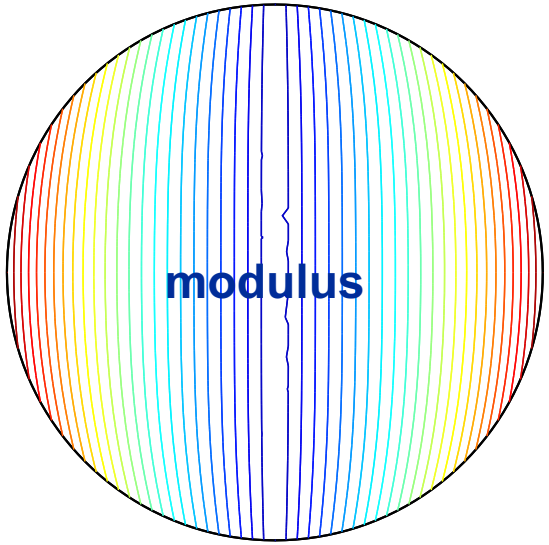


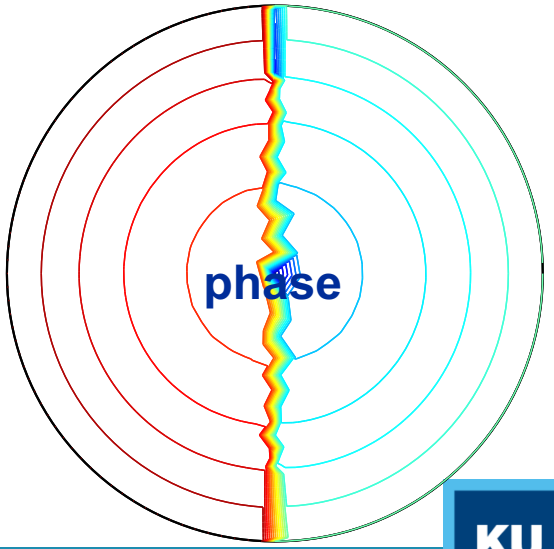






non-physical plots

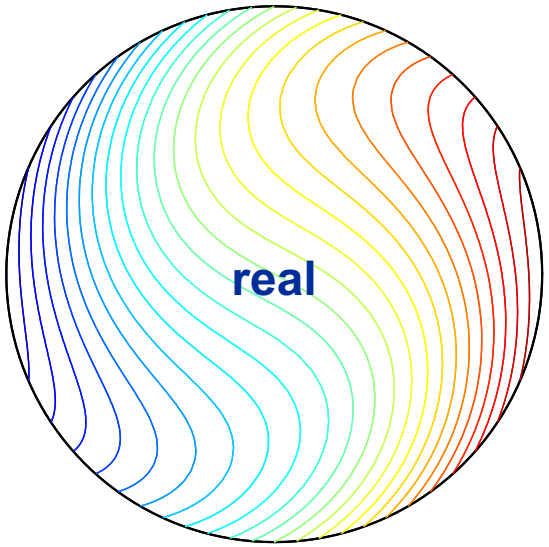


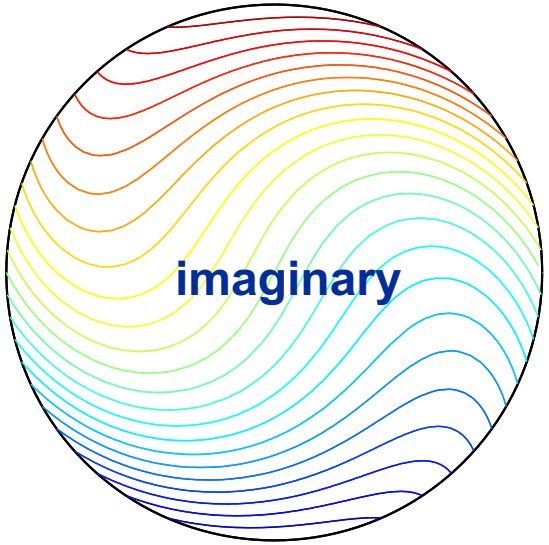


Rotating field

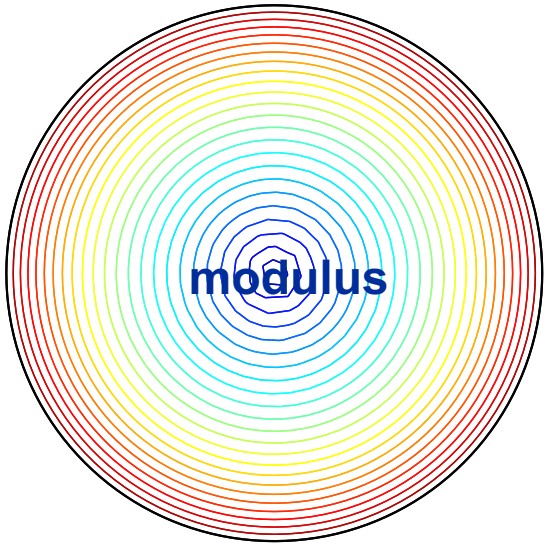
$$\boldsymbol{a}_z(R,\theta) = \alpha e^{-\imath\theta} \qquad a_z(R,\theta,t) = \alpha\cos(\omega t - \theta)$$

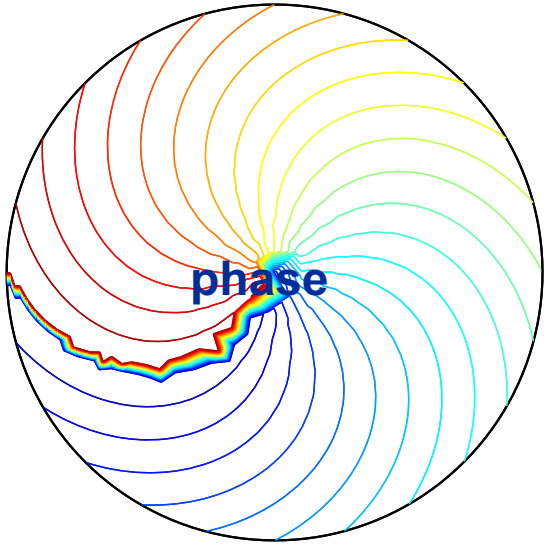
$$\boldsymbol{a}_z(x,y) = \alpha\left(\cos(\theta) - \imath\sin(\theta)\right) = \alpha\left(\frac{x}{R} - \imath\frac{y}{R}\right) \qquad \boldsymbol{b} = -\imath\frac{\alpha}{R}\hat{x} + \frac{\alpha}{R}\hat{y}$$

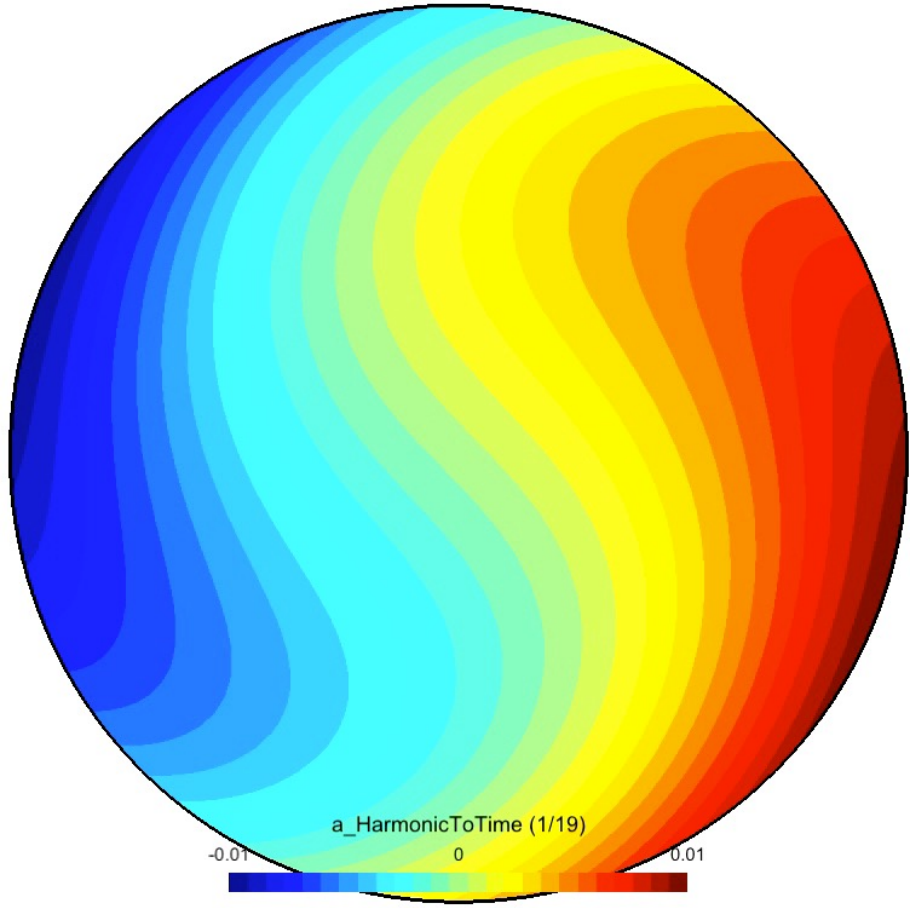




non-physical plots

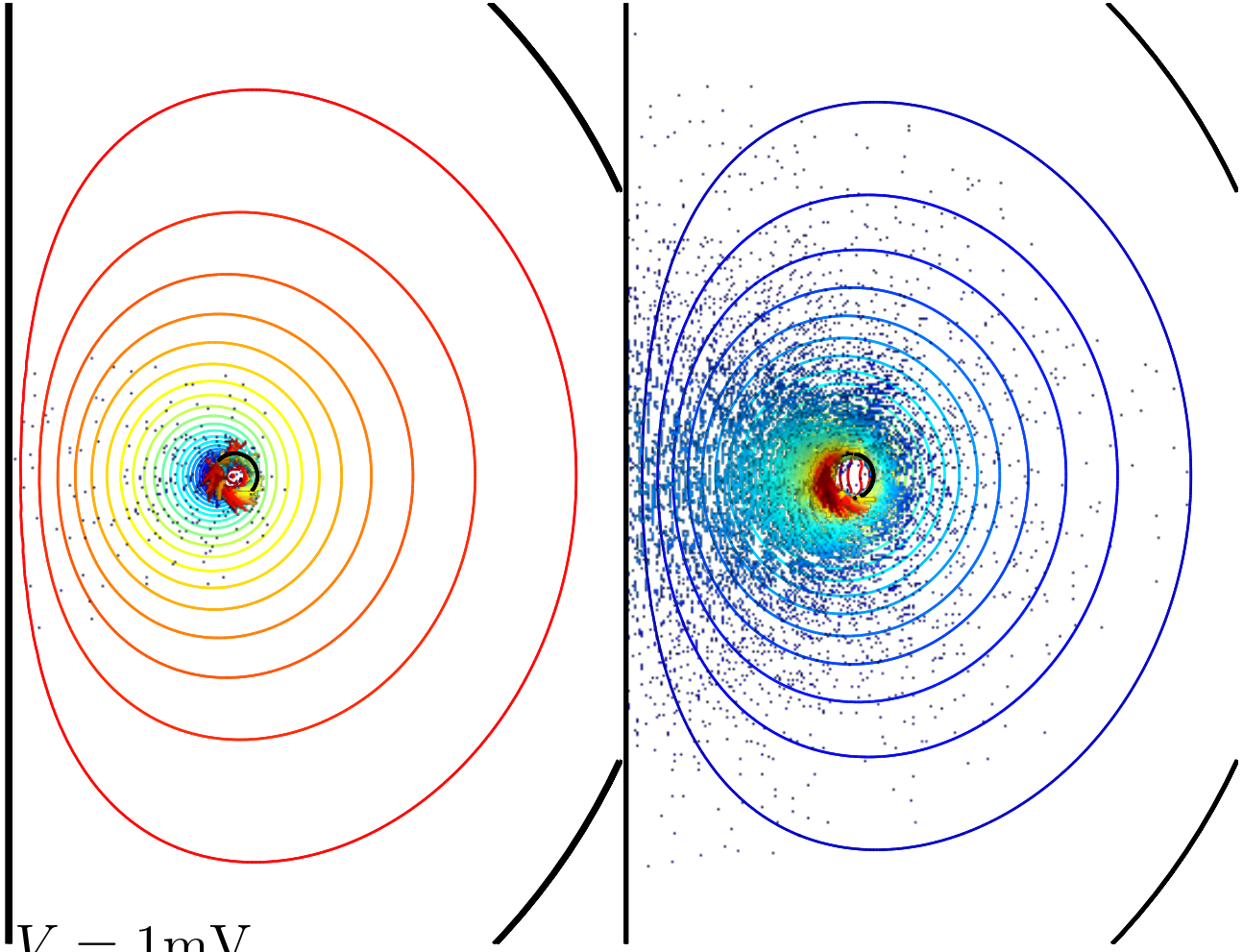






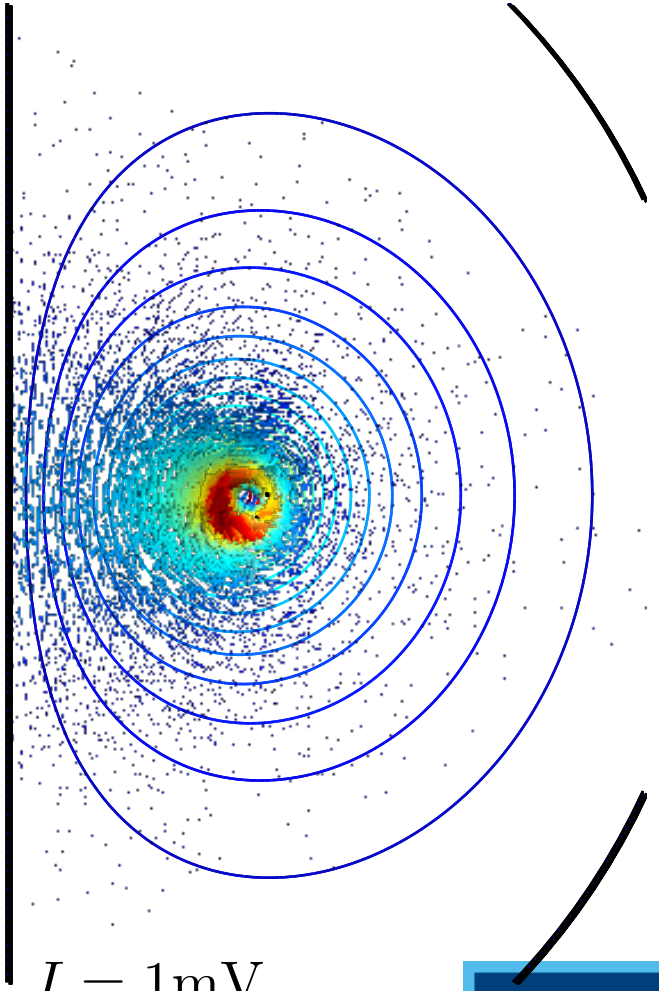
Coil: massive vs stranded conductor

Massive conductor



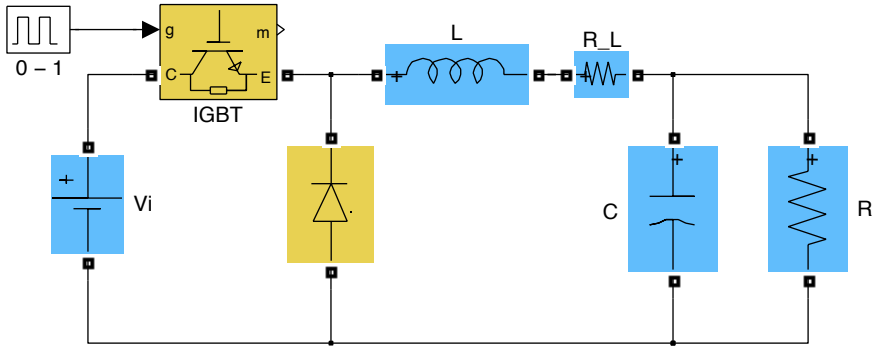
$V = 1\mathrm{mV}$

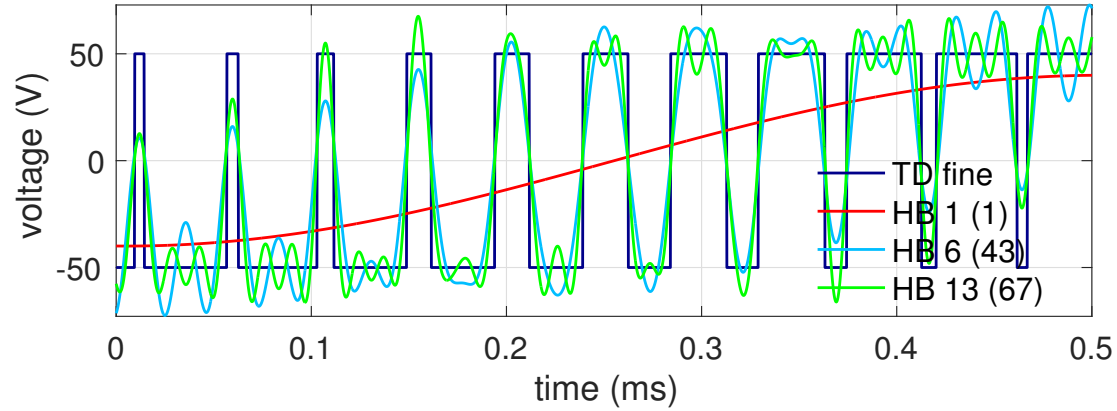
Stranded conductor

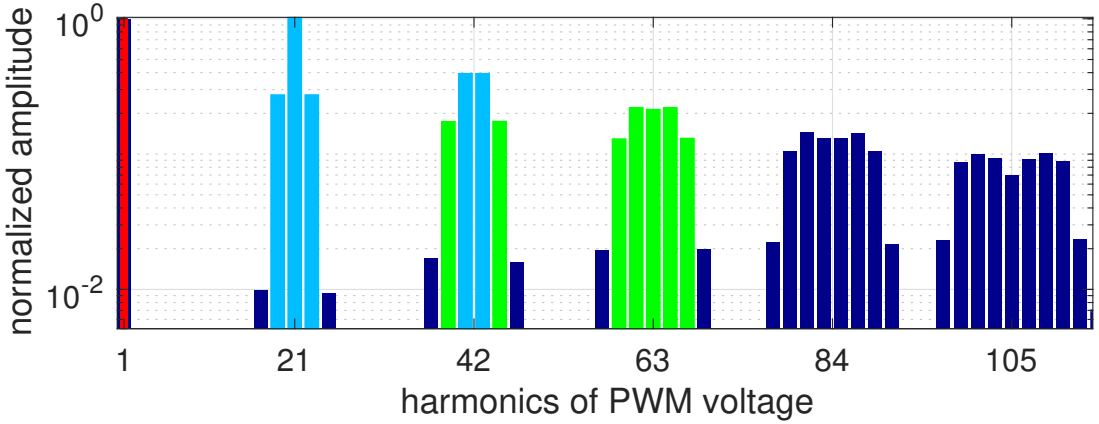


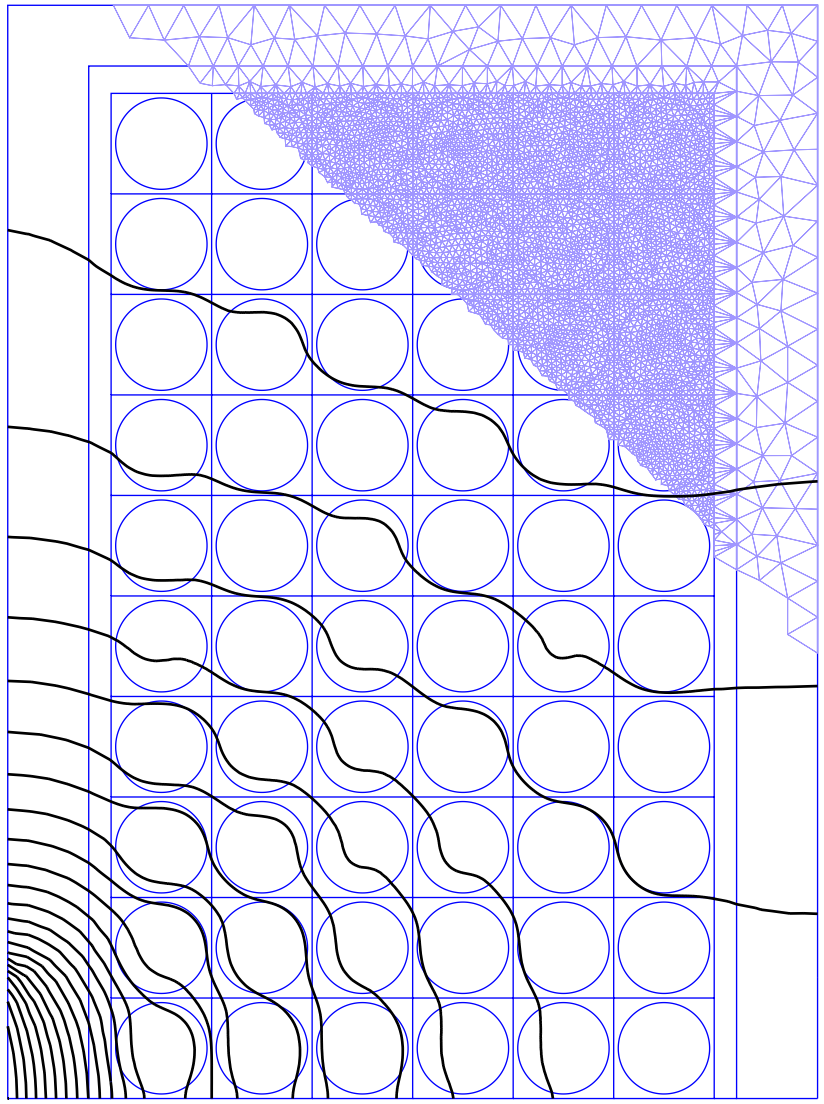
$I = 1\mathrm{mV}$

PWM


0 – 1
g
m
C
E
IGBT
L
R_L
Vi
C
R


voltage (V)
50
0
-50
0
0.1
0.2
0.3
0.4
0.5
time (ms)
TD fine
HB 1 (1)
HB 6 (43)
HB 13 (67)


normalized amplitude
10^0
10^{-2}
1
21
42
63
84
105
harmonics of PWM voltage

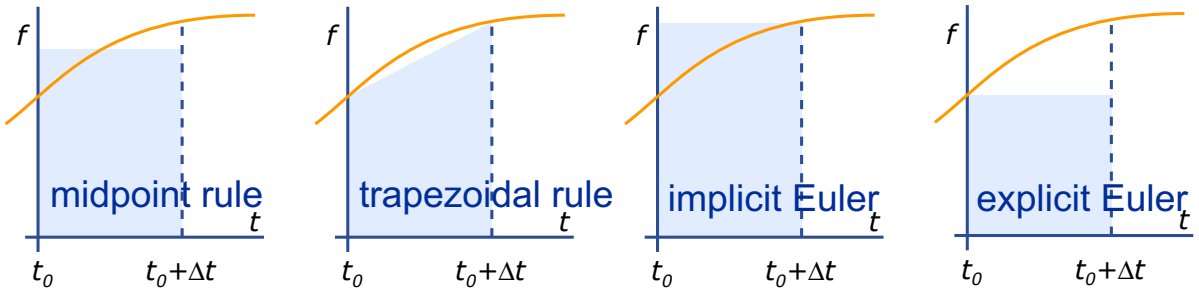


Time integration for spatially discretized PDEs

- Discretizing the spatial derivatives in PDEs, we get a (coupled) system of (nonlinear) ODEs. Methods to integrate ODEs can then be directly applied for the time integration of spatially discretized PDEs

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = f(t,\phi) \quad \text{with } \phi(t_0) = \phi_0 \iff \int_{t_n}^{t_{n+1}} \frac{\mathrm{d}\phi}{\mathrm{d}t}\mathrm{d}t = \phi_{n+1} - \phi_n = \int_{t_n}^{t_{n+1}} f(t,\phi)\mathrm{d}t$$

- Four simple schemes to estimate the time integral by approximate quadrature
  - Explicit or forward Euler $\phi_{n+1} - \phi_n = f(t_n, \phi_n)\Delta t$
  - Implicit or backward Euler $\phi_{n+1} - \phi_n = f(t_{n+1}, \phi_{n+1})\Delta t$
  - Midpoint rule (basis leapfrog) $\phi_{n+1} - \phi_n = f(t_{n+1/2}, \phi_{n+1/2})\Delta t$
  - Trapezoid rule (basis Crank-Nicholson) $\phi_{n+1} - \phi_n = \frac{1}{2}\left[f(t_n, \phi_n) + f(t_{n+1}, \phi_{n+1})\right]\Delta t$



Theta methods

- We consider methods of the form,

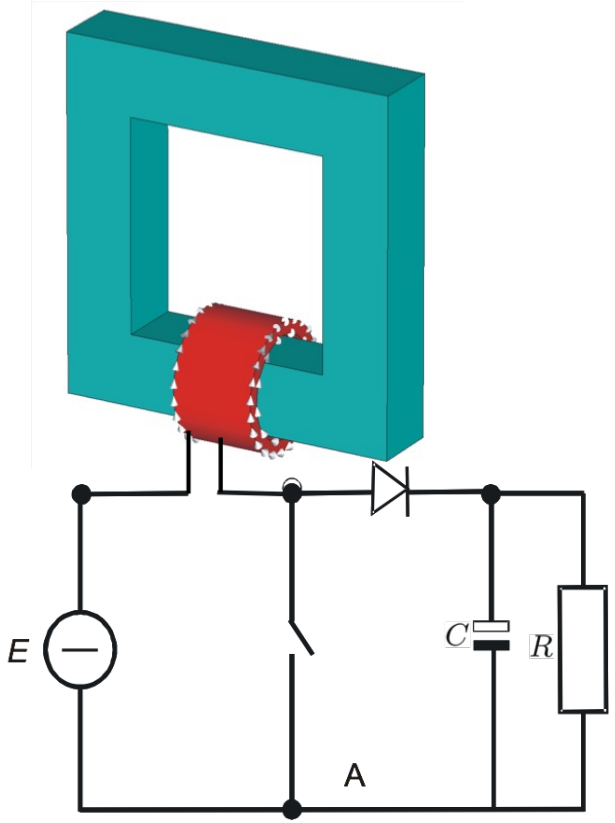
$$\phi_{n+1} = \phi_n + \Delta t\left[\theta f(t_{n+1}, \phi_{n+1}) + (1-\theta) f(t_n, \phi_n)\right], \quad n = 0, 1, \ldots$$

  with parameter $\theta \in [0, 1]$
- If $\theta = 0$, we recover explicit or forward Euler method
- If $\theta \in (0, 1]$, then the theta method is implicit: Each time step requires the solution of N (nonlinear) algebraic equations for the unknown
- If $\theta = 1$, we recover the implicit or backward Euler method (damps quickly numerical oscillations)
- If $\theta = 0.5$, we have the trapezoidal rule (Crack-Nicholson). Theoretically optimal for linear problems, but it may oscillate
- If $\theta = 2/3$, we have the so-called Galerkin choice, derived from a weighted residual approach instead of finite differences
- Solution of nonlinear algebraic equations can be done by iteration

Runge-Kutta methods and Multistep/multipoint methods

- Higher accuracy in time by using available information (known values of the derivative in time, i.e. the RHS) at more time steps. Two main approaches:
  - Runge-Kutta Methods
    - Additional points between t_n and t_{n+1}, for computational convenience
    - Difficulty: n^{th} order RK requires n evaluations of the first derivative (RHS of PDE) => more expensive
  - Multistep/Multipoint Methods
    - Additional points are at past time steps at which data has already been computed, hence for comparable order, less expensive than RK methods
    - These methods are difficulty to start
- For a given order, RK methods are more accurate and stable than multipoint methods

Adaptive time step
Step-up converter



$$\text{error} = \phi_{n+1} - \tilde{\phi}_{n+1}$$

γ acceleration factor (e.g. 1.2)
ϵ_{tol} prescribed tolerance

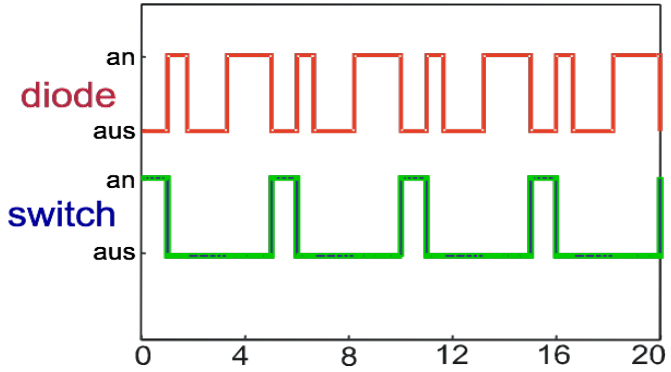
ξ safety factor (e.g. 0.9)
p order

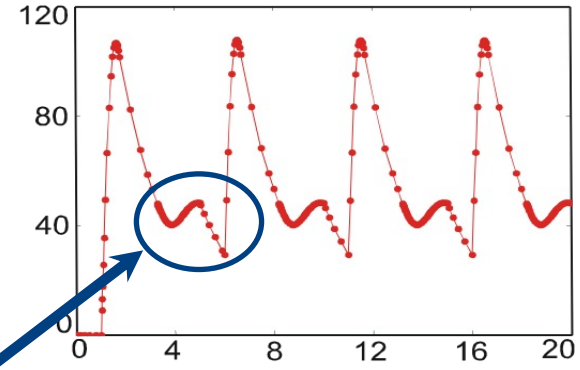
$$\| \text{error} \| > \gamma \epsilon_{tol}$$

yes → **reject time step**

no → **time step predictor**

$$\Delta t_{n+1} = \xi \left(\frac{\epsilon_{tol}}{\| \text{error} \|^p} \right)^{1/p}$$

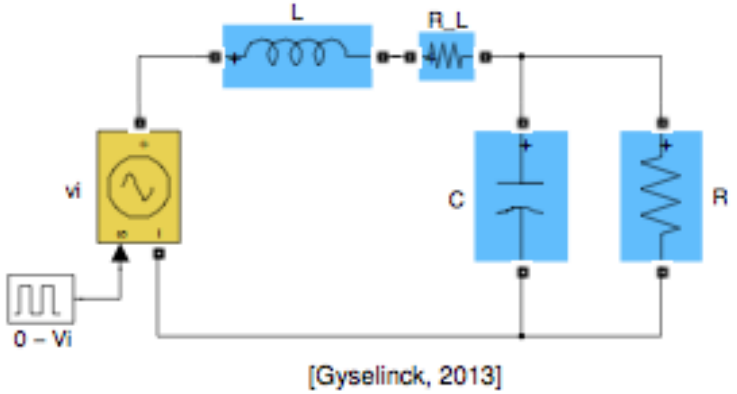




smaller time steps

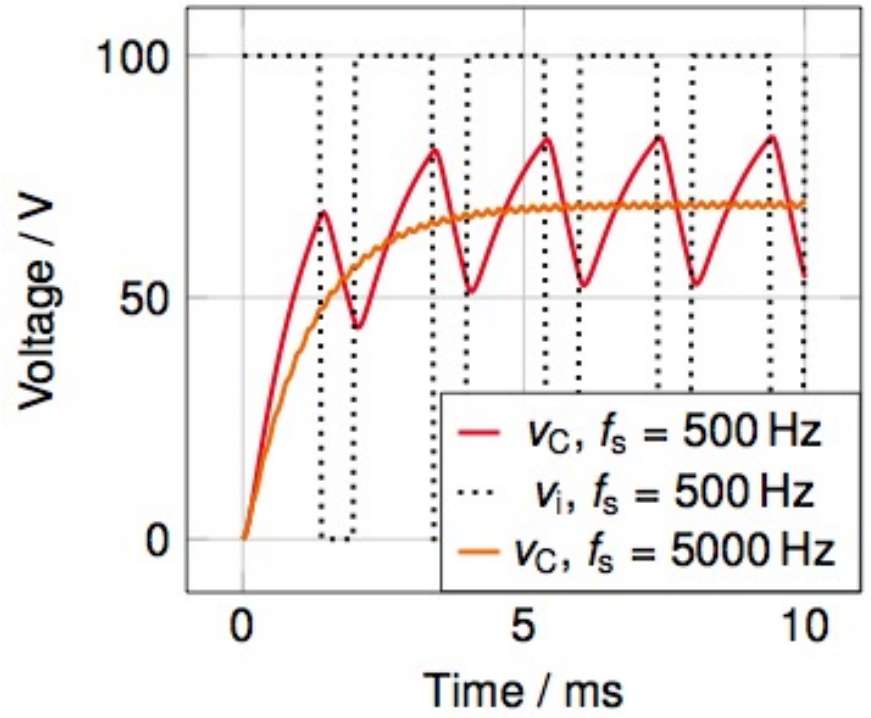
[Benderskaya, Clemens, De Gersem]

Multi-rate partial differential equations MPDEs



[Gyselinck, 2013]

$$\mathbf{A}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) + \mathbf{B}\,\mathbf{x}(t) = \mathbf{i}_s(t)$$



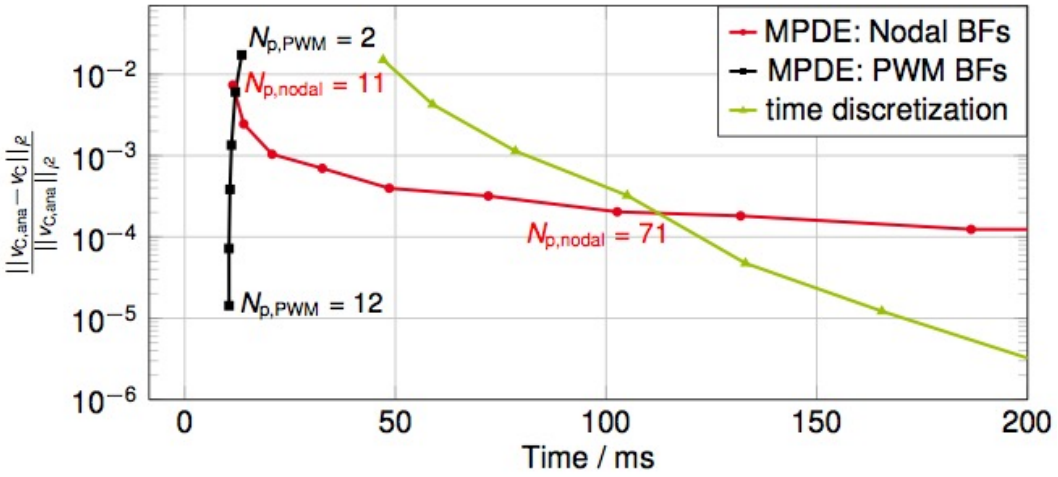
- DC to DC converter
- Output voltage controlled by duty cycle of PWM
- Voltage is stepped down while current is stepped up
- Higher frequency => smaller ripples

Disadvantages of classical time discretization:

- fastest change in signal has to be resolved
- the higher the frequency, the slower the simulation

Splitting solution in fast & slowly varying parts
⇒ Multirate Partial Differential Equation (MPDE)
Using a-priori knowledge: periodic pulsed excitation

$$\mathbf{A}\left(\frac{\partial\hat{\mathbf{x}}(t_1,t_2)}{\partial t_1} + \frac{\partial\hat{\mathbf{x}}(t_1,t_2)}{\partial t_2}\right) + \mathbf{B}\,\hat{\mathbf{x}}(t_1,t_2) = \hat{\mathbf{i}}_s(t_1,t_2)$$



Types of errors

Accuracy improvement

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Types of errors

- Errors in the formulation of the problem to be solved
  - Simplifications/assumptions of the mathematical **model**
- Errors in the **input data**, given e.g. by measurements, lack of knowledge of material parameters (no provided by manufacturer)...
- Approximation errors
  - **Discretization** errors
  - **Convergence** error (iterations, nonlinear iterations)
  - Discretization/convergence errors may be assessed by an analysis of the method used
- Round-off errors appear everywhere in a numerical computation due to the **finite precision arithmetic**. This kind of errors are quite erratic

Error estimation

- Error is problem (field/formulation) dependent. It can be estimated
  - *A priori*: function of the finite element formulation and type, associated functional spaces, mesh quality (geometric)
  - *A posteriori*: once the solution is available, it may involve some post-processing
- Error estimator may differ locally
  - e.g. (non-)conducting region
  - e.g. (non-)magnetic region
- Adequate scaling is necessary to be able to compare (relative error preferred)
- Examples of error estimators:

  local field values, local field values weighted with energy, energy densities, losses, continuity conditions, …
- Error propagation is an issue in computations with several stages

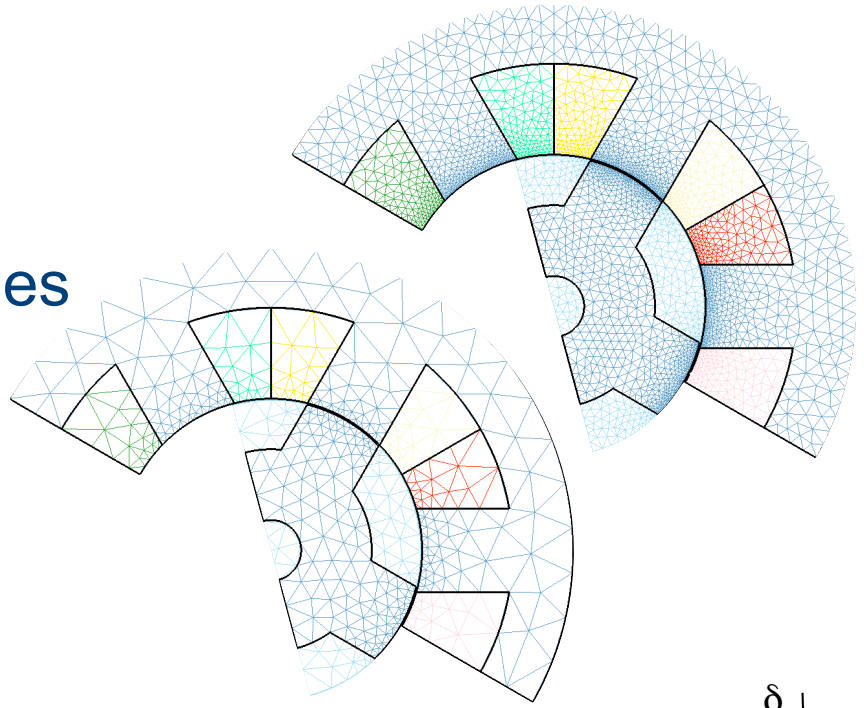
Accuracy enhancement

When error in certain region is too large, we may improve accuracy by

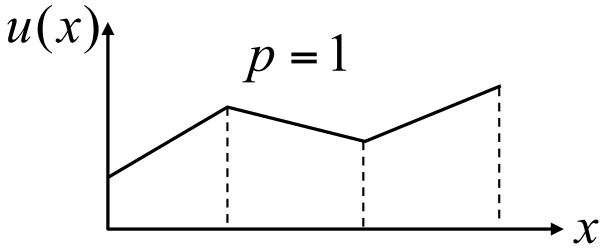
- h-adaptation: local mesh refinement around critical areas (element size h↓)
- p-adaptation: higher-order finite elements (polynomial degree p↑)
  - reformulate problem with alternative element type
  - use of “hierarchical elements” $\epsilon \approx O\left(h^{p+1}\right)$
- hp-adaptation/hp-FEM: combination elements of variable size (h) and polynomial degree (p)
- The finite element method converges exponentially fast when the mesh is refined using a suitable combination h-refinements (dividing elements into smaller ones) and p-refinements (increasing their polynomial degree)

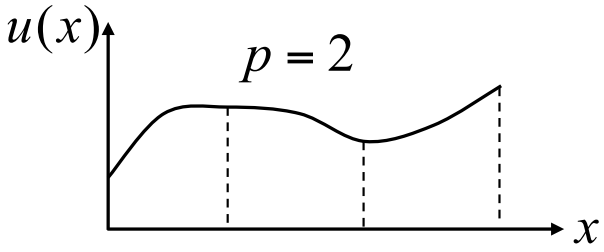
Adaptation/refinement examples

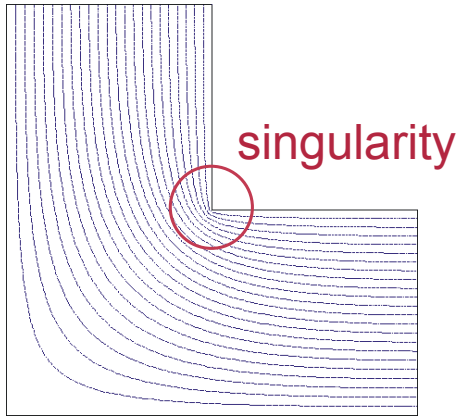
- h-adaptation == mesh refinement
  - good for singularities & material inhomogeneities
  - bad for boundary layers

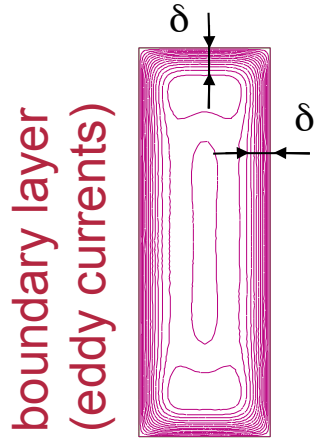


- p-adaptation == higher order elements
  - bad for singularities & material inhomogeneities
  - good for boundary layers and shocks









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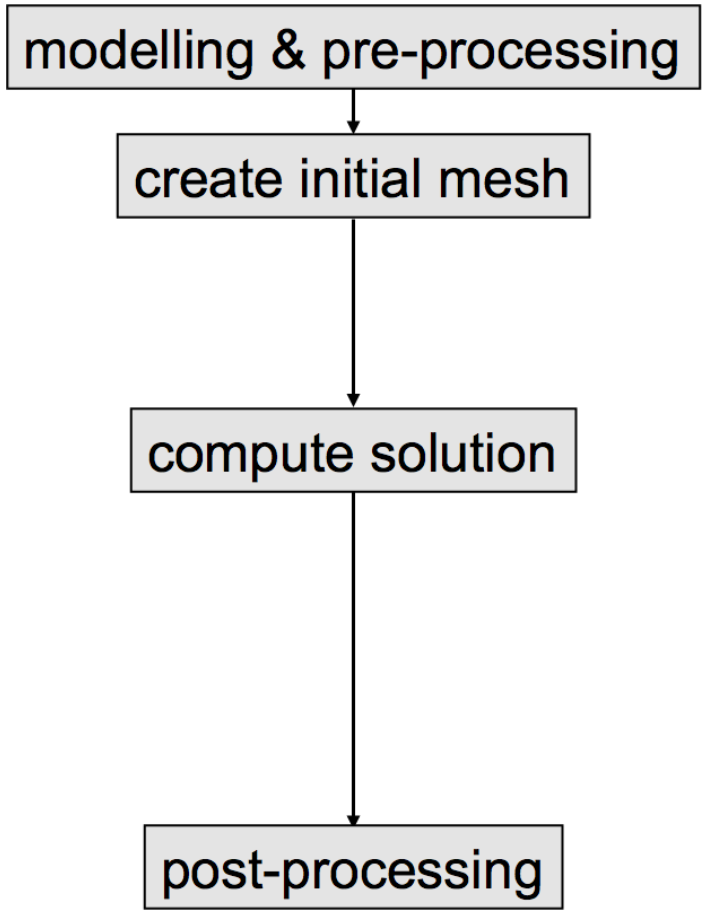
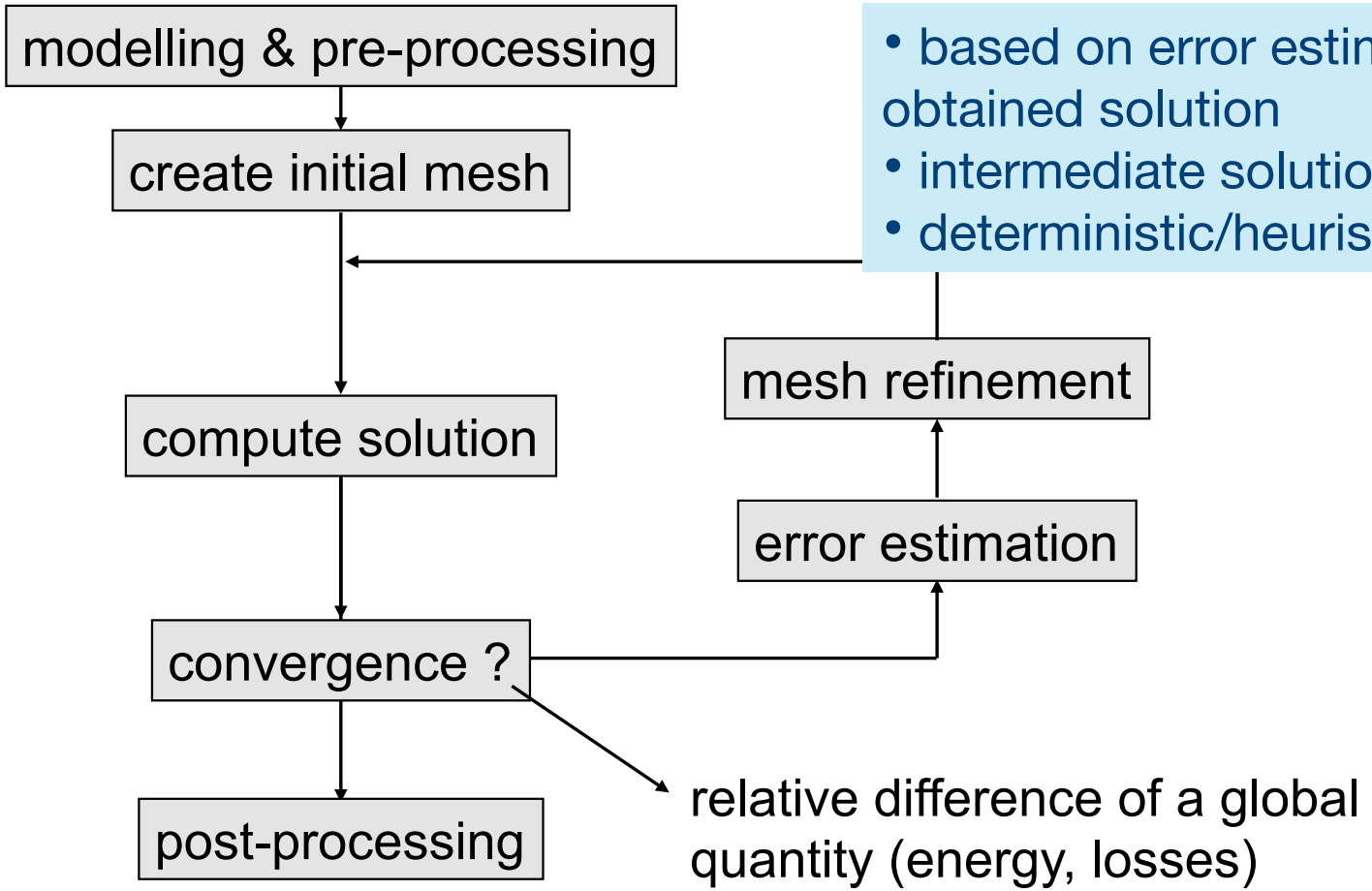
A priori mesh refinement



- based on a prediction of the behavior of the PDE
- material distribution
- sources
- boundary conditions

A posteriori mesh refinement



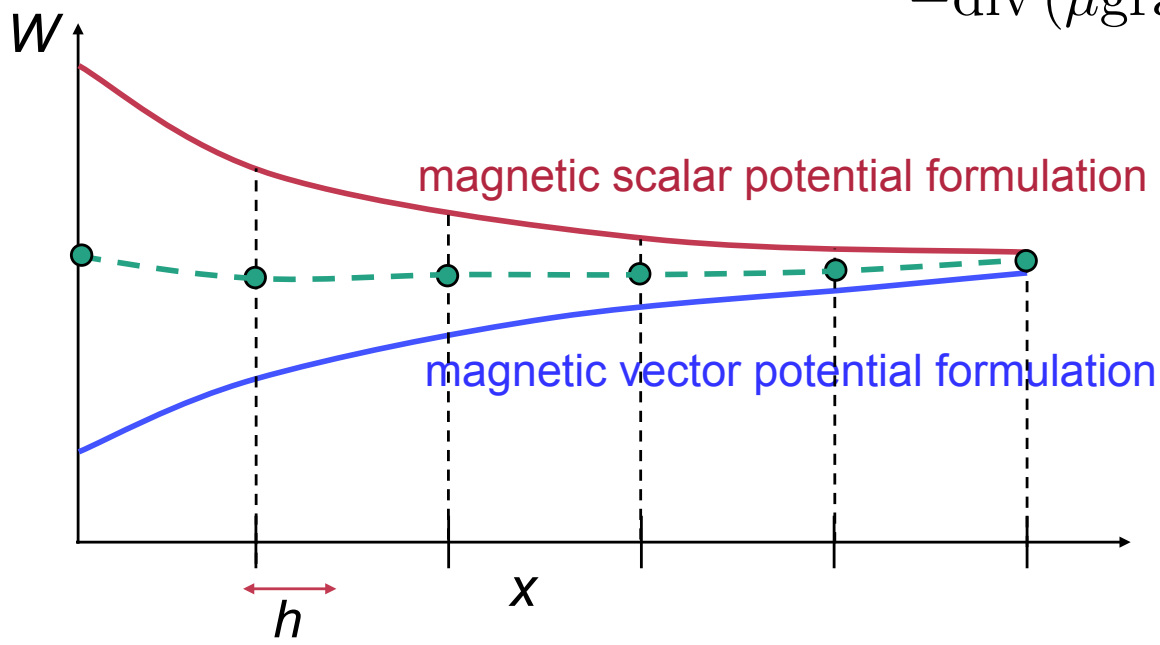
- based on error estimation on a previously obtained solution
- intermediate solutions
- deterministic/heuristic error estimators

Dual formulations

e.g. magnetostatic problem

$$-\mathrm{div}\,(\mu \mathrm{grad}\,\varphi) = -\mathrm{div}\,(\mu \boldsymbol{h}_s)$$

$$\mathrm{curl}\,\boldsymbol{h}_s = \boldsymbol{j}_s$$



$$\mathrm{curl}\,\nu\mathrm{curl}\,\boldsymbol{a} = \boldsymbol{j}_s$$

- two FE solutions per step -> expensive
- comparison of the difference in each element
- reliable error estimator
- “superconvergent” solution

$$\| \mu \boldsymbol{h}_s - \mu \mathrm{grad}\,\varphi + \mathrm{curl}\,\boldsymbol{a} \|_{\Omega_e}$$

Dual formulations

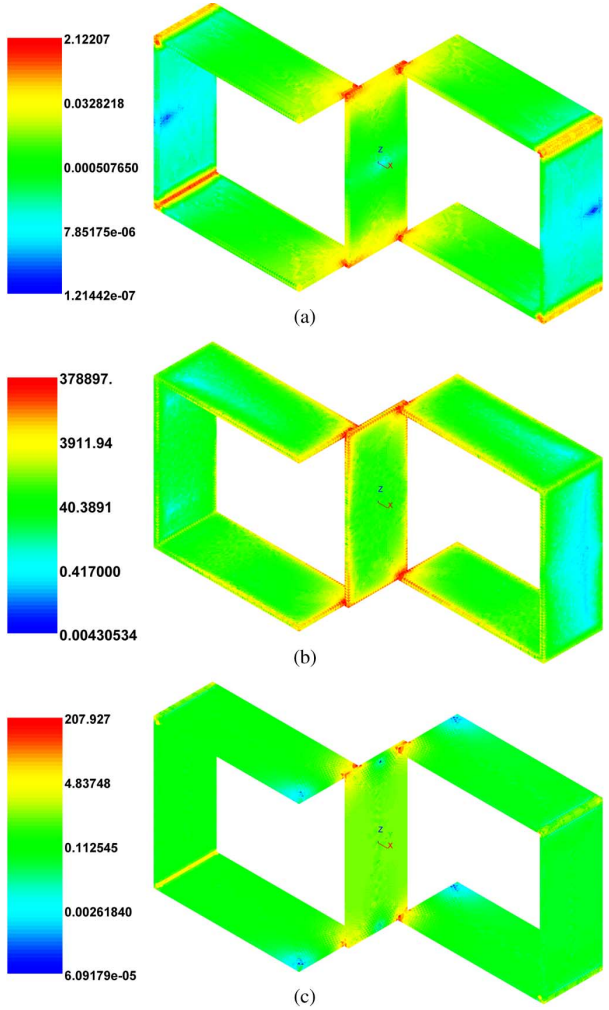


Fig. 7. Local estimator maps in iron plate for problem Team Workshop 13. (a) η_Ω; (b) $\eta_{\mathbf{A}}$; (c) $\eta_{\text{equilibrated}}$.

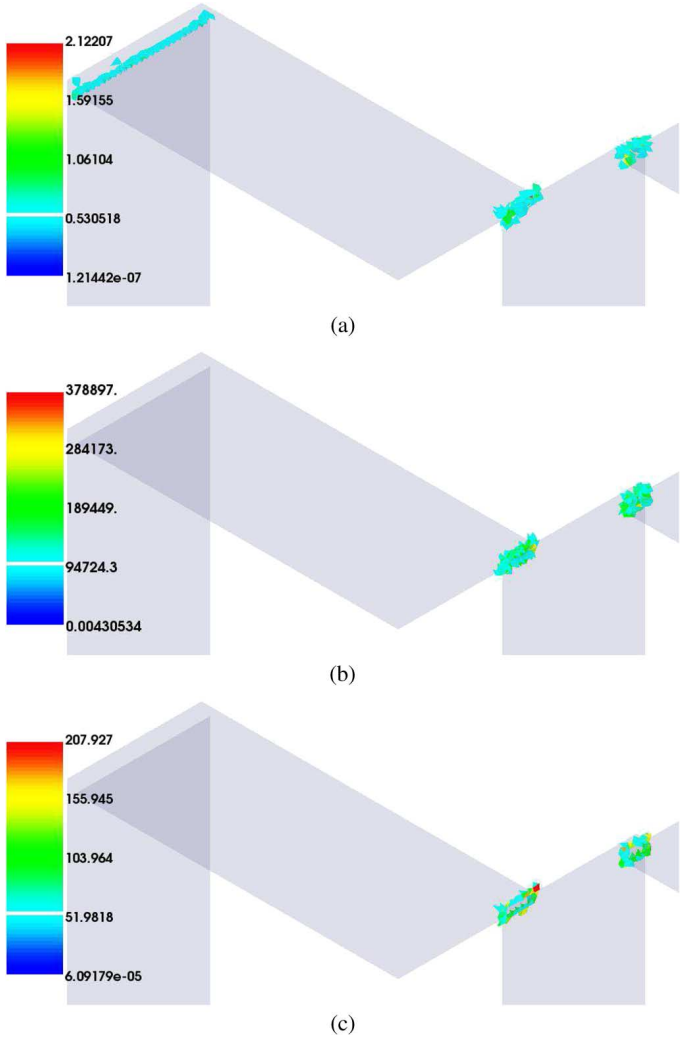


Fig. 9. Zoom out of the estimator map. (a) η_Ω; (b) $\eta_{\mathbf{A}}$; (c) $\eta_{\text{equilibrated}}$.

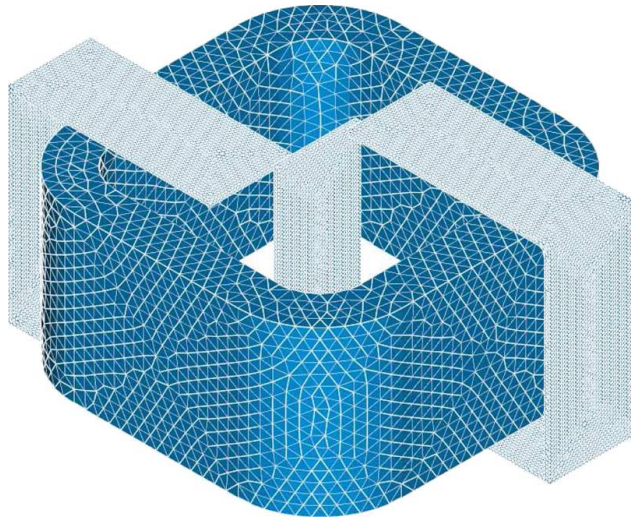


Fig. 5. Mesh of the problem Team Workshop 13.

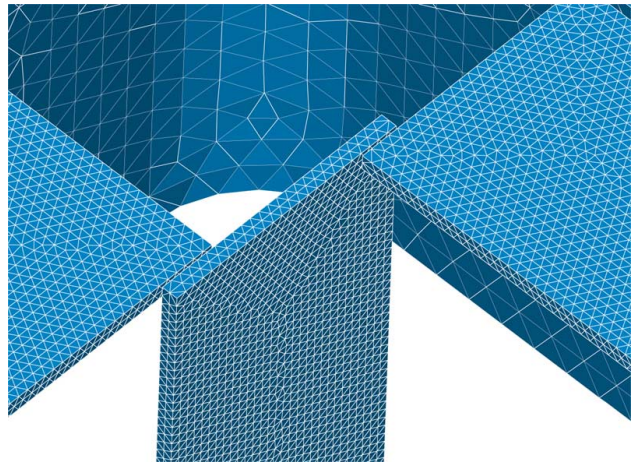
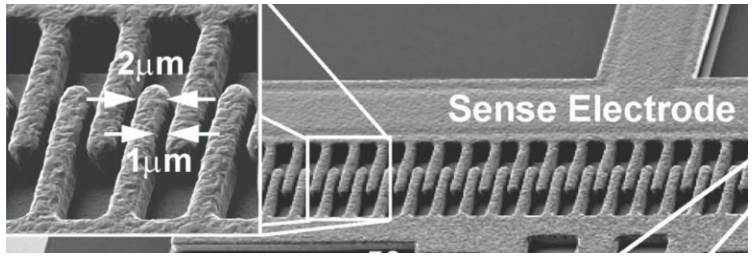


Fig. 6. Zoom of mesh of the problem Team Workshop 13.

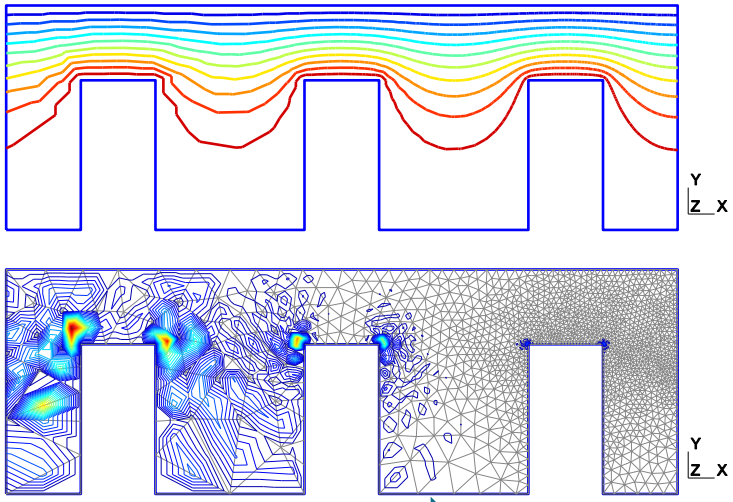
Hierarchical (or embedded) solution

- obtain solution at lower order
- compare this solution to local higher order solutions (e.g. error)
- reliable
- cheap
- does not detect singularities

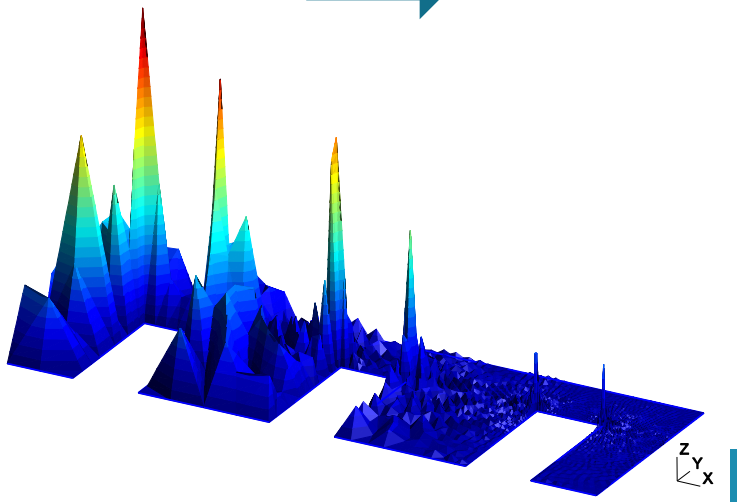
A posteriori-error estimation
Electrokinetic/electrostatic problem



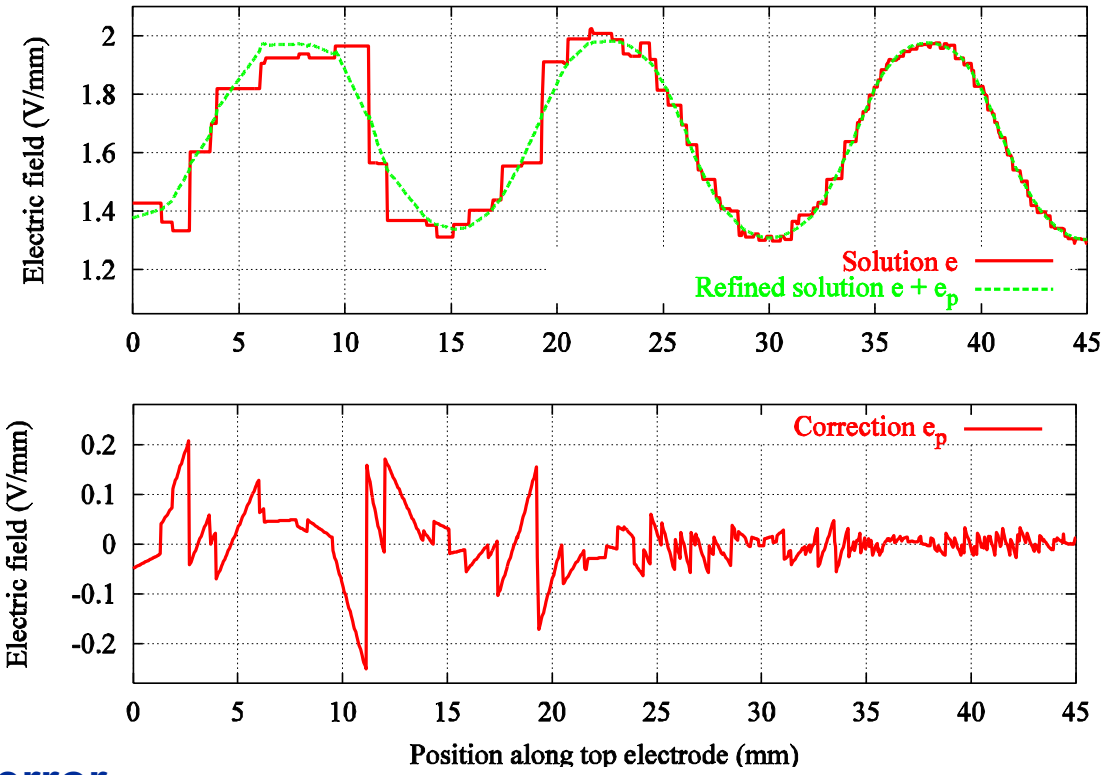
electric scalar potential



coarse mesh → fine mesh



electric field along top electrode



large error

large local correction

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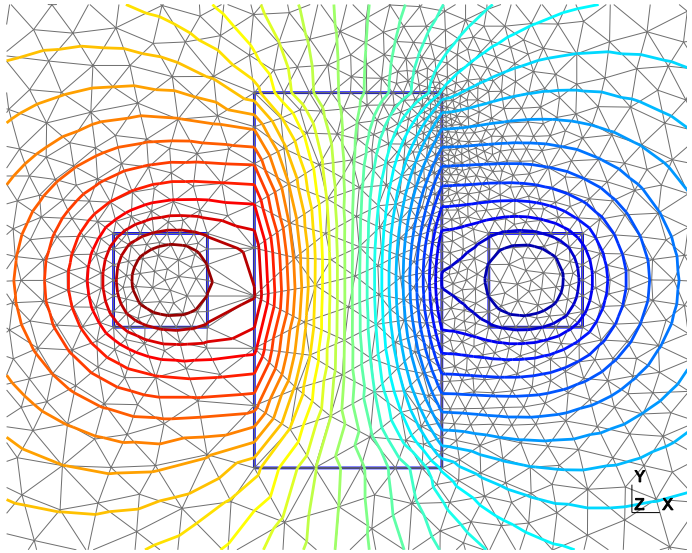
A posteriori error estimation

Magnetostatics

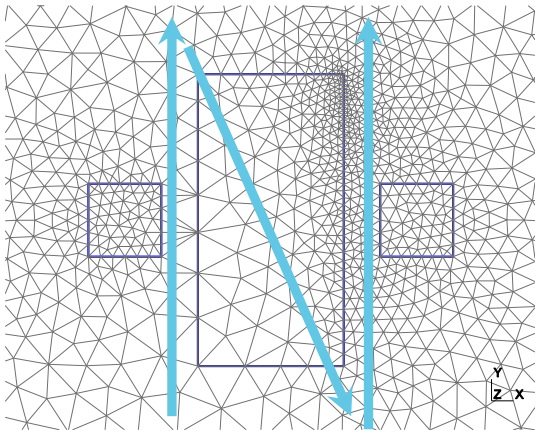
Magnetodynamics

magnetic core

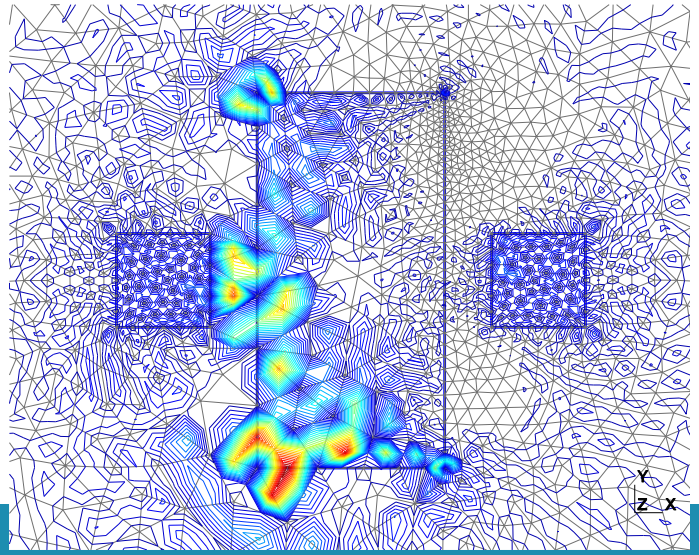


magnetic vector potential

fine mesh

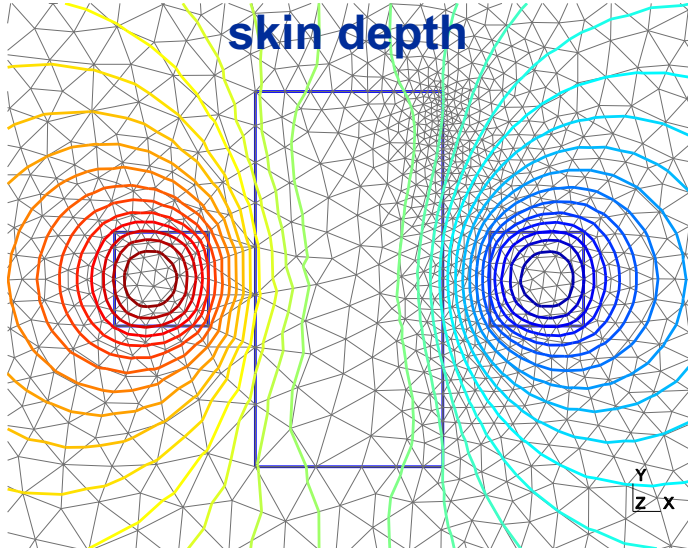


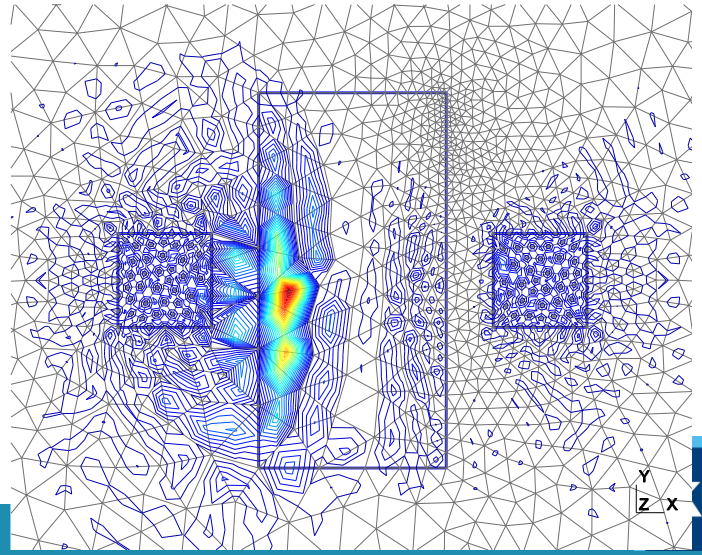
coarse mesh



large error

large local correction





conducting core

Heuristic error estimation

- mark element for refinement
  when **energy, losses, saturation** beyond threshold
- mark element for refinement
  when difference of **flux, voltage, induced current** with neighbouring elements beyond threshold
- mark element for refinement
  when **size, aspect ratio** beyond threshold
- restrict number of new elements to be created in every mesh adaptation step
- not always reliable (skilled simulation engineers)
- very cheap
- allows to manually guide mesh adaptation

Some error estimators

Region	Estimator	Discussion
ferromagnetic region (core)	$\lvert\Delta \boldsymbol{b}\rvert, \lvert\Delta \boldsymbol{b}\rvert^2, \frac{\lvert\Delta \boldsymbol{b}\rvert^2}{W}$	flux density changes (and iron losses); square form emphasizes iron losses (& smoothes ***b***)
solid conductor	$\lvert\Delta \boldsymbol{j}\rvert, \lvert\Delta \boldsymbol{j}\rvert^2, \frac{\lvert\Delta \boldsymbol{j}\rvert^2}{\sigma(T)}$	current density changes (and Joule losses), eddy currents in conducting regions->skin effect
stranded conductor	$\lvert\Delta \boldsymbol{b}\rvert, \frac{\lvert\Delta \boldsymbol{j}\rvert^2}{\sigma(T)}$	induction variation<->leakage field; stranded inductors with constant current but σ still depends on T
permanent magnet	$\lvert\Delta \boldsymbol{b}\rvert$	induction changes<->irreversible demagnetization; measure of eddy current loss in conducting magnets (& smoothes ***b***)
surrounding air, air gaps	$\lvert\Delta \boldsymbol{b}\rvert$	leakage flux, force and torque computation

Solving nonlinear problems

Successive substitution (also called Picard method)
Newton method (also called Newton-Raphson method)
Magneto-static application example

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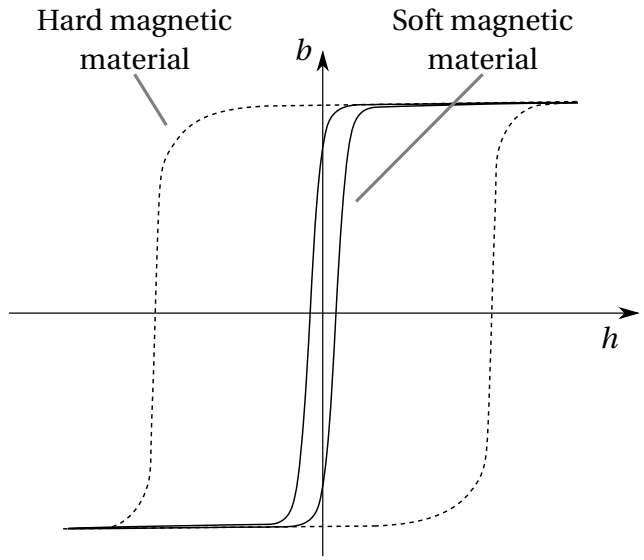
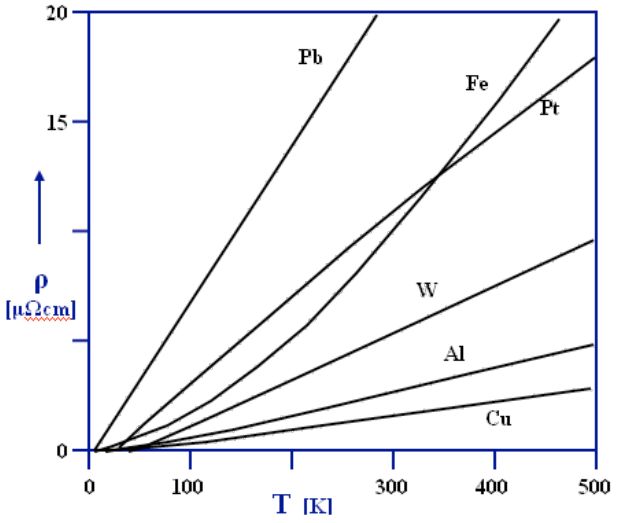
When do we have nonlinearities?

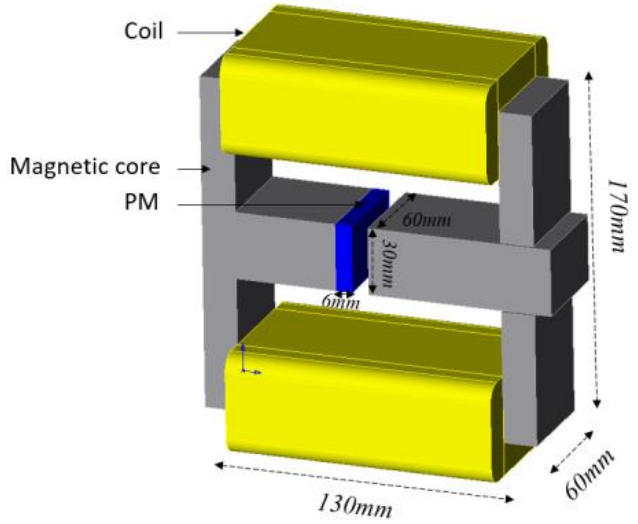
- Nonlinear **material** behavior
  - purely electromagnetic, i.e. nonlinear constitutive law
    - e.g. ferromagnetic saturation, hysteresis
  - depending on other physics
    - e.g. conductivity/permeability as a function of temperature
- Nonlinear **boundary conditions**
  - e.g. radiation conditions
- Nonlinear **terms** in multi-physical coupled models
  - e.g. electromagnetic forces in an electro/magneto-mechanical m
  - e.g. joule losses in an electro-thermal model
- Optimization problems

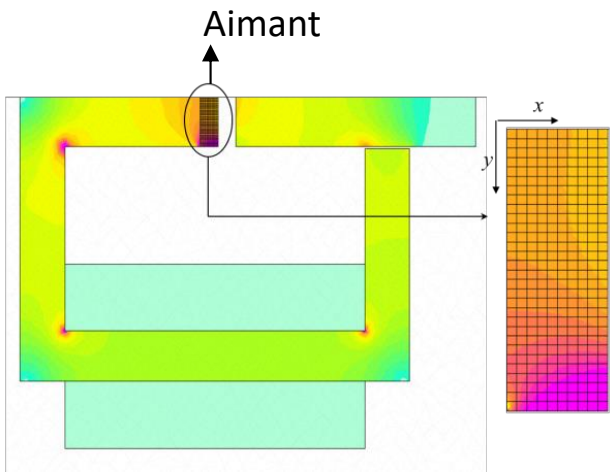


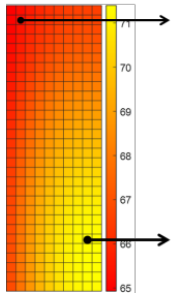
$$\rho = \rho_0(1 + \alpha(T - T_0))$$

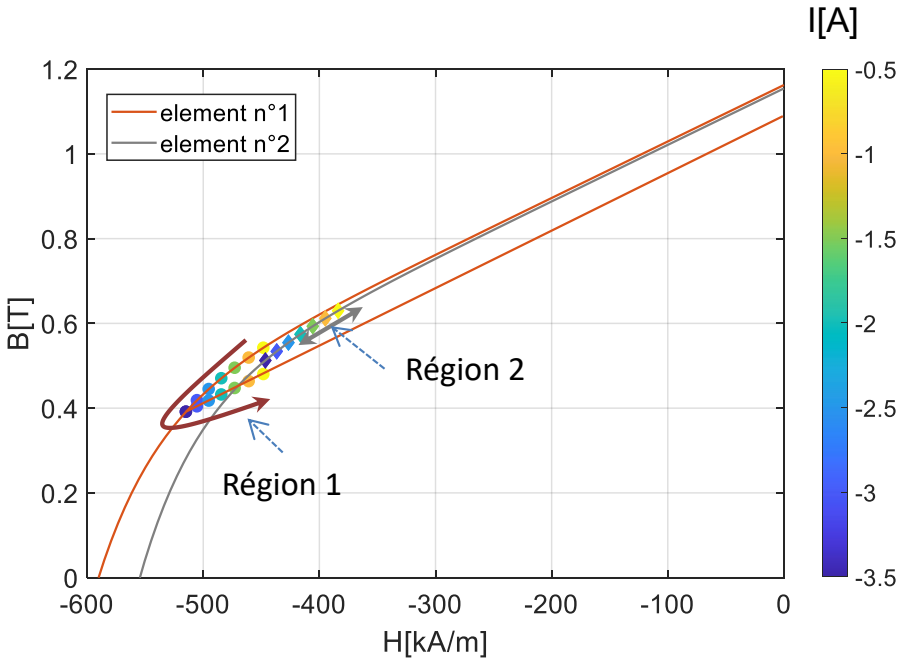


Magneto-thermal problem => Demagnetisation









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Magnetostatics

Distribution of static magnetic field due to magnets & continuous sources (DC)

"$\boldsymbol{h}$ side"

Ampère's law verified in a **strong** sense

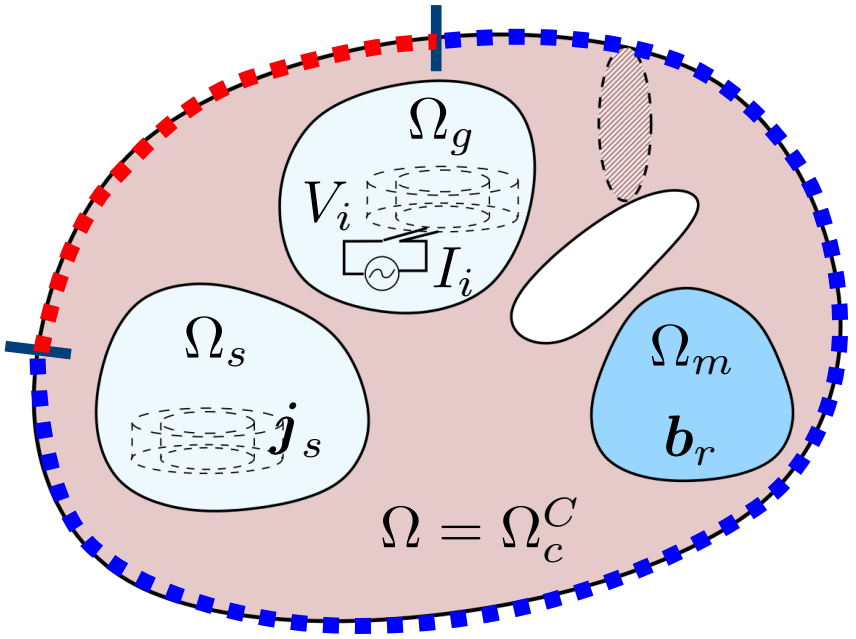
φ formulation

$\boldsymbol{h}_s$ given so that

$$\operatorname{curl} \boldsymbol{h}_s = \boldsymbol{j}_s$$

$$\operatorname{div}\left(\mu(\boldsymbol{h}_s - \operatorname{grad}\varphi)\right) = 0$$

$$\boldsymbol{h} = \boldsymbol{h}_s - \operatorname{grad}\varphi$$



$$\operatorname{curl} \boldsymbol{h} = \boldsymbol{j}_s \qquad \boldsymbol{b} = \mu\,\boldsymbol{h}(+\boldsymbol{b}_r)$$
$$\operatorname{div} \boldsymbol{b} = 0$$

"$\boldsymbol{b}$ side"

Faraday's law verified in a **strong** sense

$\boldsymbol{a}$ formulation

$\boldsymbol{j}_s$ given

$$\operatorname{curl} \nu \operatorname{curl} \boldsymbol{a} = \boldsymbol{j}_s$$

$$\boldsymbol{b} = \operatorname{curl} \boldsymbol{a}$$

Successive substitution (or fixed-point method)

$$\begin{aligned} \operatorname{curl} \nu(\boldsymbol{b}) \operatorname{curl} \boldsymbol{a} &= \boldsymbol{j}_s \\ \boldsymbol{b} &= \operatorname{curl} \boldsymbol{a} \end{aligned} \quad \Longleftrightarrow \quad G_i(\boldsymbol{a}) = 0\,, \ \forall i$$

initial value $\boldsymbol{a}_0$

for $n = 0, 1, \ldots$ till convergence

compute $\boldsymbol{b}_n = \operatorname{curl} \boldsymbol{a}_n$

evaluate $\nu(\boldsymbol{b}_n)$

solve $\operatorname{curl} (\nu(\operatorname{curl} \boldsymbol{a}_n) \operatorname{curl} \boldsymbol{a}_{n+1}) = \boldsymbol{j}_s$

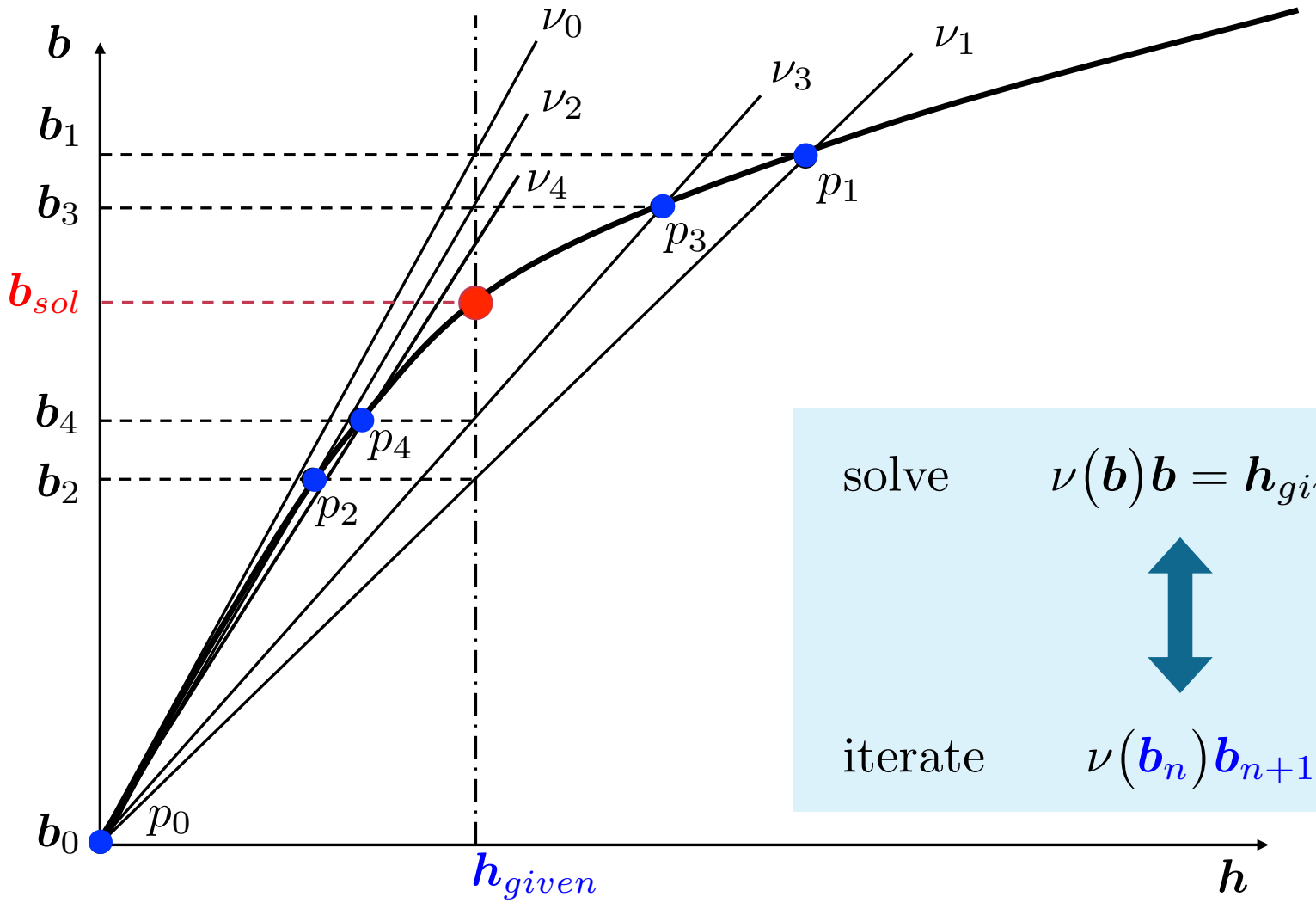
$\boldsymbol{a}_{n+1} = \lambda \boldsymbol{a}_{n+1} + (1 - \lambda) \boldsymbol{a}_n$

end

Fixed point methods

- reformulation of the problem
- linear problem at each iteration
- convergence condition
- if λ=1 Picard iteration
- linear convergence

Successive substitution (cont'd)



solve $\quad \nu(\boldsymbol{b})\boldsymbol{b} = \boldsymbol{h}_{given}$

$\Updownarrow$

iterate $\quad \nu(\boldsymbol{b}_n)\boldsymbol{b}_{n+1} = \boldsymbol{h}_{given}$

Newton method

$$G_i(\boldsymbol{a}) = 0\,,\ \forall i$$

Taylor series

$$G_i(\boldsymbol{a}^n) + \sum_j \frac{\partial G_i}{\partial a_j}(\boldsymbol{a}^n)\Delta a_j^{n+1} + \cdots = 0\,,\ \forall i$$

at each iteration
compute the correction

$$\left[\frac{\partial G_i}{\partial a_j}\right]_{\boldsymbol{a}=\boldsymbol{a}^n} \left[\Delta a_j^{n+1}\right] = -\left[G_i\right]_{\boldsymbol{a}=\boldsymbol{a}^n}$$

Newton-Raphson methods

- reformulation of the problem
- Taylor series
- Jacobian = first derivative
- solve for the correction
- quadratic convergence

$$\boldsymbol{J}(\boldsymbol{a}) = \frac{\partial \boldsymbol{G}}{\partial \boldsymbol{a}}(\boldsymbol{a}) = \left[\frac{\partial G_i}{\partial a_j}(\boldsymbol{a})\right]_{\boldsymbol{a}=\boldsymbol{a}^n}$$

Jacobian matrix

Newton method (cont'd)

solve $\quad \nu(\boldsymbol{b})\,\boldsymbol{b} = \boldsymbol{h}_{given}$

$\Updownarrow$

solve $\quad G(\boldsymbol{b}) = \nu(\boldsymbol{b})\,\boldsymbol{b} - \boldsymbol{h}_{given} = 0$

$\Updownarrow$

iterate $\quad \left.\frac{\mathrm{d}\boldsymbol{G}}{\mathrm{d}\boldsymbol{b}}\right|_{\boldsymbol{b}=\boldsymbol{b}_n} \Delta\boldsymbol{b}_{n+1} = \boldsymbol{G}\Big|_{\boldsymbol{b}=\boldsymbol{b}_n}$

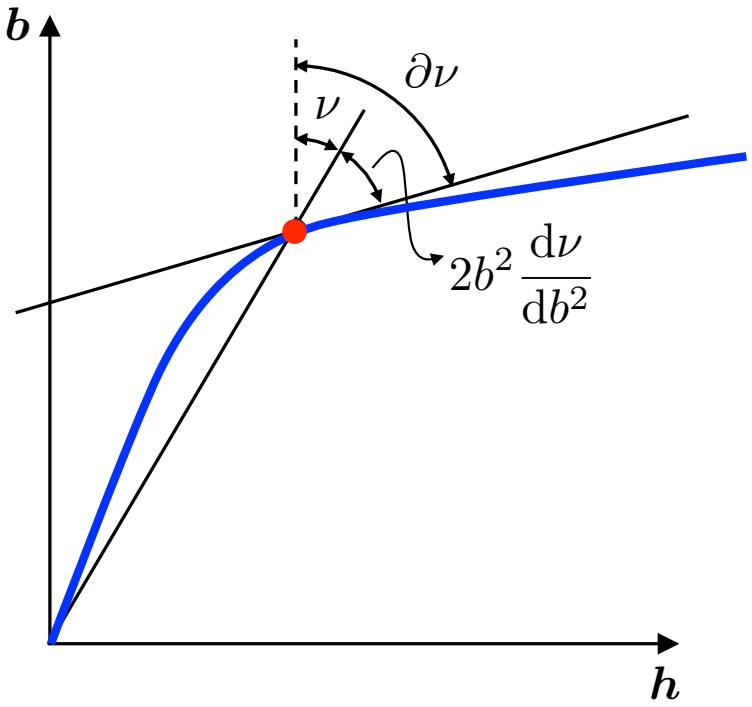
$$\left.\frac{\mathrm{d}\boldsymbol{G}}{\mathrm{d}\boldsymbol{b}}\right|_{\boldsymbol{b}=\boldsymbol{b}_n} = \frac{\mathrm{d}\nu}{\mathrm{d}\boldsymbol{b}}(\boldsymbol{b}_n)\boldsymbol{b}_n + \boxed{\nu(\boldsymbol{b}_n)}$$

reluctivity

$$= \frac{\mathrm{d}\nu}{\mathrm{d}\boldsymbol{b}^2}(\boldsymbol{b}_n)2\boldsymbol{b}_n^2 + \nu(\boldsymbol{b}_n)$$

$$= \frac{\mathrm{d}\boldsymbol{h}}{\mathrm{d}\boldsymbol{b}}(\boldsymbol{b}_n) = \boxed{\partial\nu(\boldsymbol{b}_n)}$$

differential reluctivity



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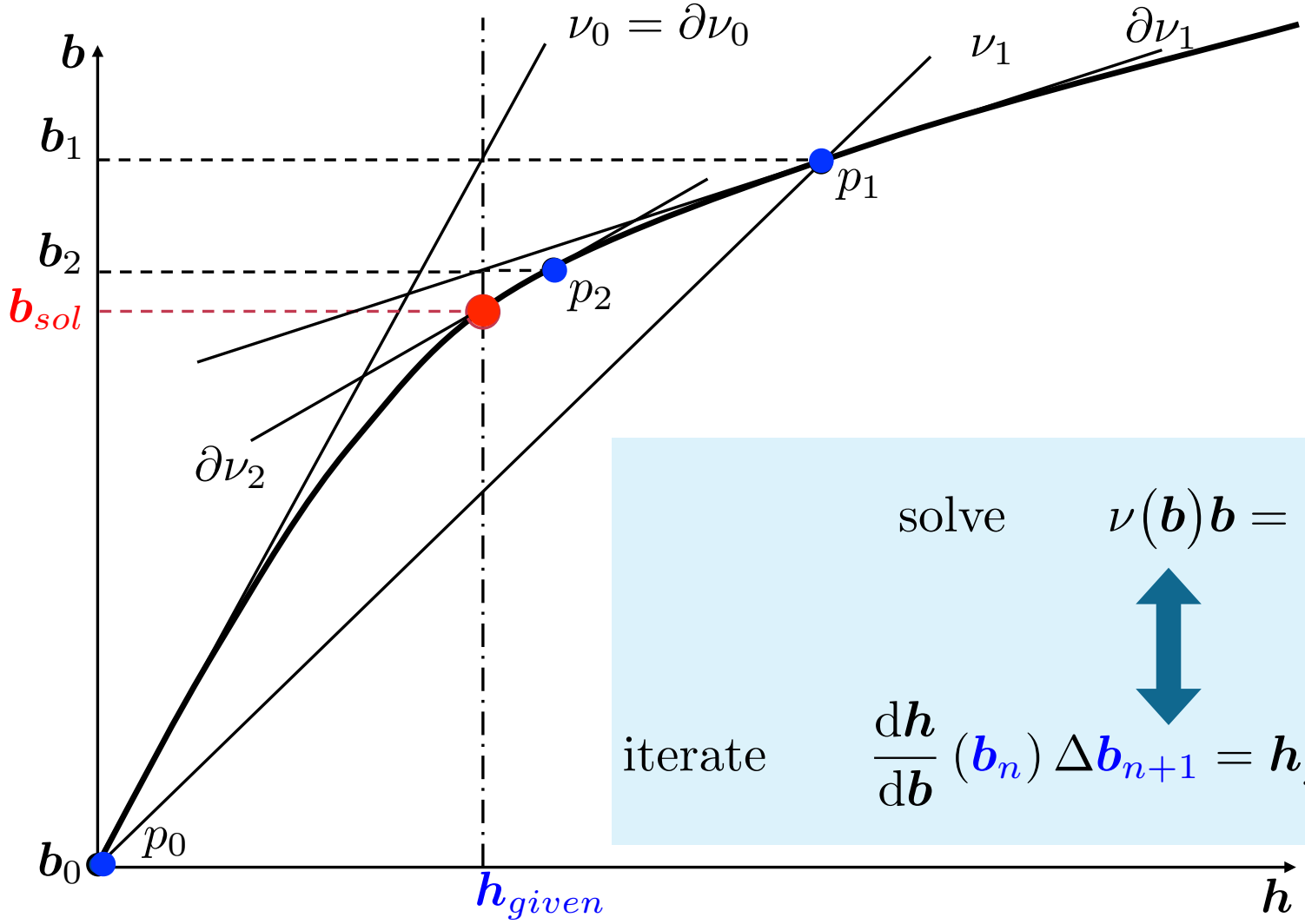
Newton method (cont'd)



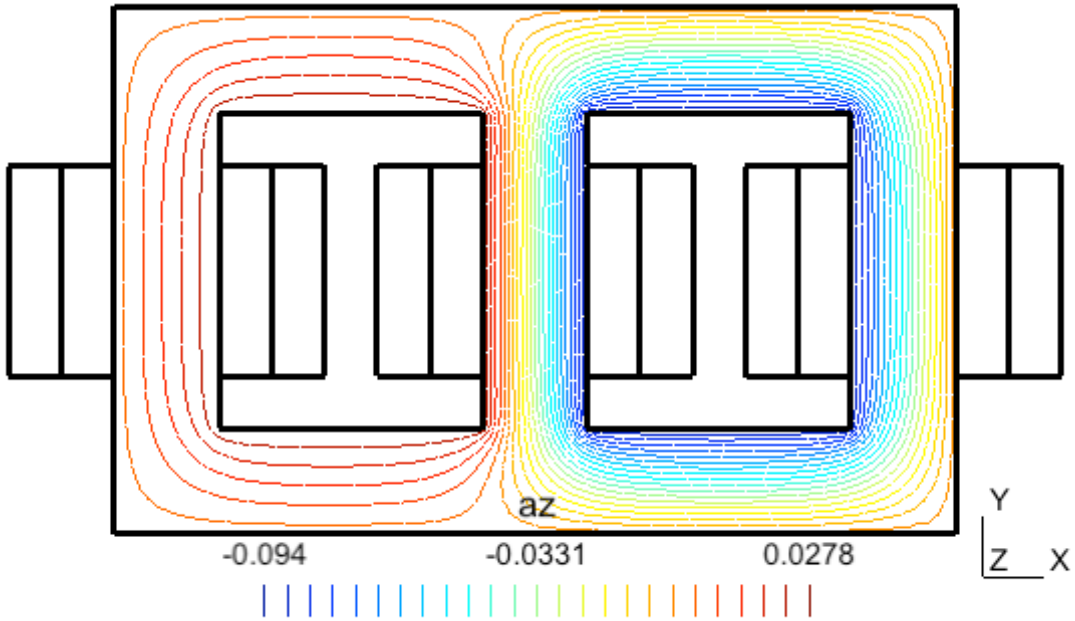
solve $\quad \nu(\boldsymbol{b})\boldsymbol{b} = \boldsymbol{h}_{given}$

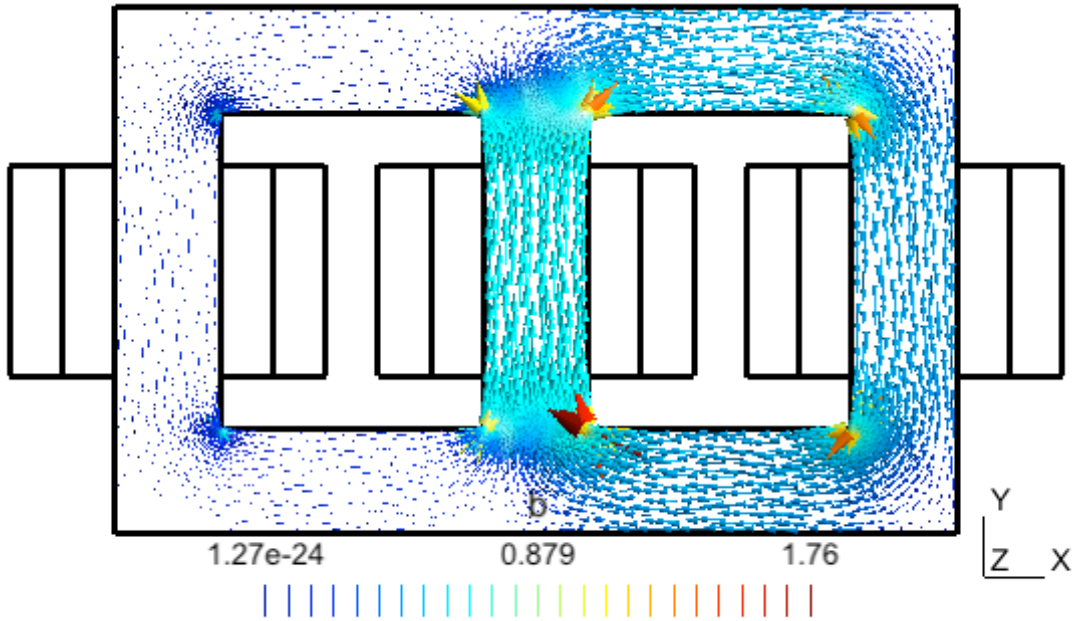
iterate $\quad \dfrac{\mathrm{d}\boldsymbol{h}}{\mathrm{d}\boldsymbol{b}}(\boldsymbol{b}_n)\,\Delta\boldsymbol{b}_{n+1} = \boldsymbol{h}_{given} - \nu(\boldsymbol{b}_n)\boldsymbol{b}_n$

Newton method (cont'd)

- graphical interpretation: we approximate the bh-curve by a tangent
- computing the Jacobian matrix maybe complicated
  - analytical expression of the derivatives not always known
  - numerical approximation by finite differences (e.g. perturbing the solution)
- quasi-Newton methods combine
  - fixed point techniques for some iterations
  - finite difference for evaluating the Jacobian
- complex harmonic systems require particular treatment when using Newton-type methods

Application — A three-phase transformer



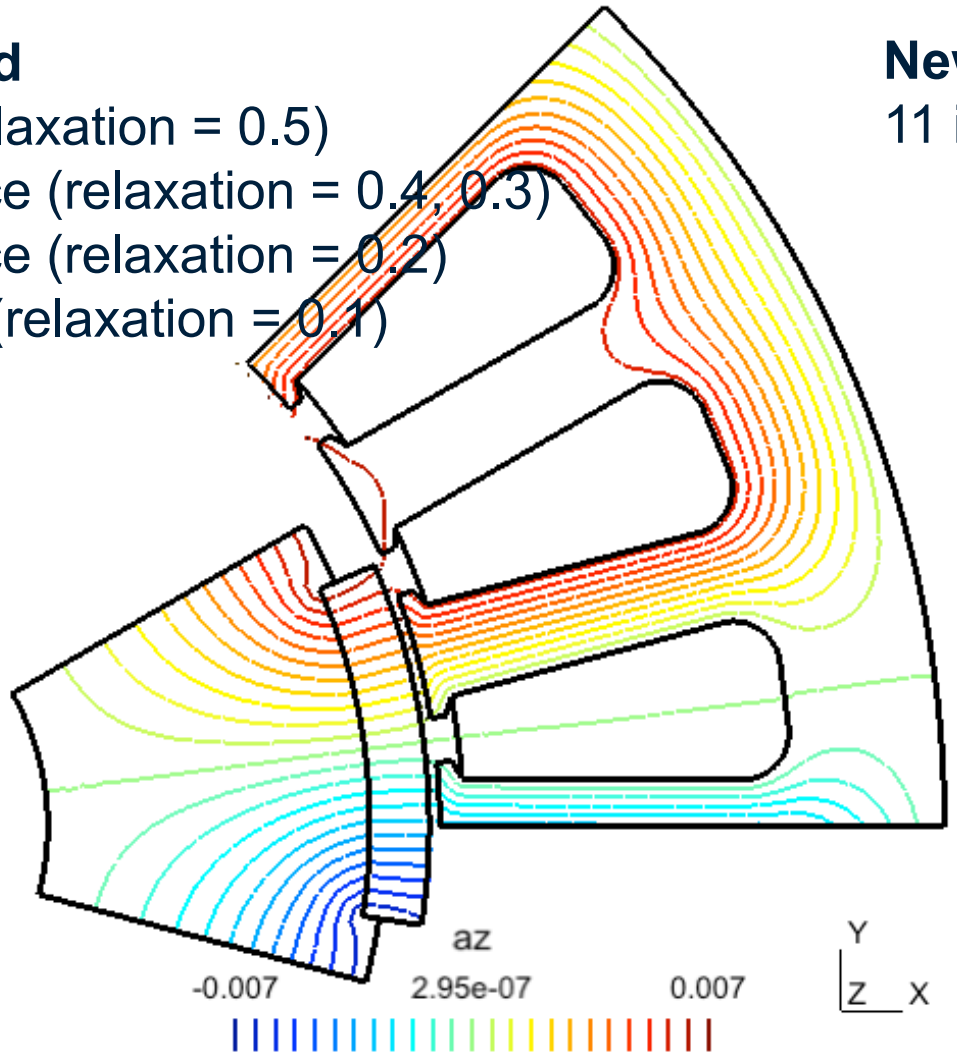


Picard method
no convergence (relaxation = 0.5)
23 iterations (relaxation = 0.3)
35 iterations (relaxation = 0.2)
73 iterations (relaxation =0.1)

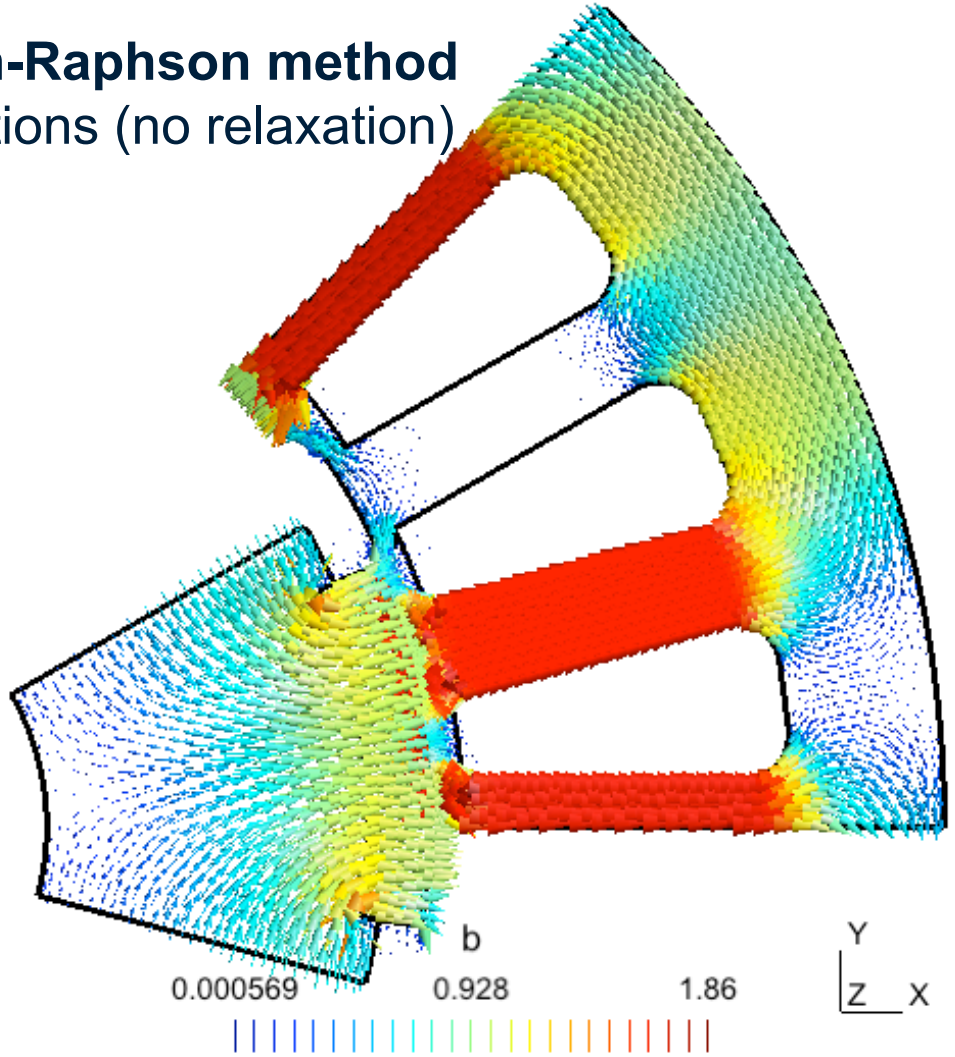
Newton-Raphson method
6 iterations (no relaxation)

Application — PM synchronous machine

Picard method
divergence (relaxation = 0.5)
no convergence (relaxation = 0.4, 0.3)
no convergence (relaxation = 0.2)
126 iterations (relaxation = 0.1)



Newton-Raphson method
11 iterations (no relaxation)



Nonlinear problems in GetDP

Resolution of nonlinear problems

For linear problems, the finite element method leads to the solution of linear systems of algebraic equations of the form:

$$\mathbf{A}\,\mathbf{x} = \mathbf{b}, \tag{1}$$

where $\mathbf{A}$ is a square matrix (e.g. the stiffness matrix of the problem), $\mathbf{b}$ takes into account potential source terms and $\mathbf{x}$ is the vector of unknowns to be calculated.

In the presence of material nonlinearities, the matrix $\mathbf{A}$ depends on the unknown field $\mathbf{x}$, and the system of equations becomes nonlinear:

$$\mathbf{A}(\mathbf{x})\,\mathbf{x} = \mathbf{b}. \tag{2}$$

Therefore, the system must necessarily be solved iteratively. Starting from an initial guess vector $\mathbf{x}_0$ (e.g. a zero vector), the following calculated values $\mathbf{x}_1$, $\mathbf{x}_2$, ... are hoped to converge to the correct solution. For the exact solution, the residual defined by $\mathbf{r}(\mathbf{x}) = \mathbf{A}(\mathbf{x})\,\mathbf{x} - \mathbf{b}$ is zero. If after p iterations, a satisfactory convergence is obtained, the iterative process is stopped. The convergence criterion could be based on some norm of the residual $\mathbf{r}(\mathbf{x}_p)$ or on the p^{th} increment $\delta\mathbf{x}_p = \mathbf{x}_p - \mathbf{x}_{p-1}$. For example, it could be :

$$\frac{||\delta\mathbf{x}_p||_\infty}{||\mathbf{x}_p||_\infty} < \varepsilon, \tag{3}$$

with ε a small dimensionless number (e.g. 10^{-6}).

Picard's method

Picard's iteration provides an easy way to handle the nonlinearity. In the Picard iterative process, a new approximation $\mathbf{x}_i$ is calculated by using a known, previous solution $\mathbf{x}_{i-1}$ in the nonlinear terms so that these terms become linear in the unknown $\mathbf{x}_i$. Therefore, the problem becomes:

$$\mathbf{A}(\mathbf{x}_{i-1})\,\mathbf{x}_i = \mathbf{b}. \tag{4}$$

The following iterative process summarizes the Picard method:

$\mathbf{x}_0 = \mathbf{0}$; // Initialization
for $i = 1, 2, 3, \ldots$ {
  $\mathbf{x}_i = \left(\mathbf{A}(\mathbf{x}_{i-1})\right)^{-1}\mathbf{b}$; // Find the new $\mathbf{x}$
}

http://onelab.info/wiki/Nonlinear_problems_in_GetDP

Post-processing of electromagnetic quantities

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Circuit parameters

resistances, inductances, capacitances

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Capacitance and resistance

- By applying basic concepts of electrostatic fields and conduction media, the capacitance and the resistance for complex geometric configurations read

$$C = \frac{\int_S \boldsymbol{d}\,\mathrm{d}S}{\oint_\gamma \boldsymbol{e}\,\mathrm{d}l} = \frac{\int_S \epsilon\,\boldsymbol{e}\,\mathrm{d}S}{\oint_\gamma \boldsymbol{e}\,\mathrm{d}l} \qquad R = \frac{\oint_\gamma \boldsymbol{e}\,\mathrm{d}l}{\int_S \boldsymbol{j}\,\mathrm{d}S} = \frac{\oint_\gamma \boldsymbol{e}\,\mathrm{d}l}{\int_S \sigma\,\boldsymbol{e}\,\mathrm{d}S}$$

- We can use the electrostatic energy for calculating *C*

$$W_e = \frac{1}{2}\int_V \boldsymbol{d}\cdot\boldsymbol{e}\,\mathrm{d}V = \frac{1}{2}\int_V \epsilon e^2\,\mathrm{d}V = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2 \qquad C = \frac{2W_e}{V^2}$$

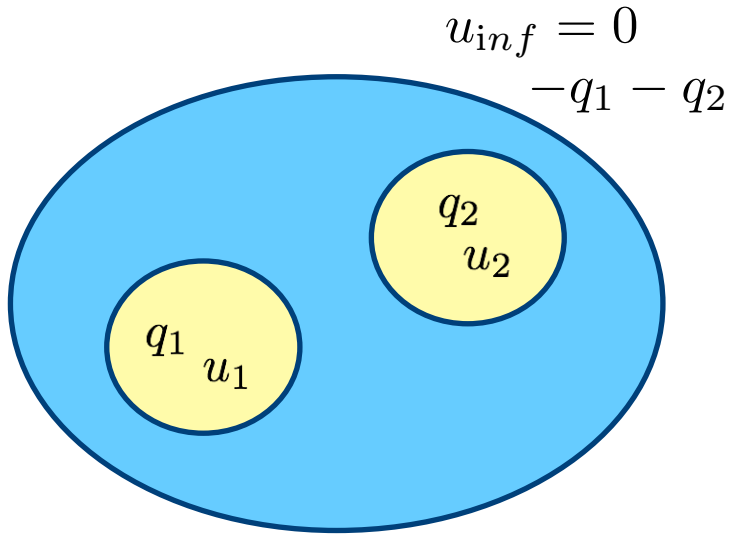
- The Joule losses or power dissipated as heat in the conductive material

$$P = \int_V \boldsymbol{j}\cdot\boldsymbol{e}\,\mathrm{d}V = \int_V \sigma e^2\,\mathrm{d}V = \frac{V^2}{R} = I^2R \qquad R = \frac{V^2}{P} = \frac{P}{I^2}$$

Capacitance matrix

Capacitance matrix

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \Longleftrightarrow \quad \mathbf{q} = \mathbf{C}\mathbf{u}$$



Electric energy

$$W_e = \frac{1}{2} \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \Longleftrightarrow \quad W_e = \frac{1}{2}\mathbf{u}^{\mathbf{T}}\mathbf{C}\mathbf{u}$$

$W_e > 0, \quad \forall \mathbf{u} \neq \mathbf{0} \Rightarrow \mathbf{C}$ symmetric and positive definite

Resistance matrix

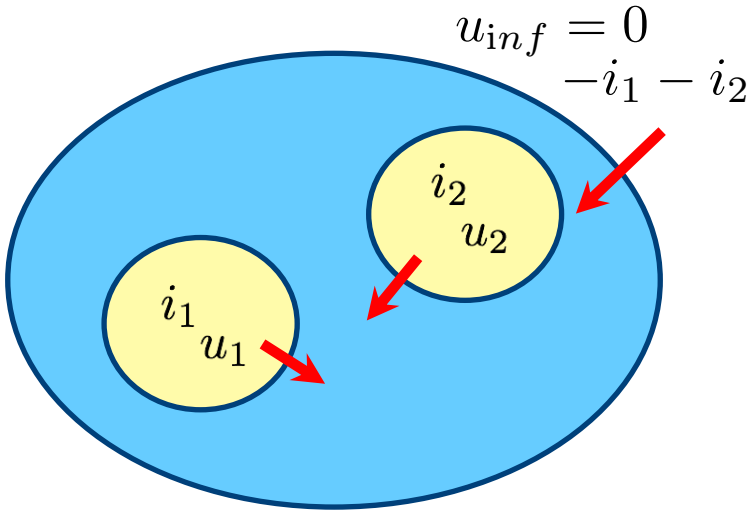
Resistance matrix

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \quad \longleftrightarrow \quad \mathbf{u} = \mathbf{R}\mathbf{i}$$

Joule losses

$$P_e = \frac{1}{2} \begin{bmatrix} i_1 & i_2 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \quad \longleftrightarrow \quad P_e = \frac{1}{2}\mathbf{i}^{\mathbf{T}}\mathbf{R}\mathbf{i}$$

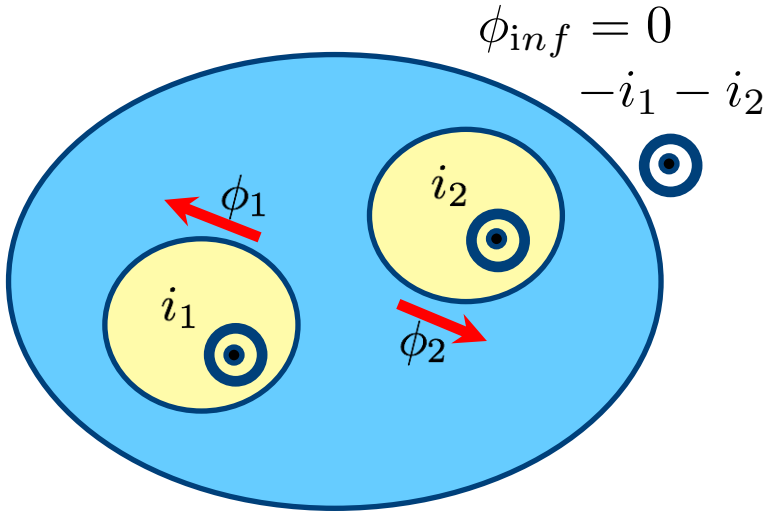
$P_e > 0, \quad \forall \mathbf{i} \neq \mathbf{0} \Rightarrow \mathbf{R}$ symmetric and positive definite



Inductance matrix

Inductance matrix

$$\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \begin{bmatrix}L_{11} & L_{12}\\ L_{21} & L_{22}\end{bmatrix}\begin{bmatrix}i_1\\ i_2\end{bmatrix} \quad \Longleftrightarrow \quad \boldsymbol{\phi} = \mathbf{Li}$$



Magnetic energy

$$W_m = \frac{1}{2}\begin{bmatrix}i_1 & i_2\end{bmatrix}\begin{bmatrix}L_{11} & L_{12}\\ L_{21} & L_{22}\end{bmatrix}\begin{bmatrix}i_1\\ i_2\end{bmatrix} \quad \Longleftrightarrow \quad W_m = \frac{1}{2}\mathbf{i^T Li}$$

$W_m > 0, \quad \forall \mathbf{i} \neq \mathbf{0} \Rightarrow \mathbf{L}$ symmetric and positive definite

Parameter extraction

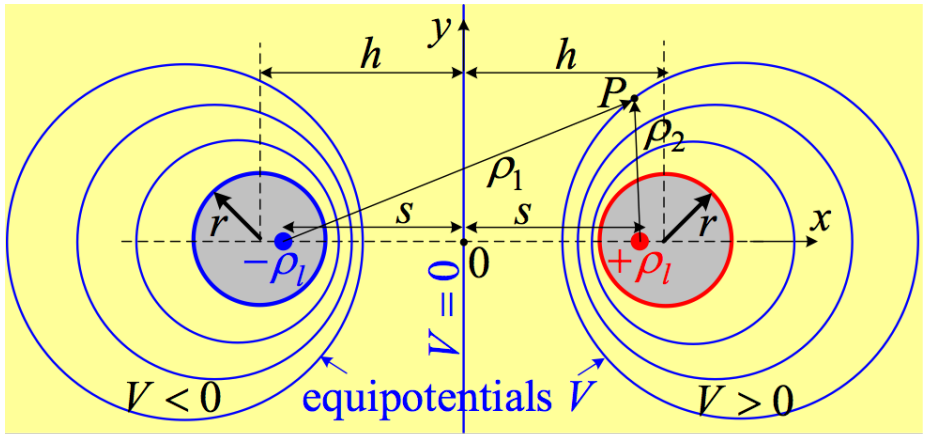
- extract C, R and L directly from the FE matrices
  (not further considered here)
- apply a few linearly independent excitations for voltages (C), currents (R) and currents (L) and compute charges (C), voltages (R) or fluxes (L)

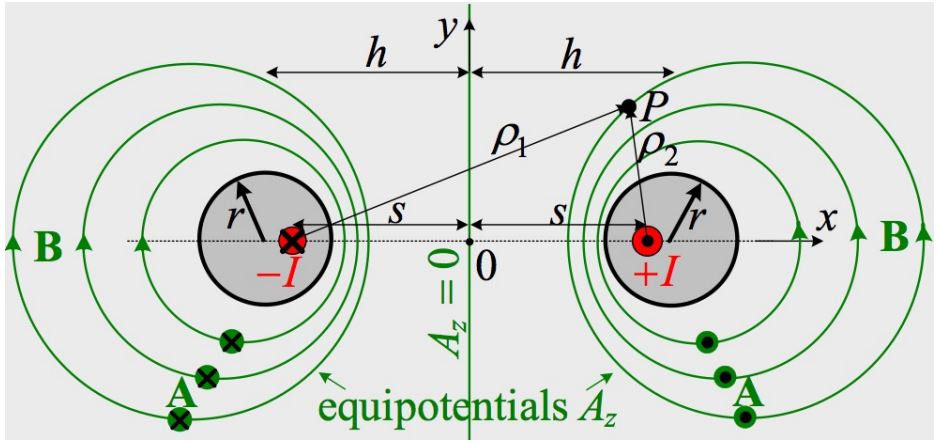
$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} 1\,\mathrm{V} \\ 0\,\mathrm{V} \end{bmatrix} \quad \Rightarrow \quad C_{11} = \frac{q_1}{1\,\mathrm{V}} \qquad C_{21} = \frac{q_2}{1\,\mathrm{V}}$$

- apply a few linearly independent excitations for voltages (C), currents (R) and currents (L) and compute the electric energy (C), the Joule losses (R) or the magnetic energy (L)

$$W_e = \frac{1}{2} \begin{bmatrix} 1\,\mathrm{V} & 0\,\mathrm{V} \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} 1\,\mathrm{V} \\ 0\,\mathrm{V} \end{bmatrix} \quad \Rightarrow \quad C_{11} = \frac{2W_e}{(1\,\mathrm{V})^2}$$

Analogy between scalar and vector potentials in a transmission line





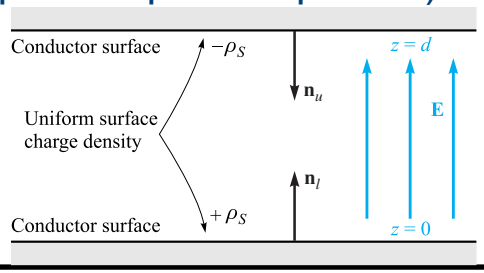
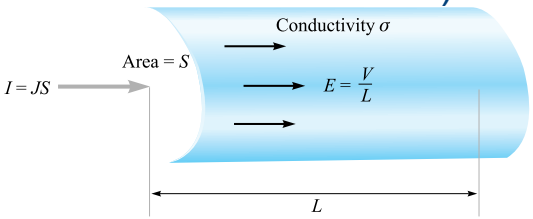
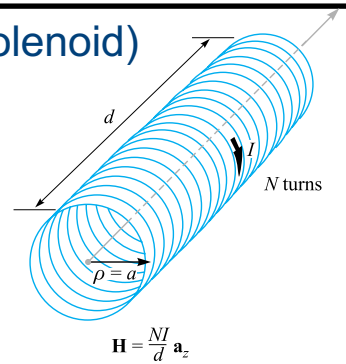
Electrostatics

$$C = \frac{\rho_l}{V_r - V_l}$$

Magnetostatics

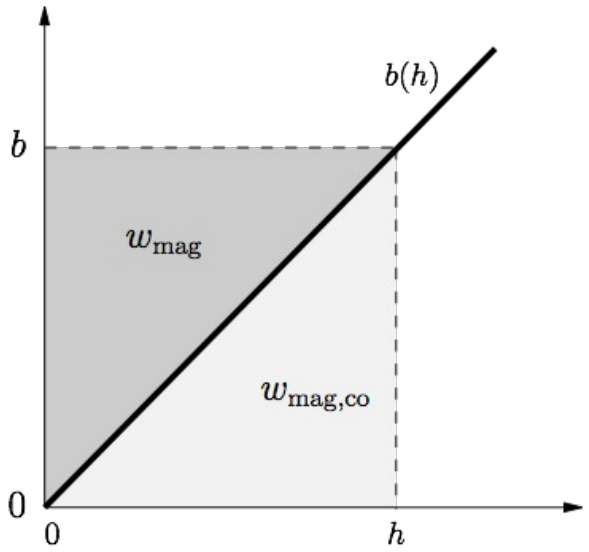
$$L = \frac{A_{zr} - A_{zl}}{I}$$

Parameter extraction

capacitance matrix	$-\operatorname{div}(\epsilon \operatorname{grad} \varphi) = 0$ $C = \dfrac{\int_S \epsilon \boldsymbol{e}\, \mathrm{d}S}{\oint_\gamma \boldsymbol{e}\, \mathrm{d}l}$	(parallel plate capacitor) $C = \epsilon \dfrac{S}{d}$ 
resistance matrix	$-\operatorname{div}(\sigma \operatorname{grad} \varphi) = 0$ $R = \dfrac{\oint_\gamma \boldsymbol{e}\, \mathrm{d}l}{\int_S \sigma \boldsymbol{e}\, \mathrm{d}S}$	(straight conductor with constant cross section) $\dfrac{1}{R} = \sigma \dfrac{S}{l}$ 
inductance matrix	$\operatorname{curl}(\nu \operatorname{curl} \boldsymbol{a}) = \boldsymbol{j}$ $L = \dfrac{N\Phi}{I} = l_z \dfrac{\int_{S_s} \boldsymbol{a}\, \boldsymbol{j}_s \mathrm{d}S}{I^2}$	(ideal solenoid) $\dfrac{1}{L} = \nu \dfrac{l_m}{S_m}$ 

Magnetic energy/Co-energy

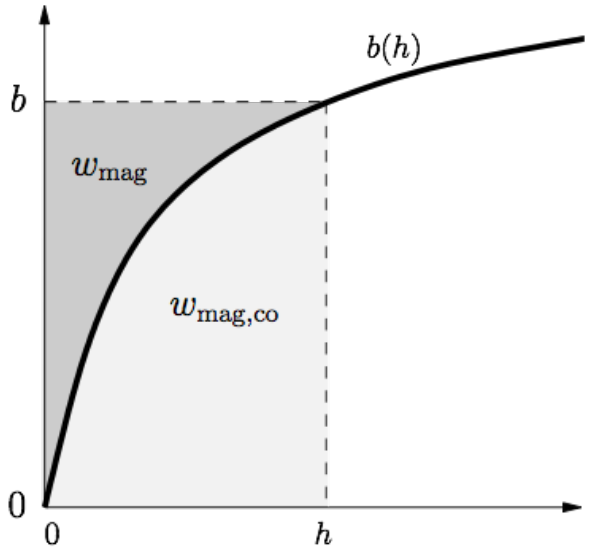
The magnetic energy and magnetic co-energy are two global quantities that can be used for the computation of inductances and forces



In the nonlinear case, energy & co-energy are NOT equal

$$W = \int_\Omega \mathrm{d}\Omega \int_0^b \boldsymbol{h}\,\mathrm{d}\boldsymbol{b}$$

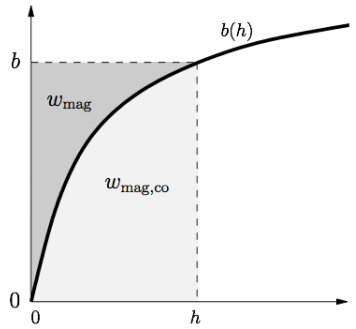
$$W_{\text{co}} = \int_\Omega \mathrm{d}\Omega \int_0^h \boldsymbol{b}\,\mathrm{d}\boldsymbol{h}$$



In the linear case, energy and co-energy are numerically equal

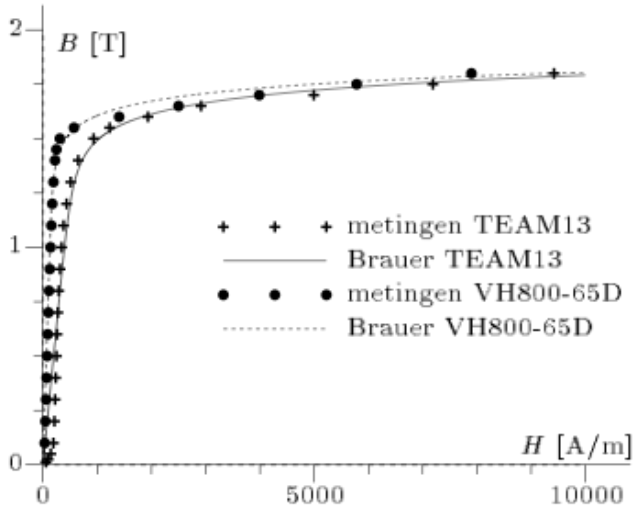
$$W = W_{\text{co}} = \frac{1}{2}\int_\Omega \boldsymbol{bh}\,\mathrm{d}\Omega = \frac{1}{2}\int_\Omega \mu\boldsymbol{h}^2\,\mathrm{d}\Omega = \frac{1}{2}\int_\Omega \frac{\boldsymbol{b}^2}{\mu}\,\mathrm{d}\Omega$$

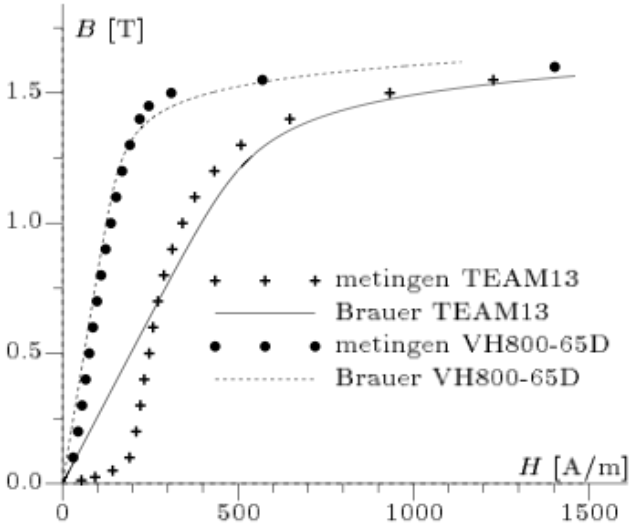
Nonlinear case co-energy>>energy



Nonlinear isotropic materials

$$\boldsymbol{h} = \nu(b^2)\boldsymbol{b} \qquad b^2 = b_x^2 + b_y^2 + b_z^2 = \boldsymbol{b}\cdot\boldsymbol{b}$$





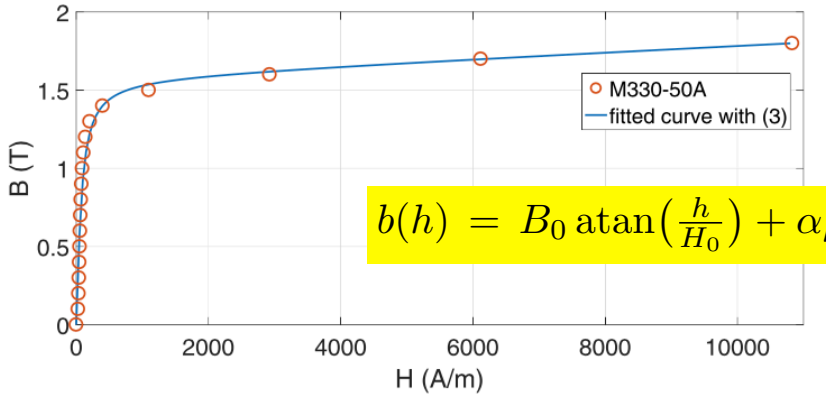
Brauer expression

$$\nu(b^2) = k_1 + k_2 e^{k^3 b^2}$$

	k_1 [m/H]	k_2 [T^{-2}]	k_3 [m/H]
TEAM13	0.3774	2.970	388.33
VH800-65D	0.0596	3.504	122.87

$$W(b) = \int_0^b h(b')\,\mathrm{d}b' = \int_0^{b^2} \frac{1}{2}\nu(b'^2)\,\mathrm{d}b'^2 = \frac{1}{2}k_1 b^2 + \frac{1}{2}\frac{k_2}{k_3}\left(e^{k_3 b^2 - 1}\right)$$

$$W_{co}(b) = \int_0^{h(b)} b(h')\,\mathrm{d}h' = hb - W(b) = \nu(b^2)b^2 - W(b)$$



$$b(h) = B_0 \operatorname{atan}\left(\frac{h}{H_0}\right) + \alpha\mu_0 h$$

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NL linear isotropic materials – differential reluctivity tensor

$$\boldsymbol{h} = \nu(b)\,\boldsymbol{b}$$

$$h_x(b_x, b_y) = h(b)\,\frac{b_x}{b}$$

$$h_y(b_x, b_y) = h(b)\,\frac{b_y}{b}$$

$$b^2 = b_x^2 + b_y^2 \qquad b = \sqrt{b_x^2 + b_y^2}$$

$$\frac{\partial h_x}{\partial b_x} = \frac{h}{b} + \frac{\mathrm{d}h}{\mathrm{d}b}\frac{\partial b}{\partial b_x}\frac{b_x}{b} - h\frac{b_x}{b^2}\frac{\partial b}{\partial b_x}$$

$$= \dots$$

$$\frac{\partial h_x}{\partial b_y} = \dots$$

$$\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{b}} = \begin{bmatrix} \frac{\partial h_x}{\partial b_x} & \frac{\partial h_x}{\partial b_y} \\ \frac{\partial h_y}{\partial b_x} & \frac{\partial h_y}{\partial b_y} \end{bmatrix} = \nu(b) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \left(\frac{\mathrm{d}h}{\mathrm{d}b} - \nu(b)\right) \frac{1}{b^2} \begin{bmatrix} b_x^2 & b_x b_y \\ b_x b_y & b_y^2 \end{bmatrix}$$

3D implementation in GetDP:

$1 = b vector

```
57 ¶
58     //dhdb_v[] = nu_sc[Norm[$1]] * TensorDiag[1,1,1] + ¶
59     //         ( dhdb_sc[Norm[$1]] - nu_sc[Norm[$1]] ) * SquDyadicProduct[$1]/SquNorm[$1];¶
60 ¶
```

#: registers

```
4     dhdb_v[] =  (Norm[$1]#111) ?  ¶
5        nu_sc[#111]#222 * TensorDiag[1,1,1] + (dhdb_sc[#111] - #222) * SquDyadicProduct[$1]/#111^2¶
6        : nu_sc[#111];¶
```

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Simple winding model

n more or less regularly spaced series-connected turns

flux linkage (Vs) in 2D model

$$\Psi_{2D} = L_z \int_{\Omega_w} j_{z,1A} \cdot a_z \, \mathrm{d}\Omega$$

$$V = RI + \frac{\mathrm{d}}{\mathrm{d}t}\left(\Psi_{2D} + L_{3D} I\right)$$

end-turns

$$\boldsymbol{SA} = \boldsymbol{KI}$$

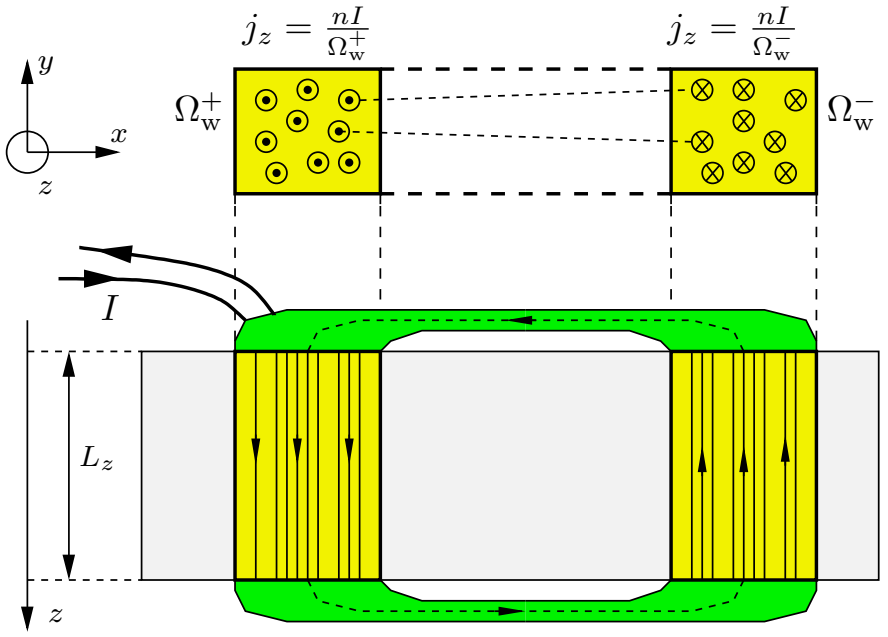
$$\boldsymbol{J} = \boldsymbol{KI}$$

$$\Psi = L_z \boldsymbol{K}^T \boldsymbol{A}$$

$$\boldsymbol{K}_i = \int_{\Omega_w} j_{z,1A} \cdot w_i \, \mathrm{d}\Omega$$

homogenized current density in 2 coil sides at @1A

$$j_{z,1A} = \frac{\pm n}{\Omega_w^{\pm}} \quad \text{in} \quad \Omega_w = \Omega_w^+ - \Omega_w^-$$



end-turns

(same K matrix -> reciprocity (symm. inductance matrix))

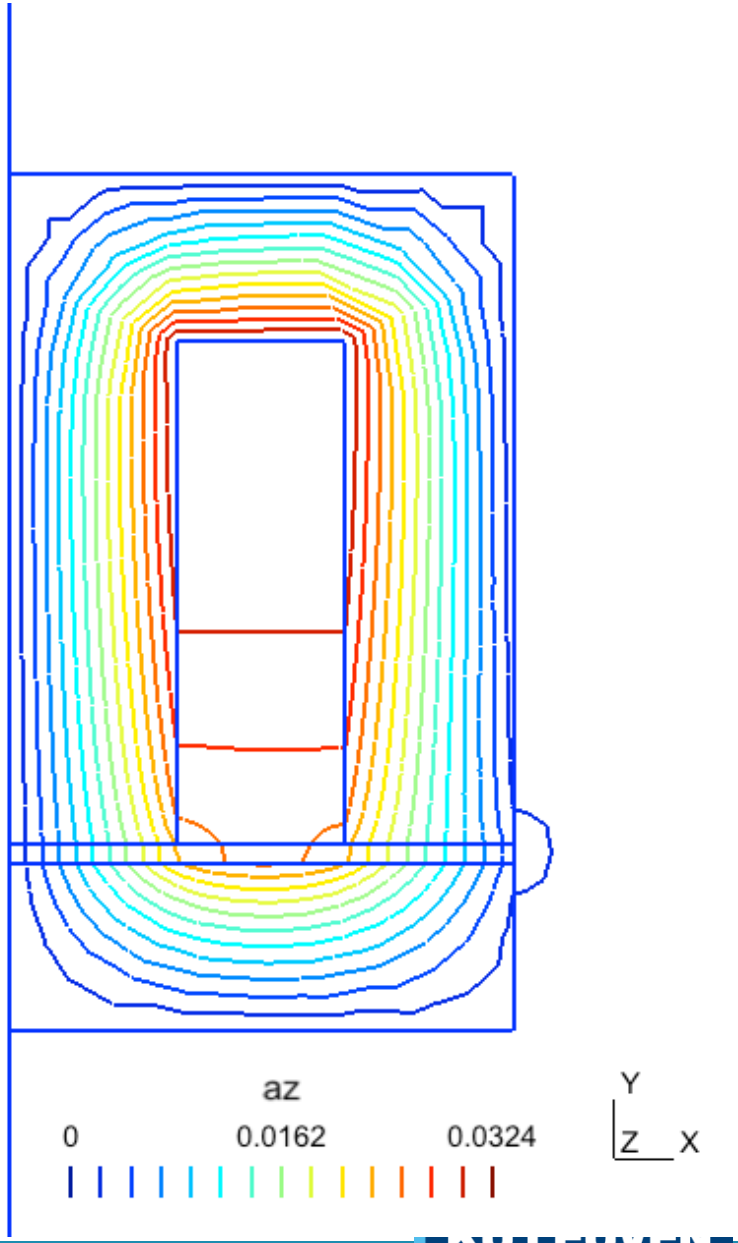
Inductance from flux linkage

- flux linkage model

$$\Phi = \int_S \boldsymbol{b}\,\mathrm{d}S = \oint_{\partial S} \boldsymbol{a}\,\mathrm{d}l$$

$$L = \frac{n\Phi}{I} = l_z \frac{\int_{S_s} \boldsymbol{a}\,\boldsymbol{j}_s \mathrm{d}S}{I^2}$$

```
{ Name Flux ; Value { Integral {
      [ SymmetryFactor*LengthAlongZ*Idir[]*NbWires[]/SurfCoil[]* CompZ[{a}] ] ;
        In Inds  ; Jacobian Vol ; Integration I1 ; }
}}
{ Name Inductance_from_Flux ; Value { Term { Type Global;
      [ $flux*1e3/II ] ; In DomainDummy ; } } } // register $flux contains the Flux
```



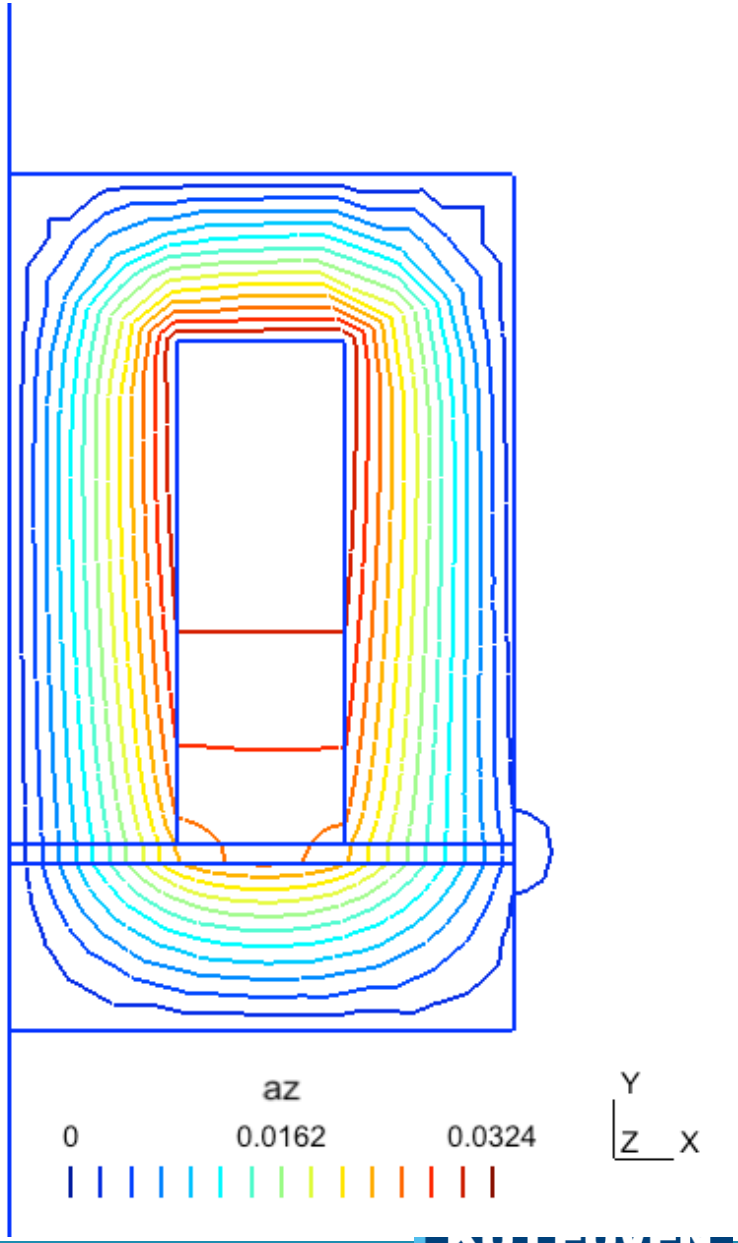
Inductance from energy

- energy model - linear material

$$W = \int_{\Omega} \mathrm{d}\Omega \int_{0}^{b} \boldsymbol{h}\, \mathrm{d}\boldsymbol{b}$$

$$L = \frac{2W}{(nI)^2}$$

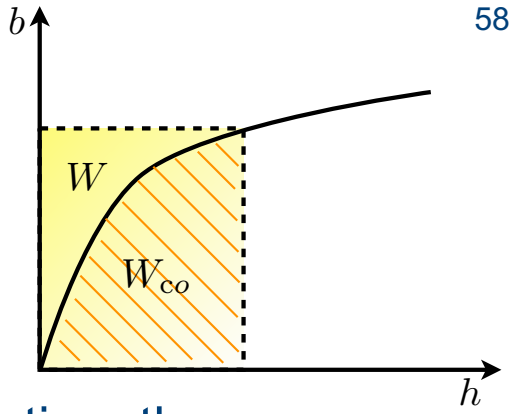
```
{ Name MagEnergy ; Value { Integral {
      [ SymmetryFactor*LengthAlongZ* 1/2 *nu[{d a}]*{d a}*{d a} ] ;
      In Domain ; Jacobian Vol ; Integration I1 ; }
} }
{ Name Inductance_from_MagEnergy ; Value { Term { Type Global; // in mH
      [ 2*$wmag*1e3/(II*II) ] ; In DomainDummy ; } } }
      // register $wmag contains the energy
```



Inductance from co-energy

- co-energy model — nonlinear materials

$$W = \int_\Omega \mathrm{d}\Omega \int_0^b \boldsymbol{h}\,\mathrm{d}\boldsymbol{b} \qquad W_{\mathrm{co}} = \int_\Omega \mathrm{d}\Omega \int_0^h \boldsymbol{b}\,\mathrm{d}\boldsymbol{h}$$

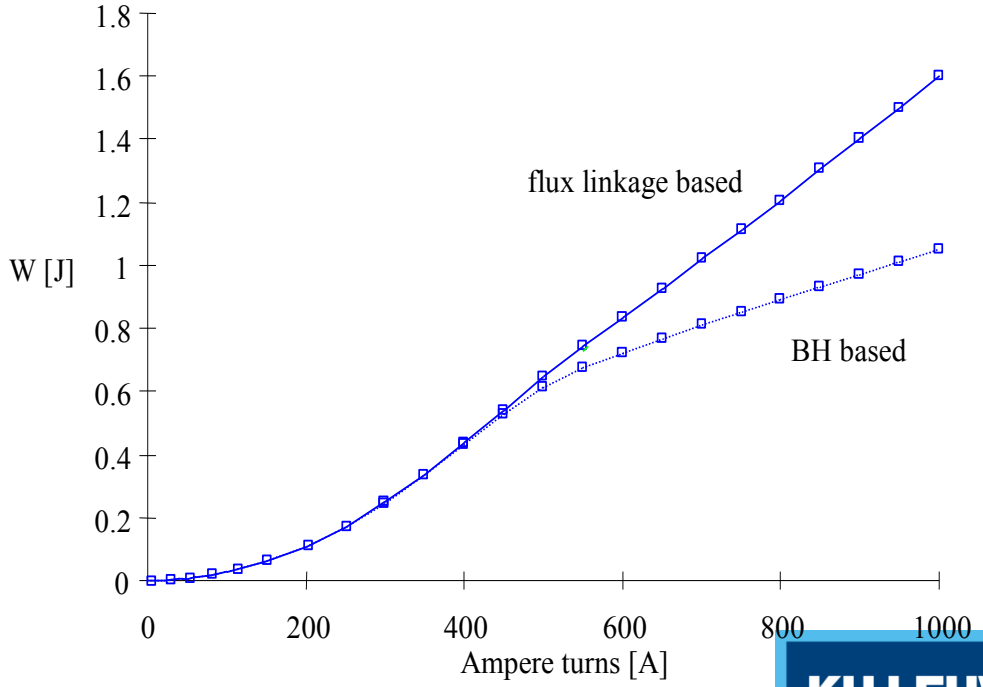


- flux linkage $L = \dfrac{n\Phi}{I} = l_z \dfrac{\int_{S_s} \boldsymbol{a}\,\boldsymbol{j}_s \mathrm{d}S}{I^2}$

Definition of the inductance?
Given the nonlinearity, its value is no longer constant, it depends on the current

Key quantity in nonlinear transient problems: Differential inductance $\dfrac{\partial L(I)}{\partial I}$

underestimation: if saturation, the variation of b becomes negligible energy is bounded, co-energy not



Force computation
Virtual work, Maxwell stress tensor

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Magnetic force computation
Generalities

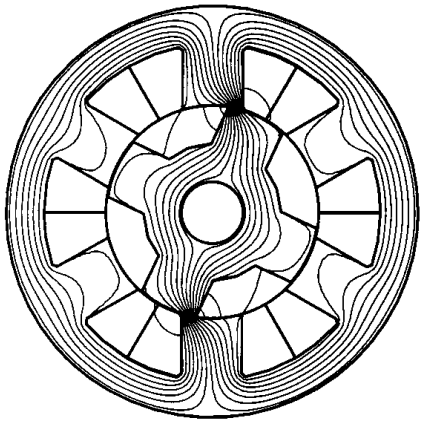
the magnetic field exerts a force on

- current (in coils, in conducting material in general) -> volume distribution, in N/m^3
- on (ferro-) magnetic material -> volume and surface distribution, N/m^3 and N/m^2

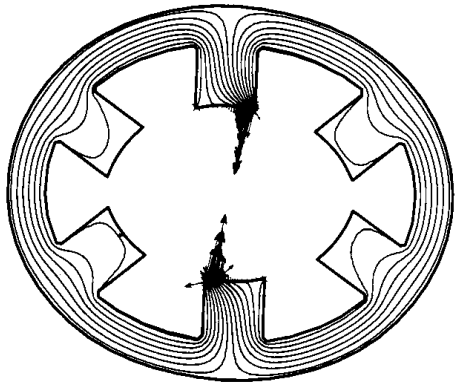
total magnetic force/torque computation methods

- volume/surface integration of Lorentz (Laplace) force density,
  for non-magnetic material only!
- numerical differentiation (finite difference) of energy or co-energy w.r.t. position
- local application of virtual-work principle
  (virtual distortion of layer of elements around volume/surface)
- surface/line integration of Maxwell stress tensor
  + approaches for averaging

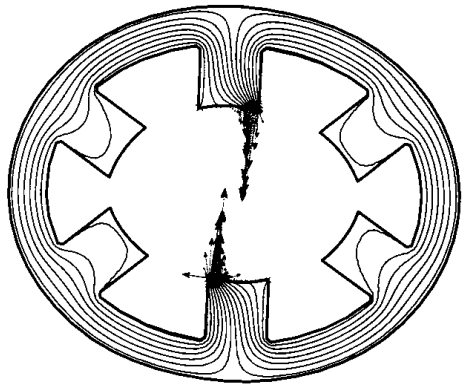
Magnetic force computation
Force distribution vs total force/torque



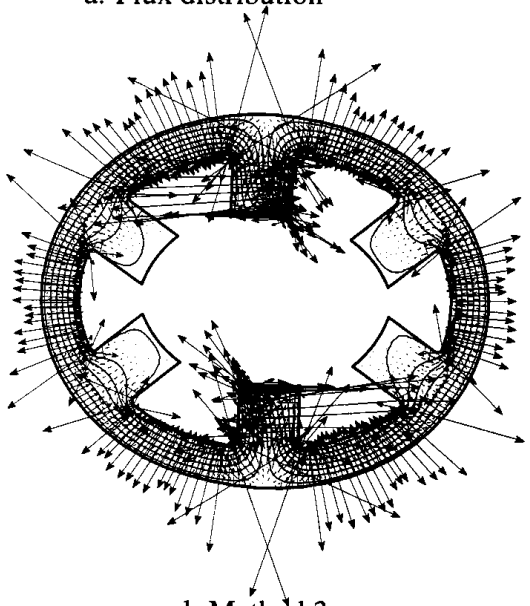
a. Flux distribution



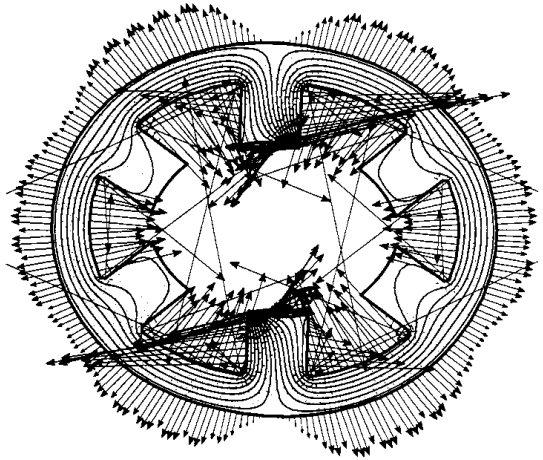
b. Method 1



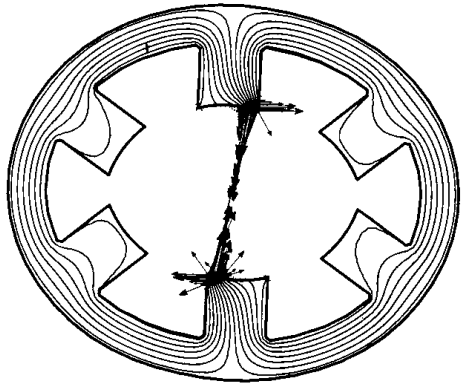
c. Method 2



d. Method 3



e. Method 4



f. Method 5

Application to an SRM in [Vandevelde2001]

Lorentz force

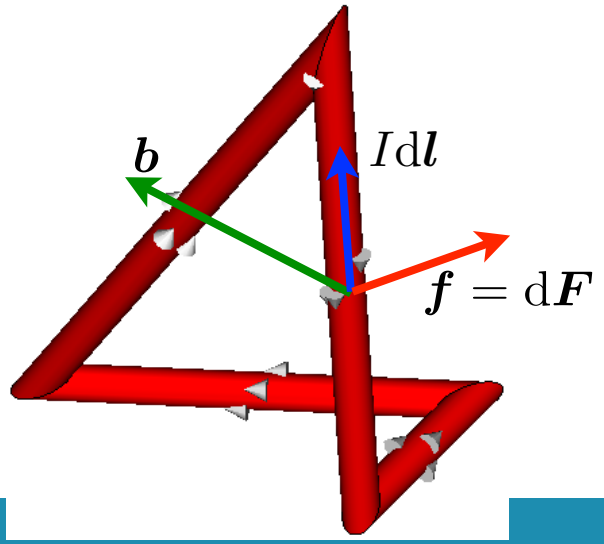
Lorentz force is simple to apply to compute the magnetic force acting on a current-carrying wire. Given the local value of current density and flux density, we can find the local force distribution and the total force:

$$\boldsymbol{f} = \boldsymbol{j} \times \boldsymbol{b} = \operatorname{curl} \boldsymbol{h} \times \boldsymbol{b} \quad (\mathrm{N/m^3})$$

$$\boldsymbol{f}_{2D} = \boldsymbol{j} \times \boldsymbol{b} = -j_z b_y \hat{\boldsymbol{x}} - j_z b_x \hat{\boldsymbol{y}}$$

$$\boldsymbol{F} = \int_{\Omega^*} \boldsymbol{j} \times \boldsymbol{b} \, \mathrm{d}\Omega \quad (\mathrm{N})$$

integration over a volume
of **non-magnetic conducting material**

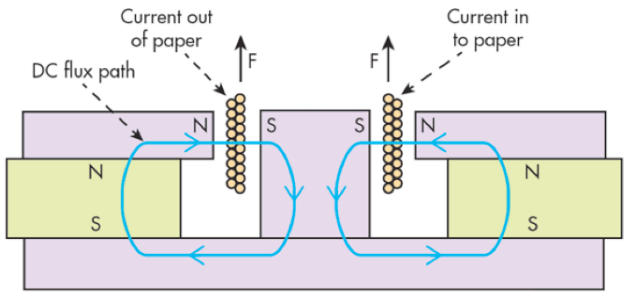


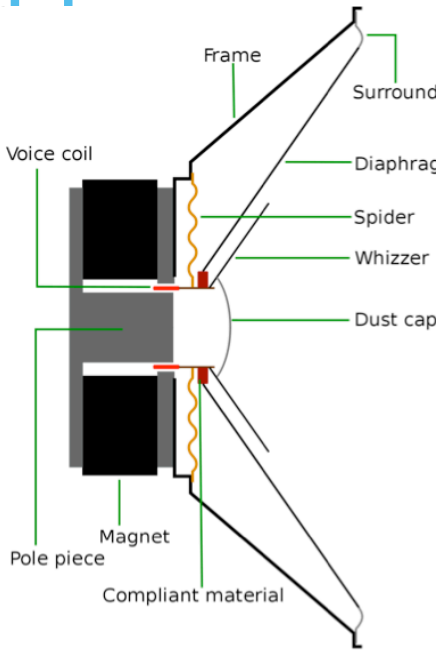
The total force on a close conductor is then obtained by integration, i.e.

$$\boldsymbol{F} = \int_{\gamma} I \mathrm{d}\boldsymbol{l} \times \boldsymbol{b}$$

Lorentz force – applications

Axisymmetric loudspeaker





exercise:
calculate axial force

- 10 turns
- 0.1 A
- radius of 1 cm
- b of 1.1 T

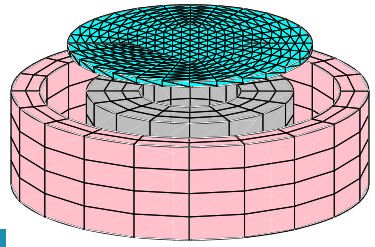
$$\boldsymbol{f} = \boldsymbol{j} \times \boldsymbol{b} = \operatorname{curl} \boldsymbol{h} \times \boldsymbol{b} \quad (\mathrm{N/m^3})$$

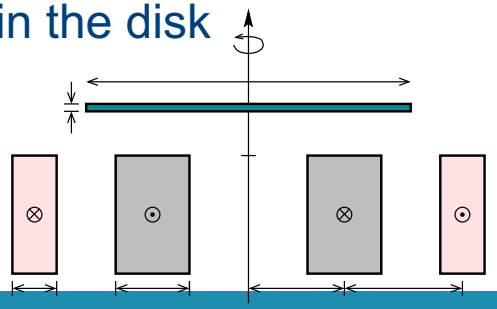
$$\boldsymbol{F} = \int_{\Omega^*} \boldsymbol{j} \times \boldsymbol{b} \, \mathrm{d}\Omega \quad (\mathrm{N})$$

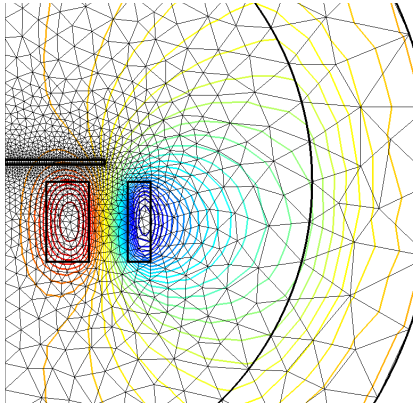
$$\boldsymbol{f}_{2D} = \boldsymbol{j} \times \boldsymbol{b} = -j_z b_y \hat{\boldsymbol{x}} - j_z b_x \hat{\boldsymbol{y}}$$

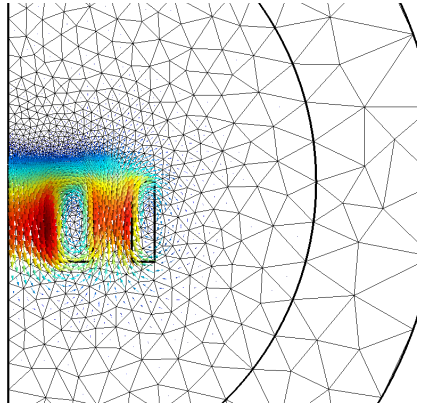
Levitation device – TEAM 28

- Aluminum disk above 2 concentric coils
- 50Hz sinusoidal current in two coils
- Induced currents in the disk









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Lorentz force – applications
TEAM 28 levitation device

50 Hz

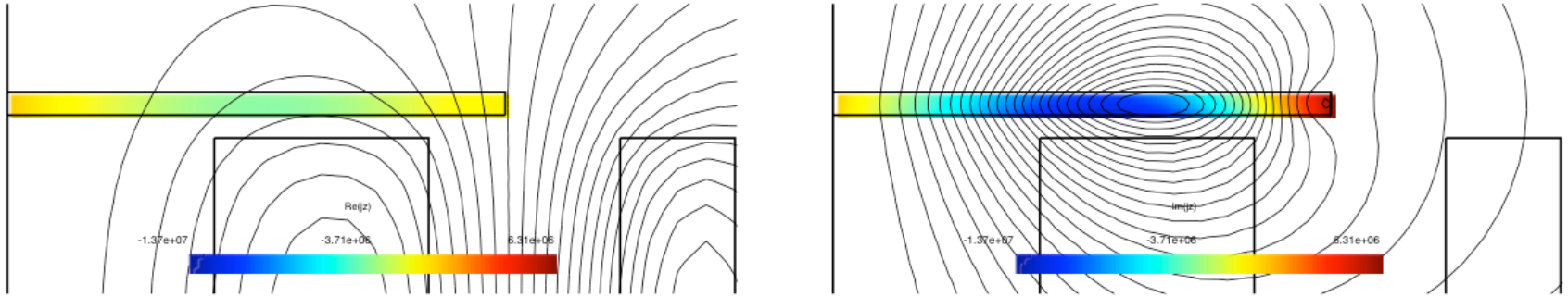


Fig. 1 Flux lines and induced current density in the plate, real (left) and imaginary (right) parts, with $f = 50\,\mathrm{Hz}$, $z_{\mathrm{pl}} = 3\,\mathrm{mm}$ and $\underline{\imath} = 20\,\mathrm{A}$

The levitating force on the aluminum plate can be computed by Lorentz force

1 kHz

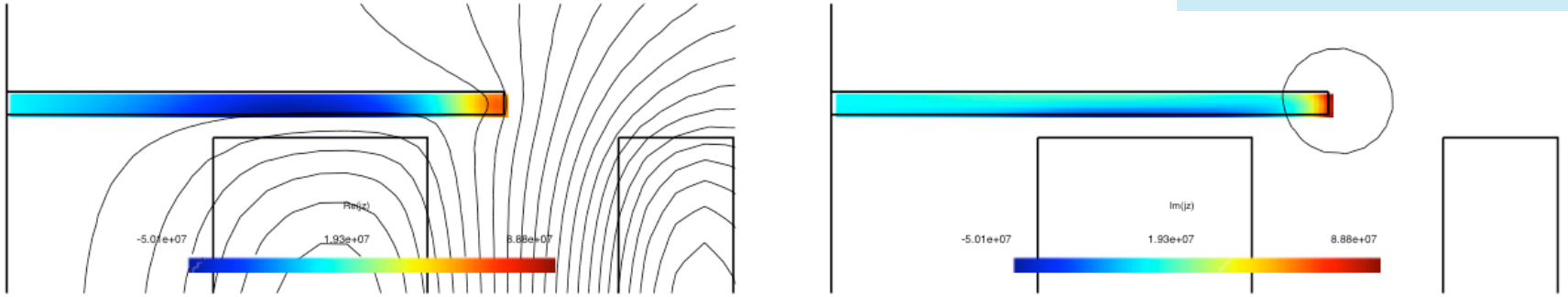


Fig. 2 Flux lines and induced current density in the plate, real (left) and imaginary (right) parts, with $f = 1\,\mathrm{kHz}$, $z_{\mathrm{pl}} = 3\,\mathrm{mm}$ and $\underline{\imath} = 20\,\mathrm{A}$

Results in [Sabariego2018]

Magnetic force computation – Virtual work

Principle used in conservative energy systems

It is based on the comparison of the energy balance between two different positions, supposing a virtual movement along the direction where the force is computed

$$\boldsymbol{F} = -\left.\frac{\partial W_{\text{mag}}}{\partial \boldsymbol{S}}\right|_{\Psi=cte} = \left.\frac{\partial W_{\text{mag,co}}}{\partial \boldsymbol{S}}\right|_{I=cte}$$

system with just 1 coil and just 1 generalised position DOF

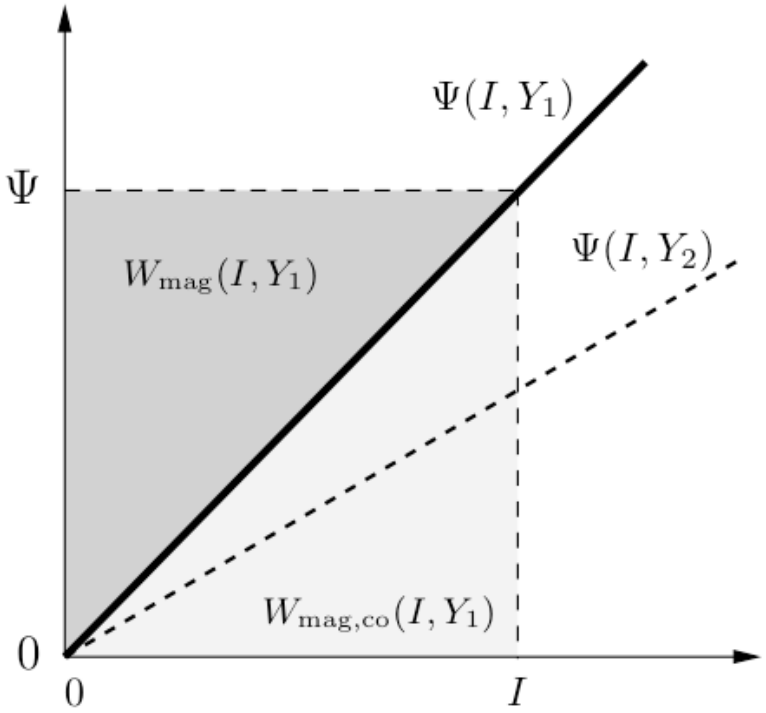
- linear coordinate Y (m) and force (N)
- angular coordinate θ (rad) and torque (Nm)
- …

$$F_Y(I,Y) = -\left.\frac{\partial W_{\text{mag}}}{\partial Y}\right|_{\Psi=cte} = \left.\frac{\partial W_{\text{mag,co}}}{\partial Y}\right|_{I=cte}$$

$$T_\theta(I,\theta) = -\left.\frac{\partial W_{\text{mag}}}{\partial \theta}\right|_{\Psi=cte} = \left.\frac{\partial W_{\text{mag,co}}}{\partial \theta}\right|_{I=cte}$$

Magnetic force computation
Virtual work – linear case

Linear case (no saturation, no PMs, no induced currents)



$$W_{\mathrm{mag}} = \int_{\Omega} w_{\mathrm{mag}} \mathrm{d}\Omega$$

FEA

$$\Psi(I,Y) = L(Y)I$$

$$W_{\mathrm{mag}} = W_{\mathrm{mag,co}} = \frac{1}{2}L(Y)I^2$$

$$F_Y(I,Y) = \left.\frac{\partial W_{\mathrm{mag}}}{\partial Y}\right|_{I=cte} = \frac{1}{2}I^2\frac{\mathrm{d}L}{\mathrm{d}Y}$$

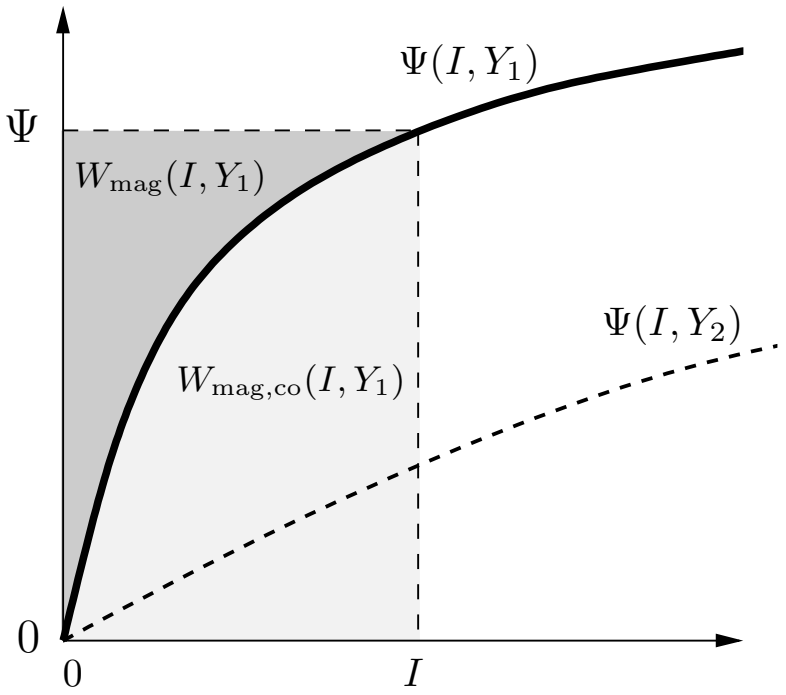
general force associated to the generalised position

Approximation by two independent FE solutions

$$F_Y(I,Y) = \left.\frac{\partial W_{\mathrm{mag}}}{\partial Y}\right|_{I=cte} \approx \frac{W_{\mathrm{mag}}(I,Y+\Delta Y/2) - W_{\mathrm{mag}}(I,Y-\Delta Y/2)}{\Delta Y}$$

Magnetic force computation
Virtual work – nonlinear case

Nonlinear case (saturation, no PMs, no induced currents)



$$W_{\mathrm{mag,co}} = \int_{\Omega} w_{\mathrm{mag,co}} \mathrm{d}\Omega$$ FEA

$$W_{\mathrm{mag}}(I, Y) = \int_0^{\Psi(I)} I \, \mathrm{d}\Psi'$$

$$W_{\mathrm{mag,co}}(I, Y) = \int_0^{I} \Psi(I', Y) \mathrm{d}I'$$

$$W_{\mathrm{mag}}(I, Y) + W_{\mathrm{mag,co}}(I, Y) = I\Psi(I, Y)$$

$$\begin{aligned} F_Y(I, Y) &= -\left.\frac{\partial W_{\mathrm{mag}}}{\partial Y}\right|_{\Psi=cte} \\ &= \left.\frac{\partial W_{\mathrm{mag,co}}}{\partial Y}\right|_{I=cte} \\ &\approx \frac{W_{\mathrm{mag,co}}(I, Y + \Delta Y/2) - W_{\mathrm{mag,co}}(I, Y - \Delta Y/2)}{\Delta Y} \end{aligned}$$

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Virtual work in trouble

- displacement should be enough for accuracy
- displacement should be large for stability

$$T_\theta = \left.\frac{\partial W_{\mathrm{mag,co}}}{\partial \theta}\right|_{I=\mathrm{cte}} \approx \frac{W_{\mathrm{mag,co}}(I, \theta + \Delta\theta/2) - W_{\mathrm{mag,co}}(I, \theta - \Delta\theta/2)}{\Delta\theta}$$

- typical procedure
  - compute $W_{\mathrm{mag,co}}(I, \theta_q)$ for many angle positions θ_q
  - smooth curve by e.g., FFT filtering

$$W_{\mathrm{mag,co}}(I, \theta_q) = \sum_\lambda \boldsymbol{c}_\lambda(I) e^{\imath\lambda\theta}, \qquad \text{omit all } \lambda > \lambda_{limit}$$

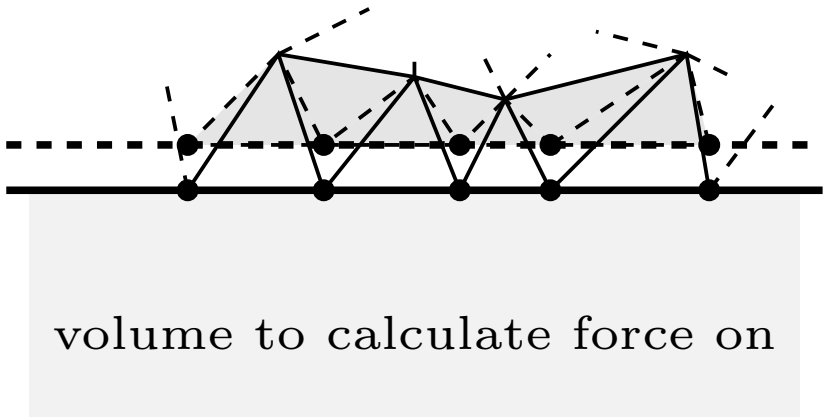
  - compute torque

$$T_\theta = \frac{\partial W_{\mathrm{mag,co}}(I, \theta)}{\partial \theta} = \sum_\lambda \imath\lambda \boldsymbol{c}_\lambda(I) e^{\imath\lambda\theta}$$

Virtual work implementation in GetDP

$$F_Y = -\frac{\partial W}{\partial Y}\bigg|_{\Psi=cte}$$

magnetic energy contained in layer of elements around the body



boundary moves virtually up

Maxwell stress tensor

- From Lorentz force we can write: $\boldsymbol{f} = \boldsymbol{j} \times \boldsymbol{b} = \operatorname{curl} \boldsymbol{h} \times \boldsymbol{b} = \nu \operatorname{curl} \boldsymbol{b} \times \boldsymbol{b}$
- developing this equation for e.g. the x component

$$f_x = \nu \left(b_z \frac{\partial b_x}{\partial z} - b_z \frac{\partial b_z}{\partial x} - b_y \frac{\partial b_y}{\partial x} + b_y \frac{\partial b_x}{\partial y} \right)$$

- adding and subtracting $\nu b_x \frac{\partial b_x}{\partial x}$ and considering $\frac{\partial}{\partial x}(b_x^2) = 2 b_x \frac{\partial b_x}{\partial x}$

$$\begin{aligned} f_x &= \nu \left(\frac{1}{2} \frac{\partial}{\partial x}(b_x^2) + b_z \frac{\partial b_x}{\partial z} + b_y \frac{\partial b_x}{\partial y} - \frac{1}{2} \frac{\partial}{\partial x}(b_x^2 + b_y^2 + b_z^2) \right) \\ &= \nu \left(\frac{\partial}{\partial x}(b_x^2 - \frac{1}{2}|\boldsymbol{b}|^2) + \frac{\partial b_x b_z}{\partial z} + \frac{\partial b_x b_y}{\partial y} - b_x \operatorname{div} \boldsymbol{b} \right) \\ &= \nu \left(\frac{\partial}{\partial x}(b_x^2 - \frac{1}{2}|\boldsymbol{b}|^2) + \frac{\partial b_x b_z}{\partial z} + \frac{\partial b_x b_y}{\partial y} \right) \end{aligned}$$

$\operatorname{div} \boldsymbol{b} = 0$

Maxwell stress tensor (cont'd)

- The remaining expression

$$f_x = \nu \left(\frac{\partial}{\partial x}(b_x^2 - \frac{1}{2}|\boldsymbol{b}|^2) + \frac{\partial b_x b_z}{\partial z} + \frac{\partial b_x b_y}{\partial y} \right)$$

- may be recognized as the divergence of a vector with components

$$\operatorname{div} T_x = \begin{cases} \dfrac{\partial T_x}{\partial x} = f_{xx} = \nu \left(b_x^2 - \dfrac{1}{2}|\boldsymbol{b}|^2 \right) \\ \dfrac{\partial T_x}{\partial y} = f_{xy} = \nu(b_x b_y) \\ \dfrac{\partial T_x}{\partial z} = f_{xz} = \nu(b_x b_z) \end{cases}$$

- a similar development can be performed for the other components of the force

Maxwell stress tensor (cont'd)

Maxwell stress tensor $\boldsymbol{T} = \frac{1}{\mu_0}\boldsymbol{b}\boldsymbol{b} - \frac{1}{2\mu_0}|\boldsymbol{b}|^2\mathbb{I}$ (N/m³)

$$\boldsymbol{T} = \nu \begin{bmatrix} b_x^2 - \frac{1}{2}|\boldsymbol{b}|^2 & b_x b_y & b_x b_z \\ b_y b_x & b_y^2 - \frac{1}{2}|\boldsymbol{b}|^2 & b_y b_z \\ b_z b_x & b_z b_y & b_z^2 - \frac{1}{2}|\boldsymbol{b}|^2 \end{bmatrix}$$

force density $\boldsymbol{f} = \operatorname{div} \boldsymbol{T}$

force $\boldsymbol{F} = \int_V \operatorname{div} \boldsymbol{T}\, \mathrm{d}V = \int_S \boldsymbol{T} \cdot \boldsymbol{n}\, \mathrm{d}S$

Using the divergence theorem,
the volume integral is reduced to a surface integral

$\boldsymbol{n}$ outward normal to boundary $S = \partial V$

Maxwell stress tensor

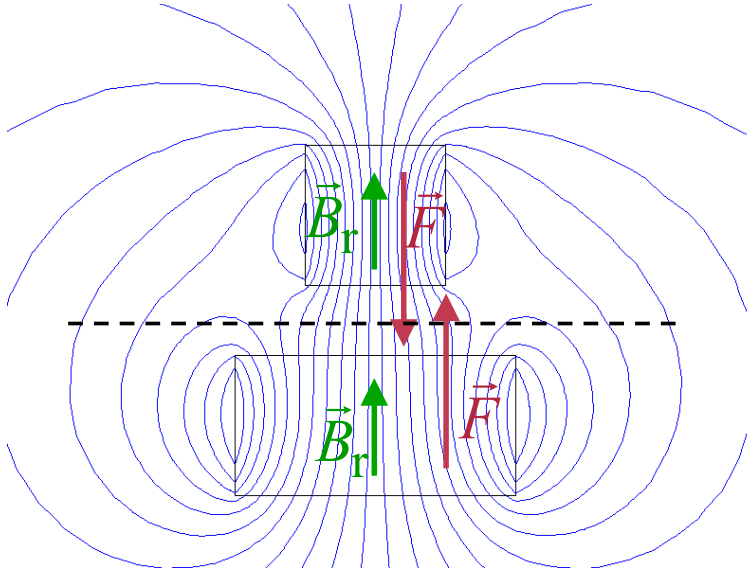
$$\boldsymbol{F} = \int_V \operatorname{div} \boldsymbol{T}\, \mathrm{d}V = \int_S \boldsymbol{T} \cdot \boldsymbol{n}\, \mathrm{d}S$$

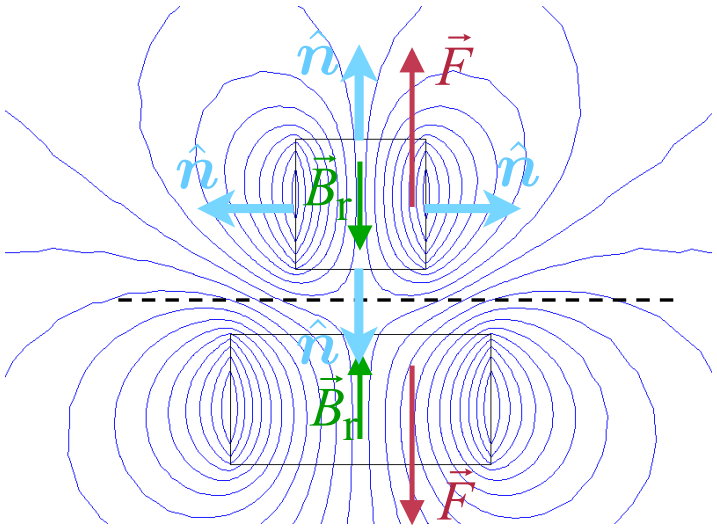
- force on a plane
- normal force

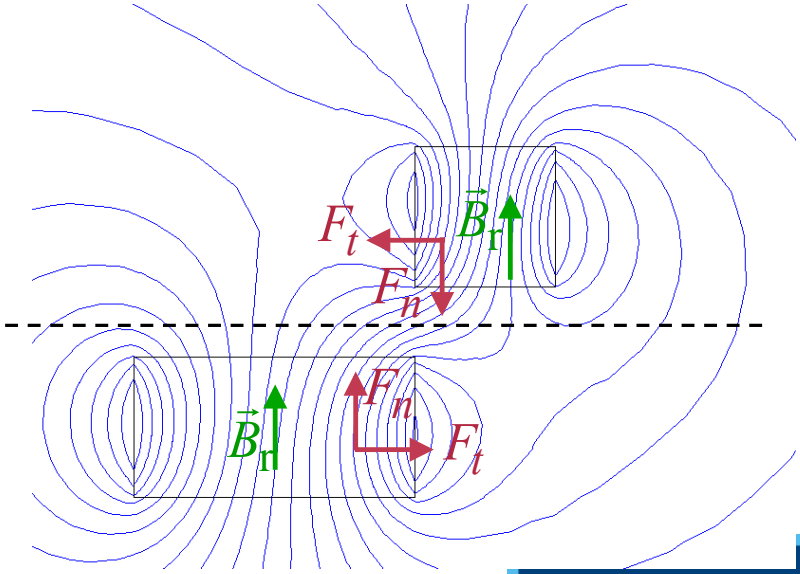
$$F_n = \int_S \frac{1}{2} \left(b_n h_n - \boldsymbol{b}_t \cdot \boldsymbol{h}_t \right) \mathrm{d}S$$

- tangential force

$$\boldsymbol{F}_t = \int_S b_n \boldsymbol{h}_t \, \mathrm{d}S$$







Maxwell stress tensor vs. virtual work

- MST needs only FE solution
- Loss of accuracy (based on derived values)
  - Introduction of a contour around domain of interest -> additional error source
  - Some round-off errors in calculation of contour coordinates
  - Problem when normal and tangential components of the local flux densities differ by orders of magnitude -> Integration may involve the subtraction of large numbers
- Accuracy improved by
  - smoothing algorithms (e.g. averaging of results on different contours)
  - different integration schemes

Eggshell method/Arkkio's method
=> Averaging Maxwell stress tensor

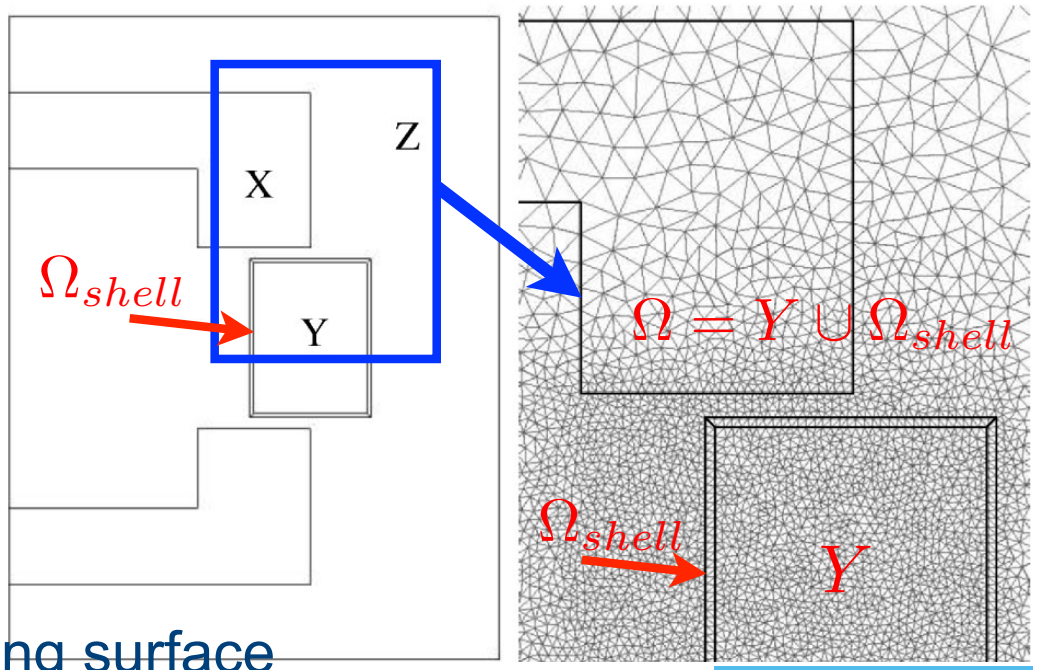
- remind that the aim of force computation is to determine the velocity/displacement of the moving parts
- eggshell region enclosing moving piece, non constant thickness $\Omega_{shell} = \Omega - Y$
- set velocity to zero in Z, confining the non-zero velocity in the shell, and in Y
- (virtual) velocity associated with displacement

$$\boldsymbol{v} = \gamma \partial_t \boldsymbol{u}\,, \quad \operatorname{grad} \boldsymbol{v} = \operatorname{grad} \gamma\, \partial_t \boldsymbol{u}$$

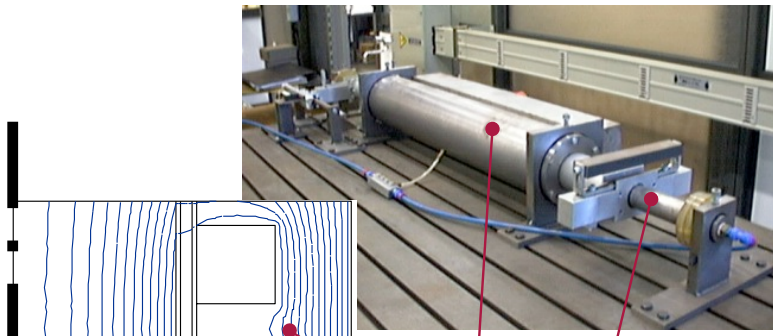
- γ is any smooth function,
  = 1 on inner shell surface
  = 0 on outer shell surface

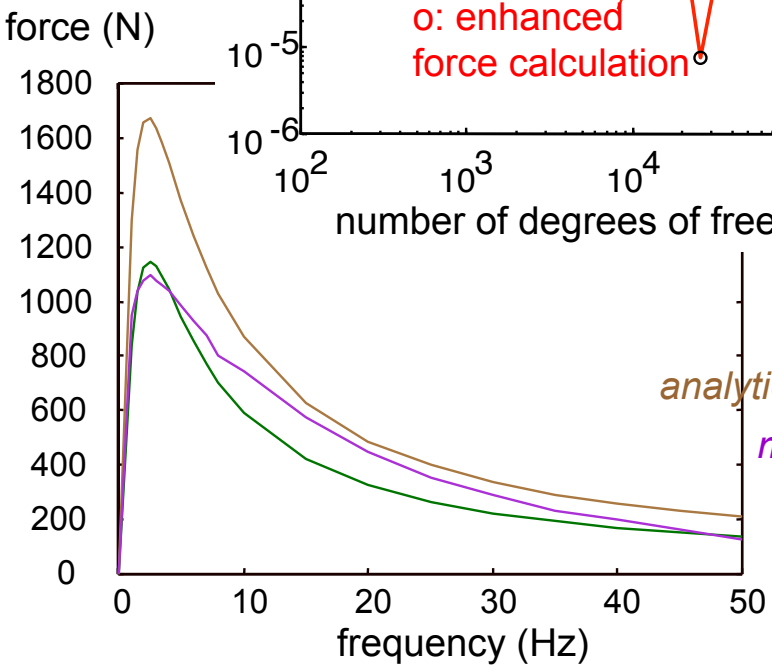
$$\boldsymbol{F} = \int_{\Omega_{shell}} \boldsymbol{T} \cdot \operatorname{grad} \gamma \, \mathrm{d}\Omega_{shell}$$

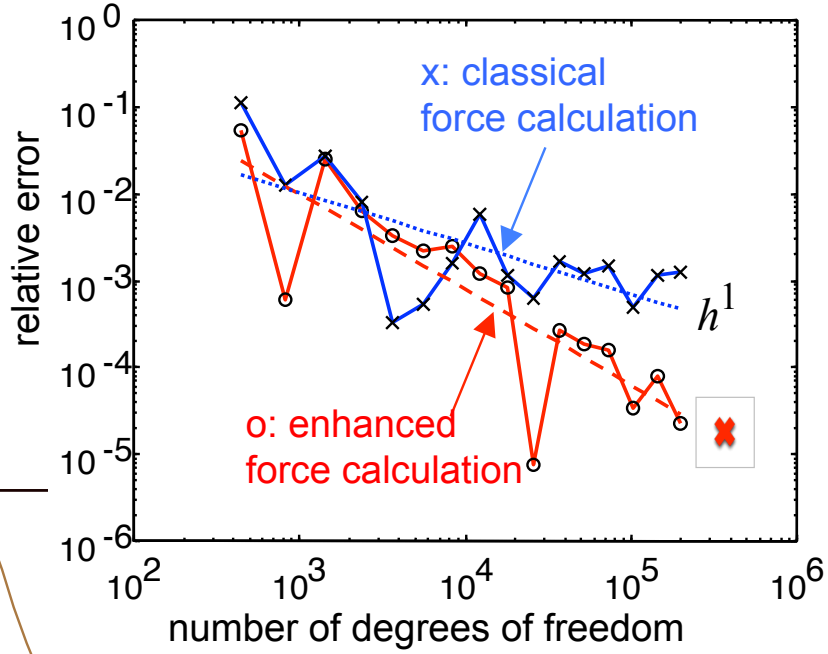
- integration of Maxwell stress tensor on an enclosing surface



Linear actuator







Force computation in GetDP
Maxwell stress tensor

$$\boldsymbol{T} = \nu \begin{bmatrix} b_x^2 - \frac{1}{2}|\boldsymbol{b}|^2 & b_x b_y & b_x b_z \\ b_y b_x & b_y^2 - \frac{1}{2}|\boldsymbol{b}|^2 & b_y b_z \\ b_z b_x & b_z b_y & b_z^2 - \frac{1}{2}|\boldsymbol{b}|^2 \end{bmatrix}$$

- PostProcessing $1= *b*

```
Tmax[] = (SquDyadicProduct[$1]-0.5*SquNorm[$1]*TensorDiag[1,1,1])/mu0 ;
{ Name F; Value { Integral {
        [ SymmetryFactor * Tmax[{d a}] * {d un} ] ; In DomainAirlayer ; Jacobian Vol ; Integration II ; } } }
 { Name Fy ; Value { Term { Type Global; [ CompY[$fmag] ] ; In DomainDummy ; } } }
// Magnetic force stored in run time variable $fmag, keeping just the Y component
// {d un} = {Grad un} => layer with arbitrary thickness, use smooth function un=y
```

- PostOperation

```
Print[ F[DomainAirlayer], OnGlobal, Format TimeTable, File  Fmax.dat , StoreInVariable $fmag, LastTimeStepOnly ];
// Save result of PostOperation in variable $fmag
 Print[ Fy, OnRegion DomainDummy, Format TimeTable, LastTimeStepOnly,
    SendToServer  Force MST [N] , Color  LightYellow ]; // If translational movement, only the vertical component
```

Losses

Joule losses, iron losses, dielectric losses...

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Electromagnetic power

- Poynting vector $\boldsymbol{s} = \boldsymbol{e} \times \boldsymbol{h}$

- Power exchanged with a volume (interior normal)

$$P = \oint_{\partial V} \boldsymbol{s} \cdot \boldsymbol{n} \, \mathrm{d}s = -\int_V \operatorname{div} \boldsymbol{s} \, \mathrm{d}v = \int_V p \, \mathrm{d}v$$

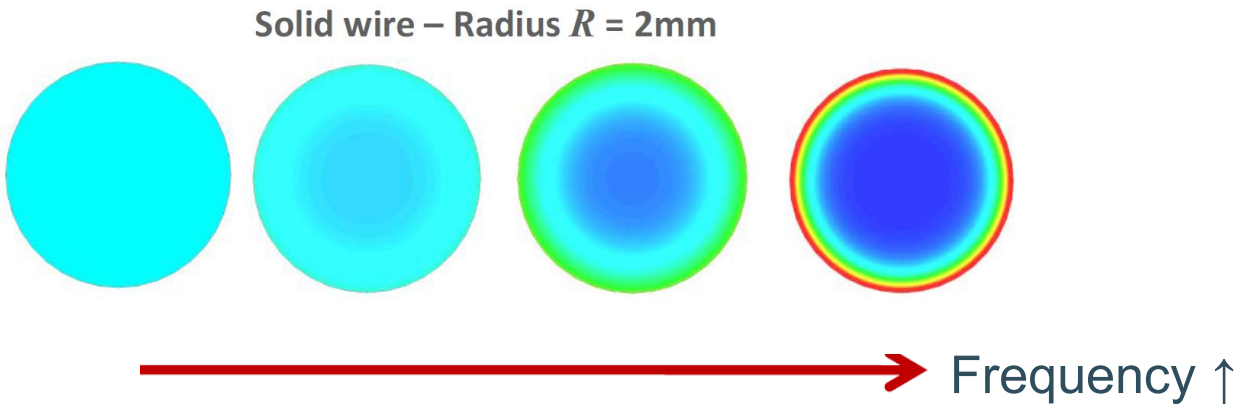
- Power density

$$p = -\operatorname{div} \boldsymbol{e} \times \boldsymbol{h} = -\boldsymbol{h} \cdot \operatorname{curl} \boldsymbol{e} + \boldsymbol{e} \cdot \operatorname{curl} \boldsymbol{h}$$
$$\Rightarrow p = \boldsymbol{h} \cdot \partial_t \boldsymbol{b} + \boldsymbol{e} \cdot \boldsymbol{j} + \boldsymbol{e} \cdot \partial_t \boldsymbol{d}$$

magnetic conduction electric

Eddy current/Joule losses

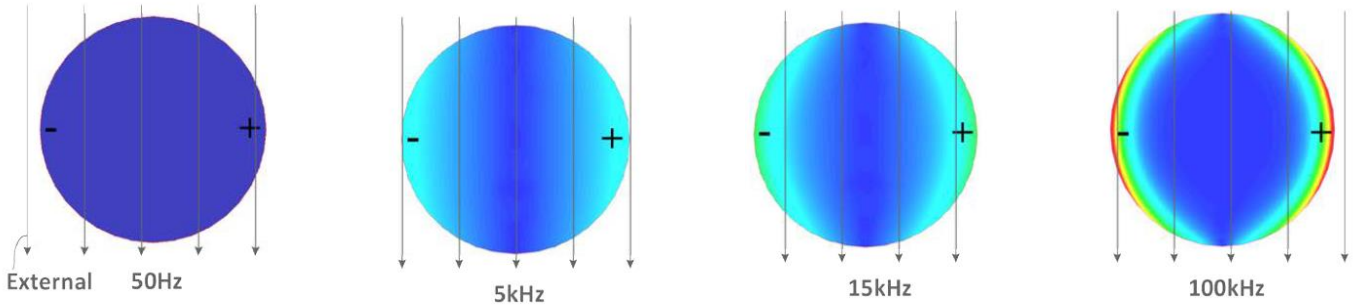
- In a conducting material submitted to time-varying field, loops of induced currents appear, i.e. eddy currents appear
- Skin effects



Current tends to flow at the surface due to skin & proximity effects

The higher the frequency, the higher the AC resistance

- Proximity effect



Eddy current/Joule losses

- In a conducting material submitted to time-varying field, loops of induced currents appear, i.e. eddy currents appear
- Skin and proximity effects
- Eddy currents in conductors generate heat and EM forces
  - Positive effects:
    - heat may be used for induction heating
    - EM forces may be used for levitation creating movement or for braking (regenerative)
  - Undesirable effects:
    - power loss in transformers (laminated cores for reducing the eddy currents)
    - hot spots in insulations

$$P_{\text{Joule}} = \frac{1}{\text{Vol}(\Omega_c)} \int_{\Omega_c} \boldsymbol{e} \cdot \boldsymbol{j} \, \mathrm{d}\Omega = \frac{1}{\text{Vol}(\Omega_c)} \int_{\Omega_c} \frac{\boldsymbol{j}^2}{\sigma} \, \mathrm{d}\Omega$$

Iron losses

In general, the magnetic losses are classified in

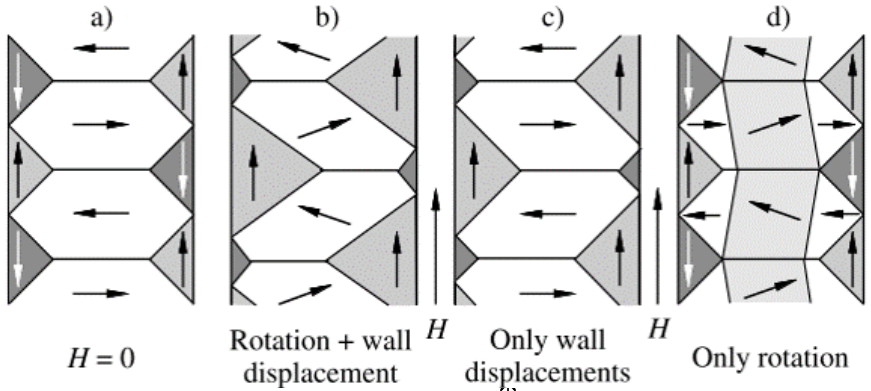
- Hysteresis losses

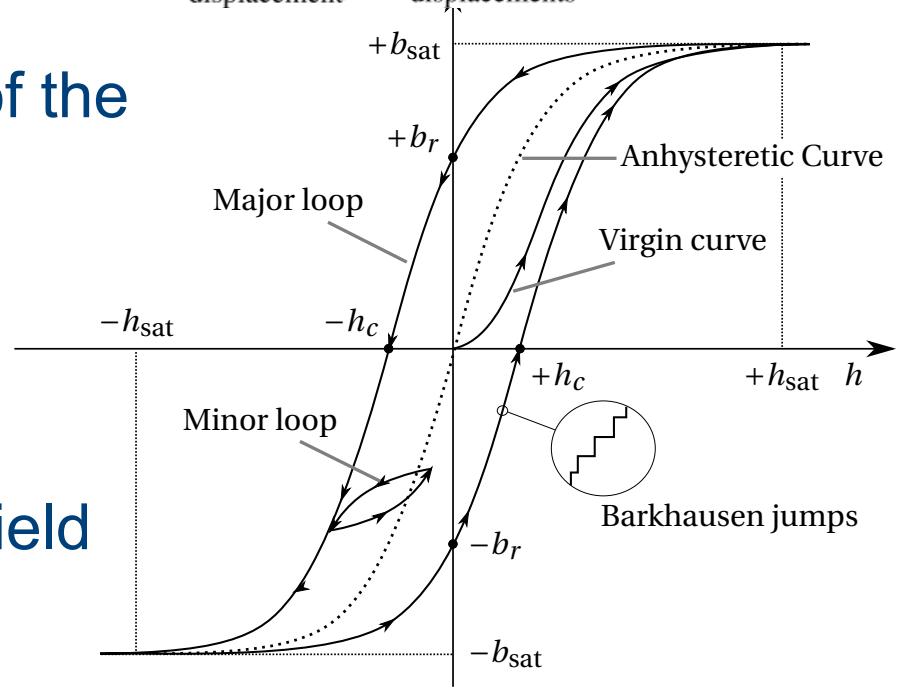
  discontinuous character at the microscopic scale of the magnetization process (energy loss by each Barkhausen jump)

- Dynamic losses
  - Eddy current losses

    conducting material submitted to time varying field
  - Excess losses or anomalous losses

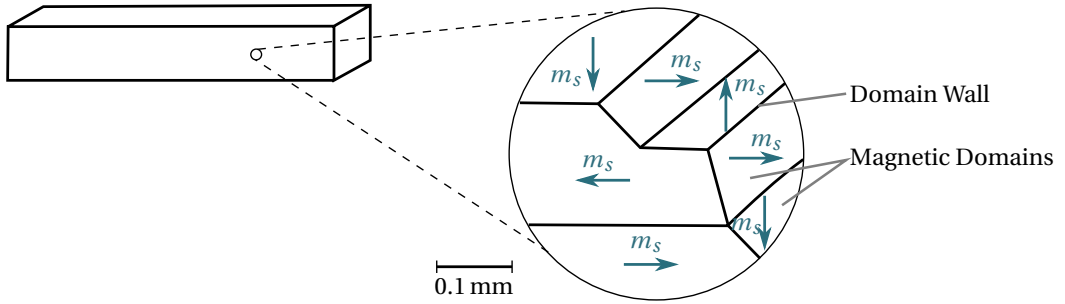
    due to the existence of magnetic domains, that enhance eddy currents in the proximities of domain walls, magnetic skin effect





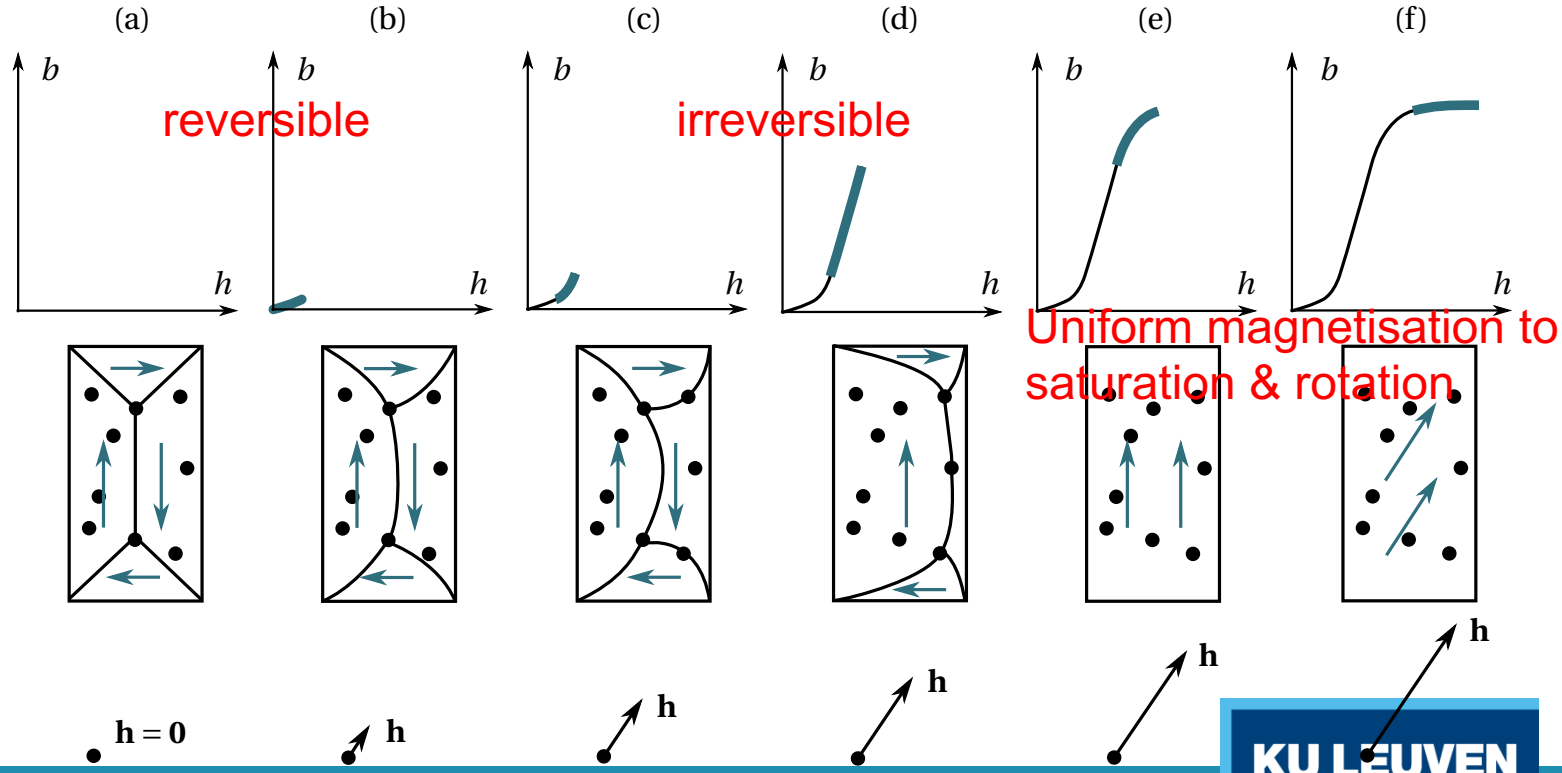
Magnetic domains and bh-curve

- Subdivision of material in magnetic domains



- Evolution of magnetisation

Displacement of the walls of magnetic domains



Minor hysteresis loops

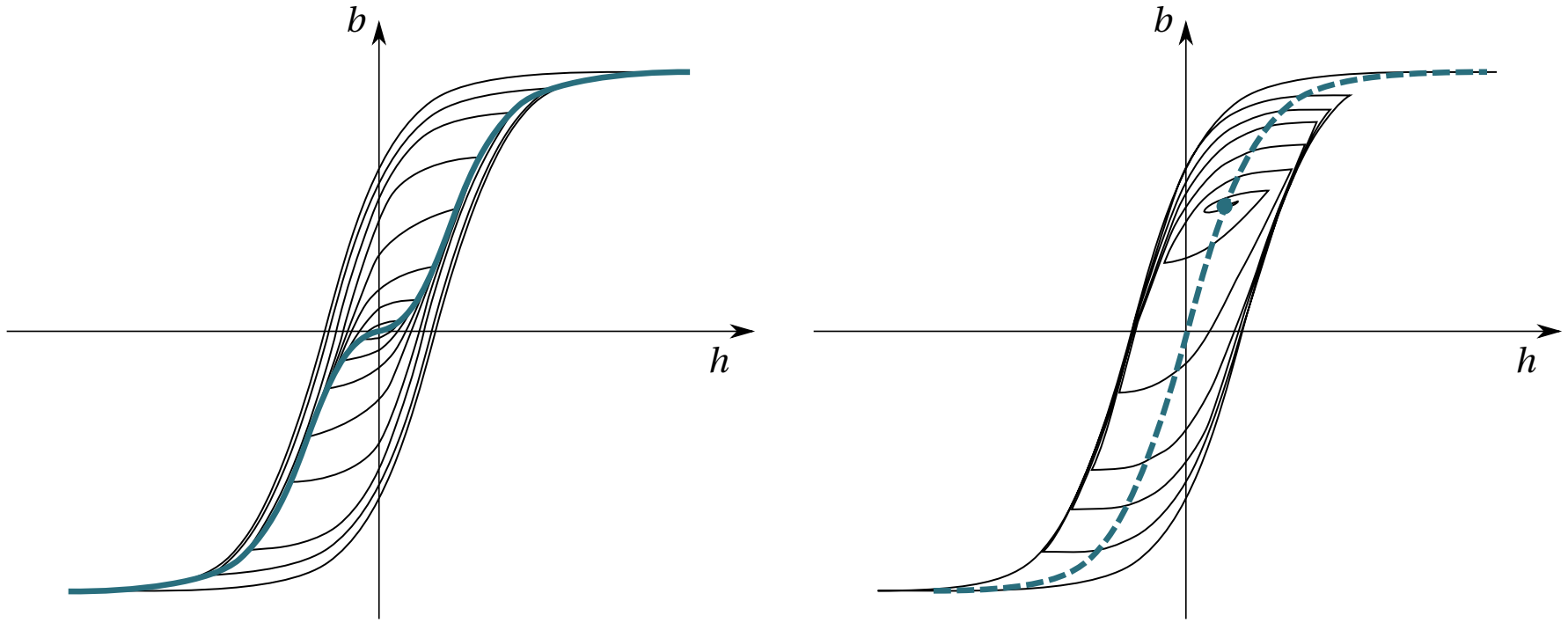


Figure 1.6: *(left): Set of symmetric minor hysteresis loops with the reversal points connections on the virgin curve, (right): Path history generated by a field with oscillating amplitude, slowly decreasing from infinity to zero, around a constant bias field value. Other final points can be reached by repeating the operation under different bias values. The locus of all these final points constitutes the anhysteretic curve (in dashed line).*

Hysteresis losses

under periodic conditions with sinusoidal source

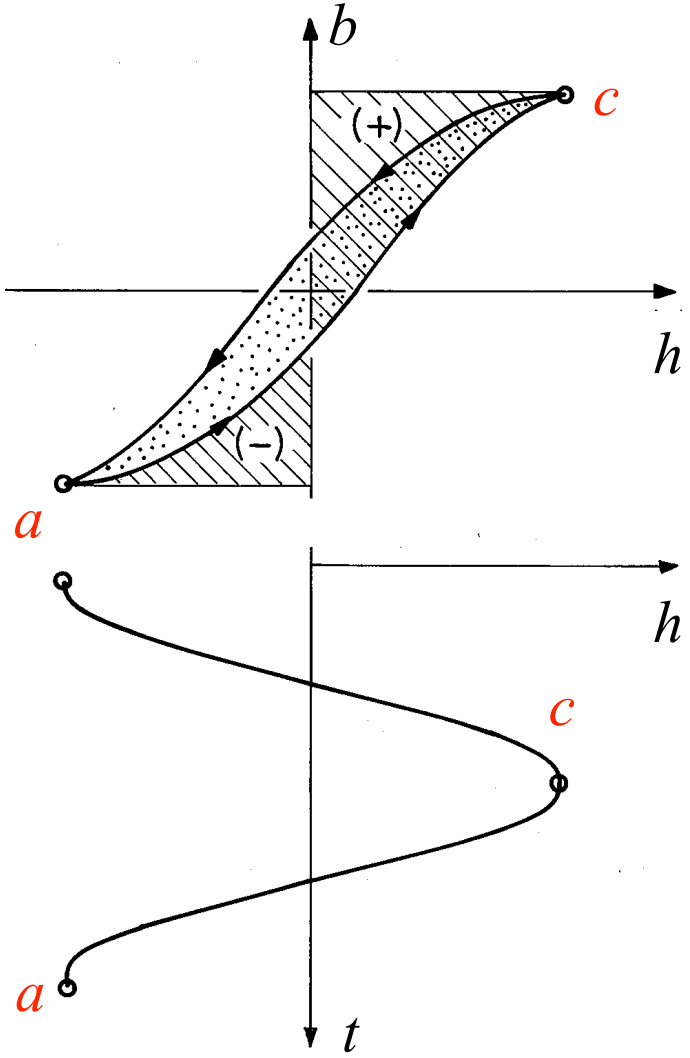
$$\int_0^{\frac{T}{2}} \boldsymbol{h} \cdot \partial_t \boldsymbol{b} \, \mathrm{d}t = \int_{b_a}^{b_c} \partial_t W_m \, \mathrm{d}t = \int_{b_a}^{b_c} \boldsymbol{h}_+ \, \mathrm{d}\boldsymbol{b}$$

$$\int_{\frac{T}{2}}^{T} \boldsymbol{h} \cdot \partial_t \boldsymbol{b} \, \mathrm{d}t = \int_{b_c}^{b_a} \boldsymbol{h}_- \, \mathrm{d}\boldsymbol{b} = -\int_{b_a}^{b_c} \boldsymbol{h}_- \, \mathrm{d}\boldsymbol{b}$$

time average power dissipation:

$$P_{hys} = \frac{1}{T}\left[\int_{b_a}^{b_c} \boldsymbol{h}_+ \, \mathrm{d}\boldsymbol{b} - \int_{b_a}^{b_c} \boldsymbol{h}_- \, \mathrm{d}\boldsymbol{b}\right]$$

the area within the hysteresis loop is the energy dissipated in one cycle



Dynamic hysteresis loops

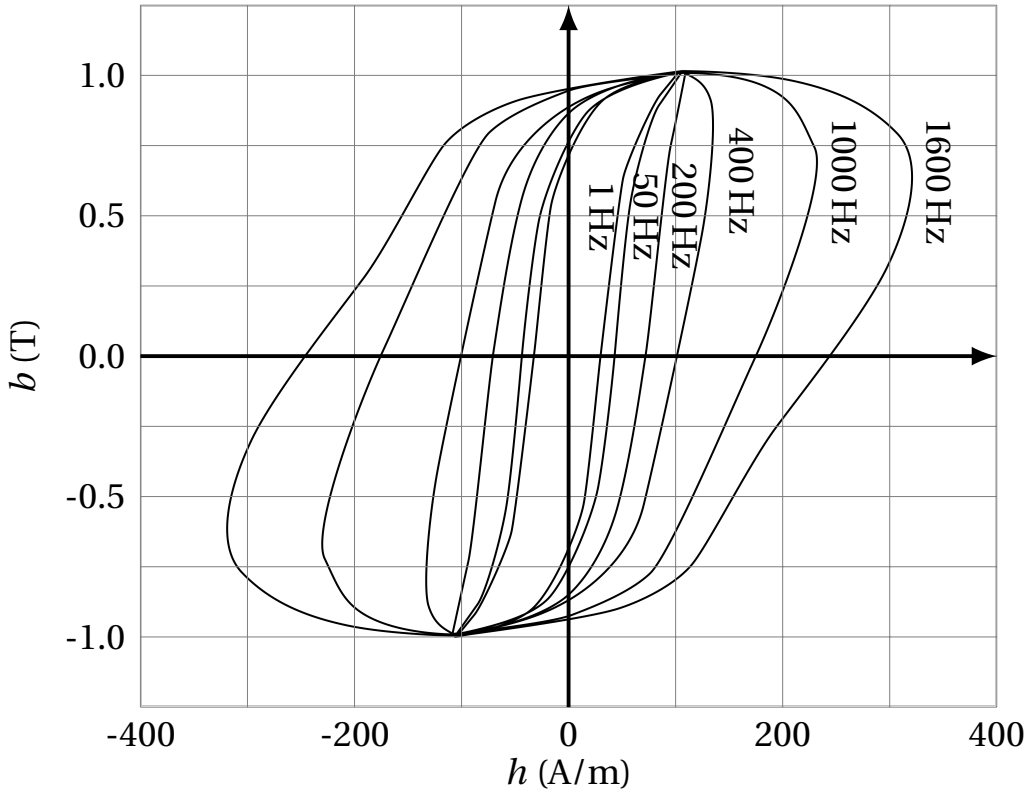


Figure 1.8: *Dynamic influence on the hysteresis loops measured on a specimen of a non-oriented iron-silicon alloy under sinusoidal induction of* 1 T *amplitude at different frequencies [26].*

Hysteresis losses and field simulation

- as a post-processing step

  e.g. by Steinmetz formula $p_{\mathrm{hyst}} = \sigma_{\mathrm{hyst}} k_{\mathrm{hyst}} \frac{f}{50\,\mathrm{Hz}} \left(\frac{|\boldsymbol{b}|}{1\,\mathrm{T}} \right)^2$

- within a time-harmonic field simulation

  complex permeability

  $b(t) = \hat{b}\cos(\omega t)$

  $h(t) = \hat{h}\cos(\omega t - \varphi)$

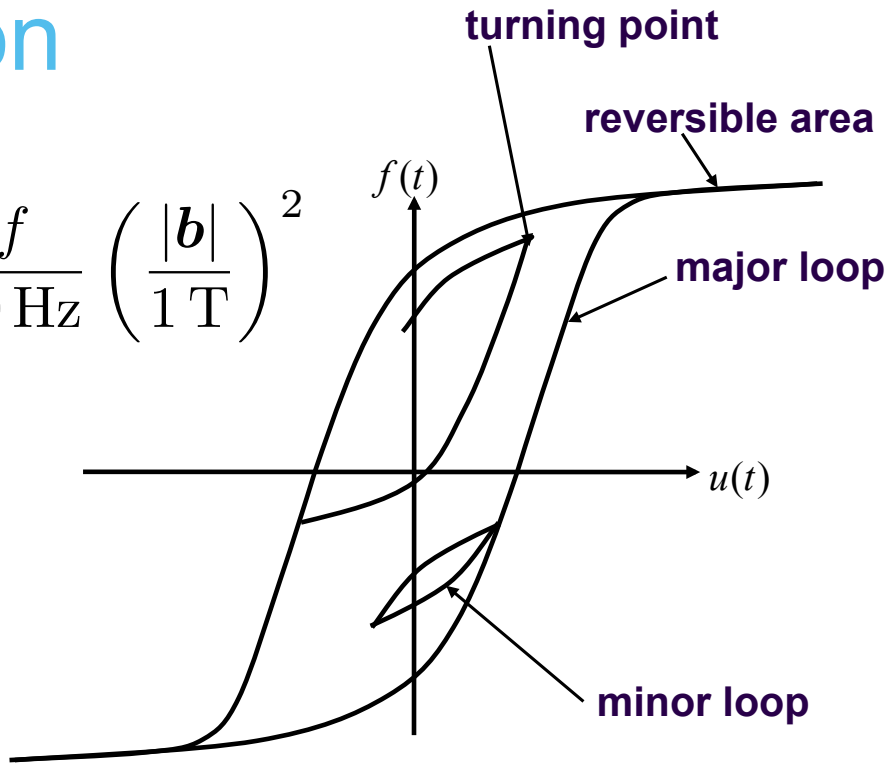
  $\boldsymbol{\mu} = \frac{\hat{b}}{\hat{h}} e^{\imath\varphi}$

- within a transient simulation by a hysteresis model

  Preisach model (purely mathematical, not limited to EM)

  Jiles-Atherton model (inspired by physics of material)

  Energy-based models (1st and 2nd law of thermodynamics to represent hysteretic behaviour)



Excess or anomalous losses

- magnetic “skin effect” is neglected => dynamic losses are bigger than so-called classical losses

$$P_{\mathrm{ex}}(t) = P_{\mathrm{dyn}}(t) - P_{\mathrm{cl}}(t)$$

- magnetization process represented as a distribution of coercive fields locally associated with magnetic objects
- if we define an excess field as part of the external field applied to compensate the field created by the movement of the magnetic objects, the excess losses can be calculated by means

$$P_{\mathrm{ex}} = \sqrt{\sigma G V_o}\frac{1}{T}\int_0^T \left|\frac{\partial b(t)}{\partial t}\right|^{1.5} \mathrm{d}t$$

- friction coefficient G is the constant of proportionality between the excess field and the rate of change of magnetic flux

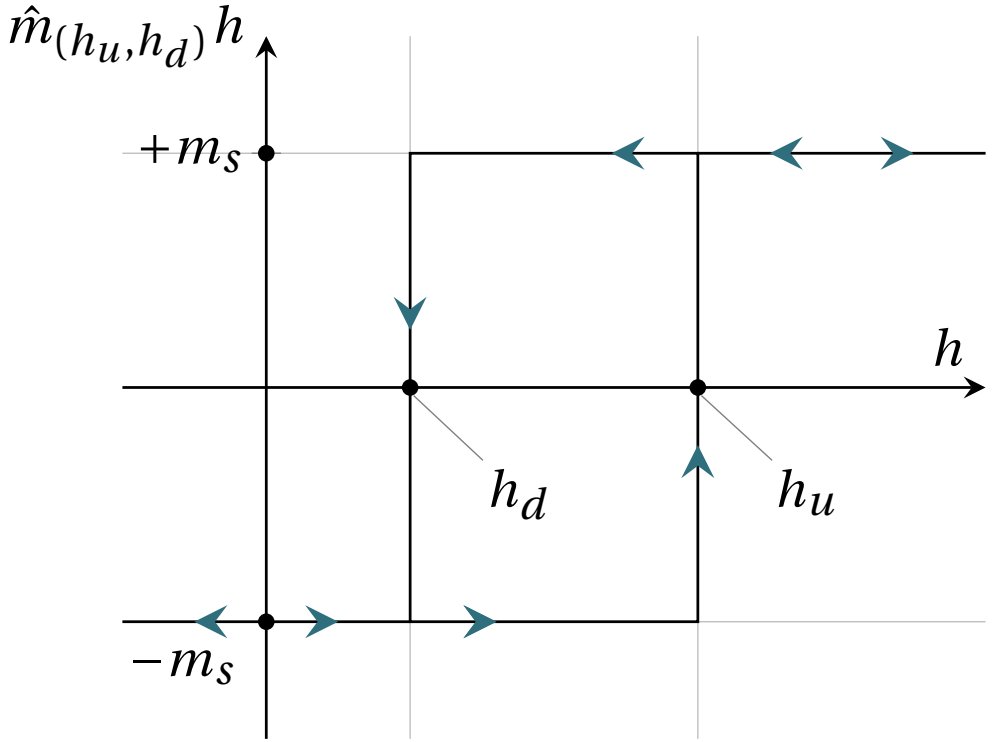
$$H_{\mathrm{ex}} = G\sigma\frac{\partial \Phi(t)}{\partial t}$$

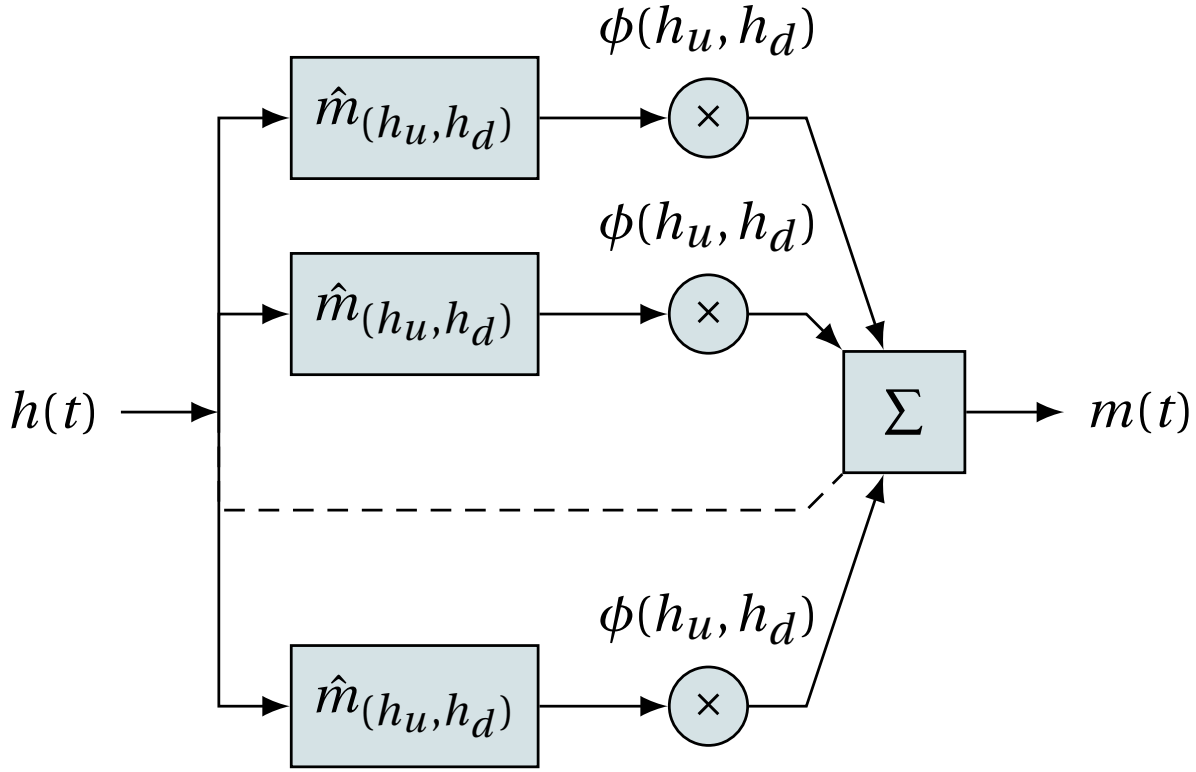
- parameter V_0 is related to the number of magnetic objects and the excess field, linear relation (experimental)

$$H_{\mathrm{ex}} = n(t)V_0$$

Preisach model

$$m(t) = \int\int_{h_u \geq h_d} \phi(h_u, h_d)\, \hat{m}_{(h_u,h_d)}\, h(t)\, dh_u dh_d \quad (*)$$





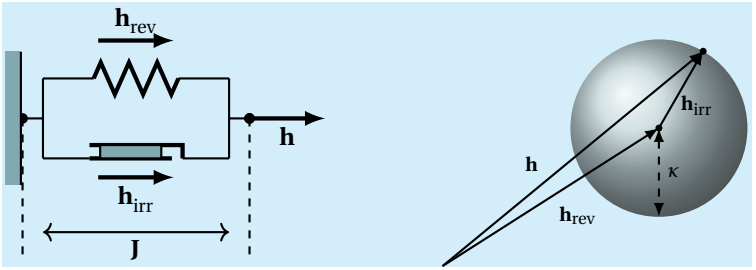
find $\boldsymbol{a}^*$ such that

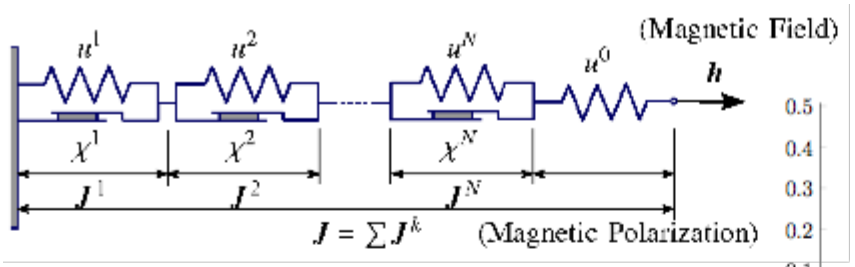
$$\int_\Gamma (\nu \operatorname{curl} \boldsymbol{a}^* \times \boldsymbol{w}_i)\, \boldsymbol{n} \mathrm{d}\Gamma + \int_\Omega \left(\nu \operatorname{curl} \boldsymbol{a}^* \cdot \operatorname{curl} \boldsymbol{w}_i + \sigma \partial_t \boldsymbol{a}^* \cdot \boldsymbol{w}_i\right) \mathrm{d}\Omega = \int_\Omega \boldsymbol{j}_s \cdot \boldsymbol{w}_i \, \mathrm{d}\Omega$$

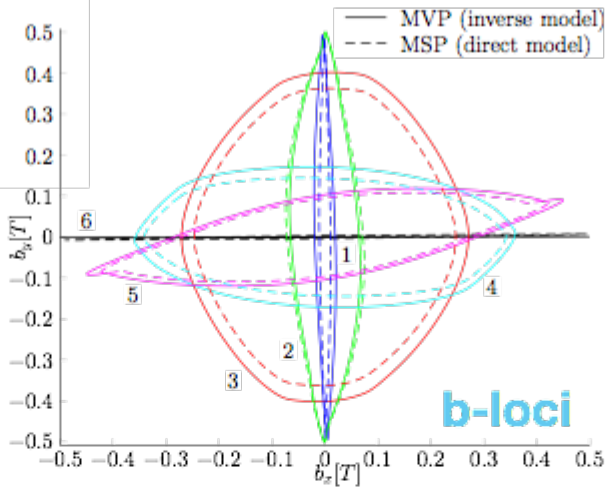
holds $\forall \boldsymbol{w}_i(\boldsymbol{x})$

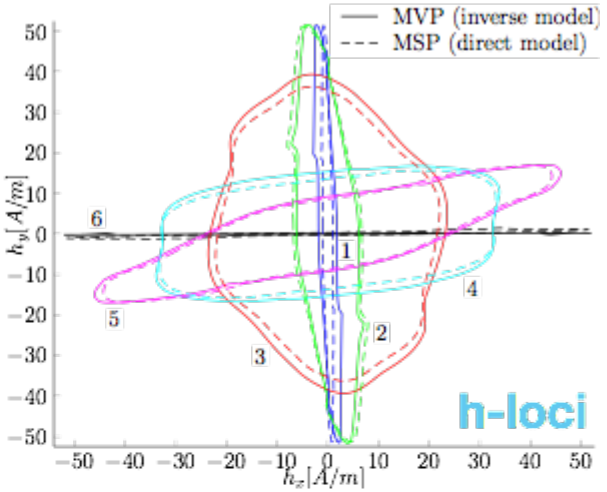
$\boldsymbol{h}(\operatorname{curl} \boldsymbol{a}^*)$ Material law is hysteretic

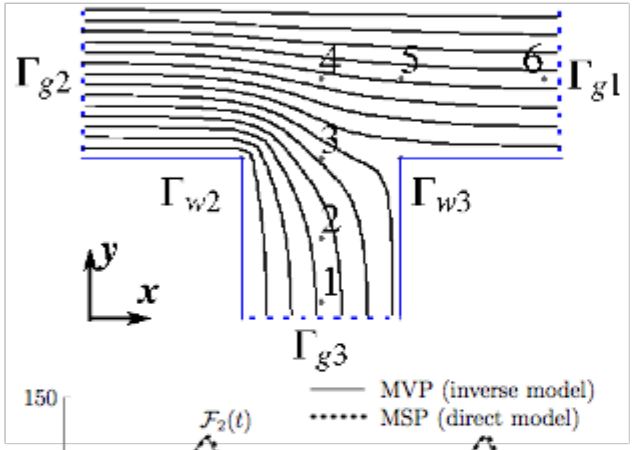
Energy consistent hysteresis model

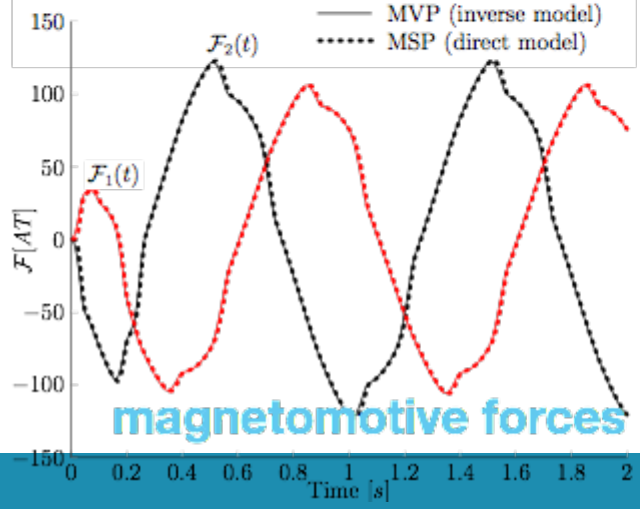


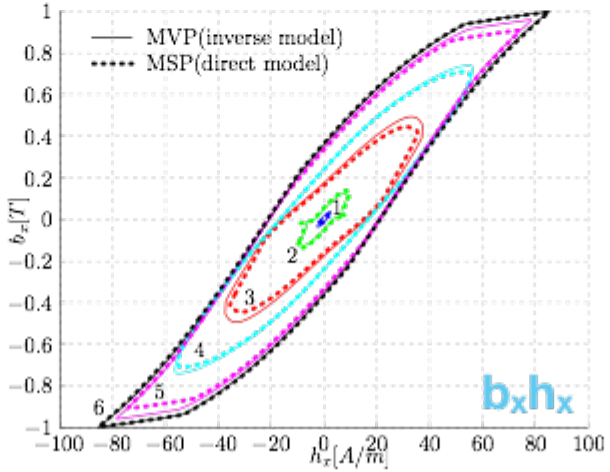


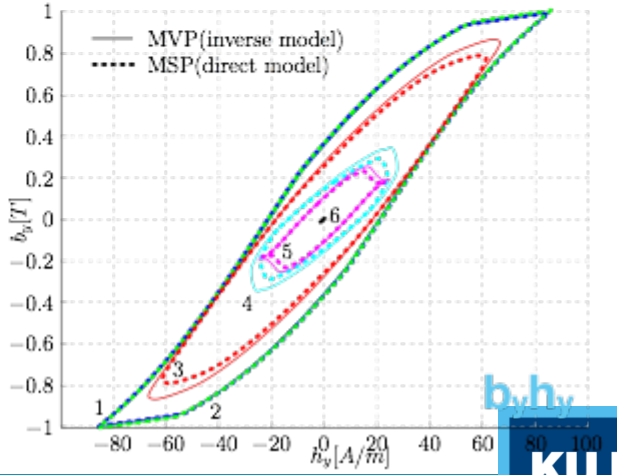






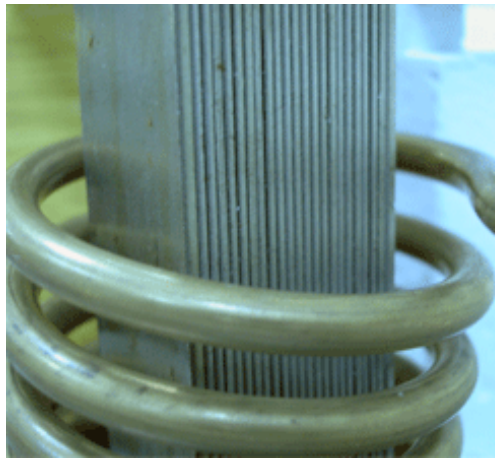


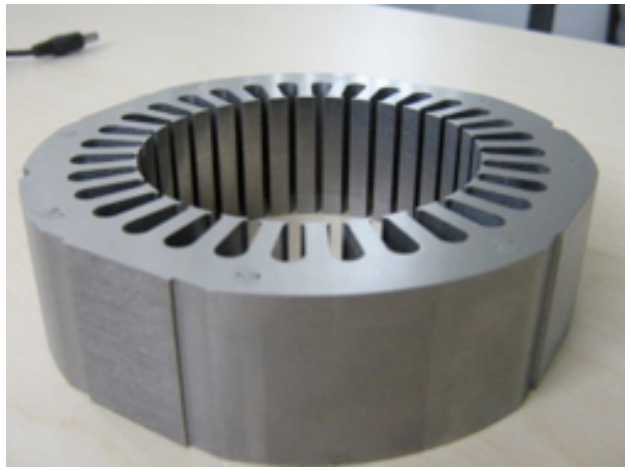


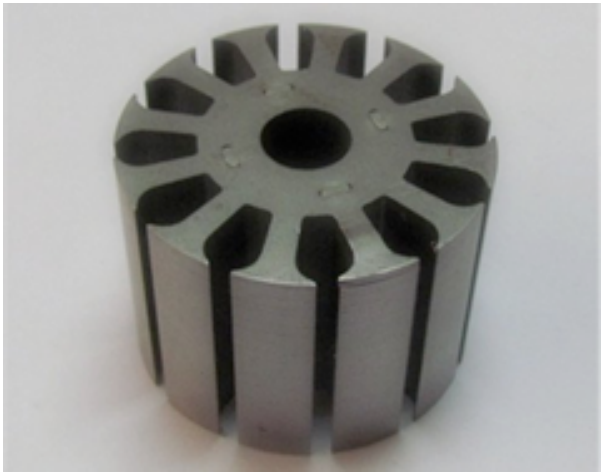


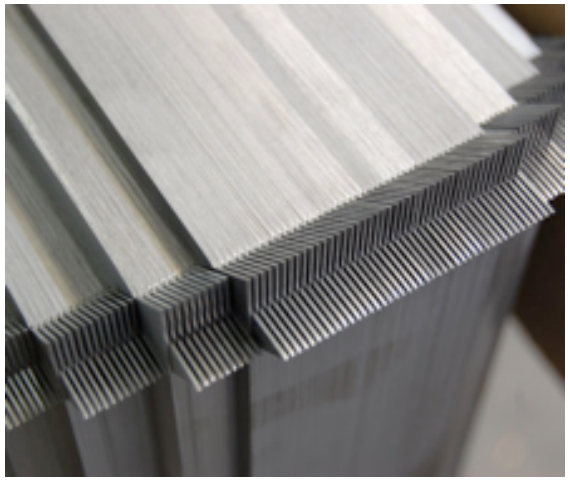
KU LEUVEN

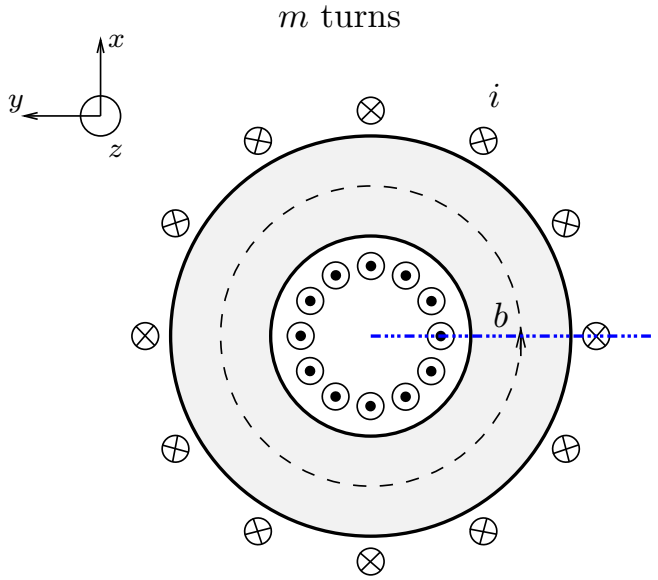
Eddy-current effects in laminated iron cores

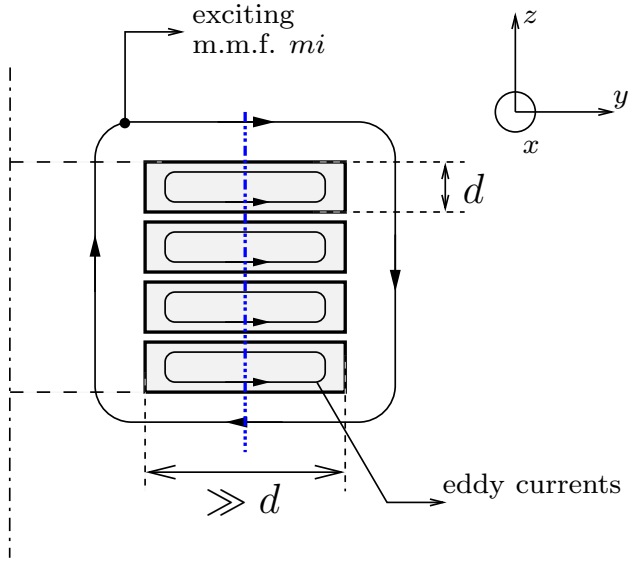








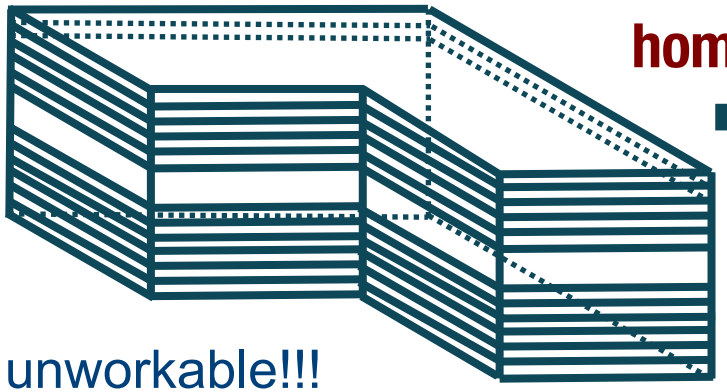




laminated core to reduce eddy current losses due to time-varying flux excitation

Modeling eddy currents in laminated iron cores

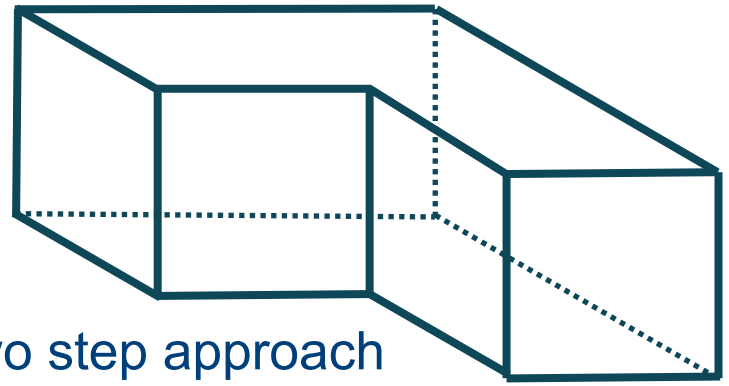
stack as a set of massive conductors with insulating layers



unworkable!!!

homogenization

stack as a single homogeneous non-conducting material



two step approach
a posteriori estimation of eddy current losses and hysteresis losses

- two step approach OK provided that the eddy current effect is negligible
- iron loss directly included in the constitutive law

$$h_{dyn}(b, \frac{db}{dt}) = h_{stat}(b) + \frac{\sigma d^2}{12}\frac{db}{dt} \quad (+ \text{ excess term})$$

- flux parallel to lamination + excess term
- flux perpendicular to laminations via anisotropic conductivity tensor

Eddy current losses in laminated cores
Classical losses

- In a lamination of thickness *d*, under a periodic field of period *T*, these losses are given by

$$P_{\mathrm{cl}} = \frac{\sigma d^2}{12}\frac{1}{T}\mathrm{Vol}(\Omega_l)\int_0^T \left(\frac{\partial b(t)}{\partial t}\right)^2 \mathrm{d}t$$

- For non-sinusoidal fluxes, these losses can be calculated by using a Fourier expansion of the induction and summing the effects of all the harmonics:

$$P_{\mathrm{cl}} = \frac{\sigma \pi^2 f^2 d^2}{6}\sum_k k^2 b_k^2$$

$\mathrm{Vol}(\Omega_l) =$ volume of the lamination

$b_k =$ amplitude of the kth harmonic

Dielectric loss

- dielectric constant => storage of electric energy in material (via polarization)

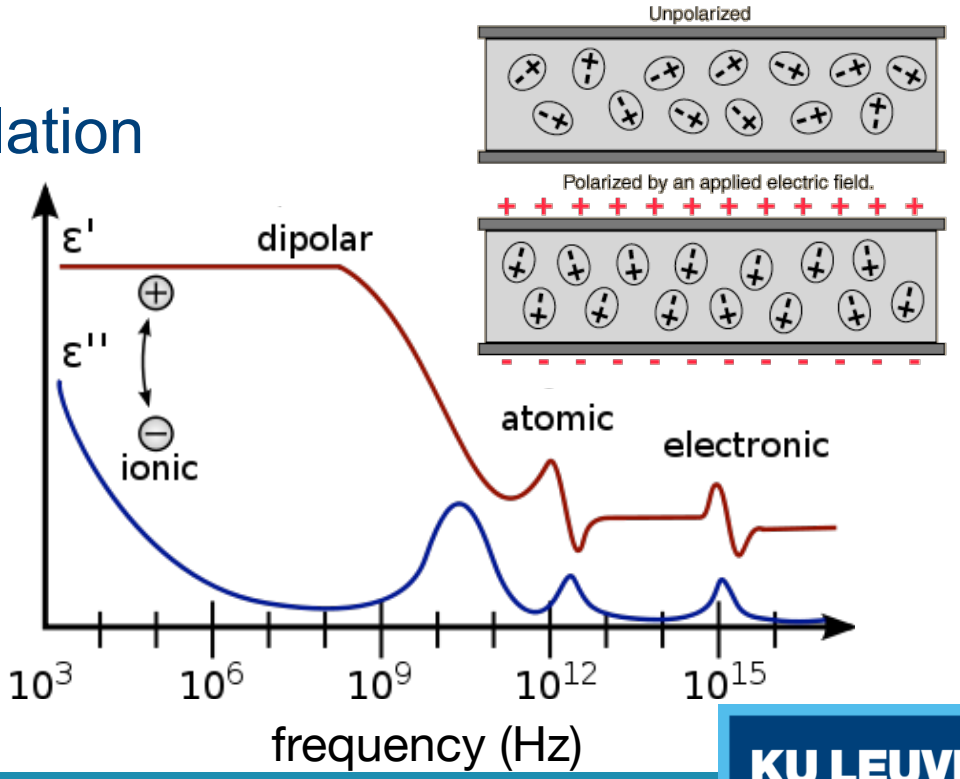
$$\boldsymbol{d} = \epsilon_0 \boldsymbol{e} + \boldsymbol{p} = \epsilon_0(\boldsymbol{e} + \chi_e \boldsymbol{e}) = \epsilon_0(1 + \chi_e)\boldsymbol{e} = \epsilon_0 \epsilon_r \boldsymbol{e}$$

- electrical conductivity => losses of electric energy in material (heat dissipation)
- complex dielectric constant $\boldsymbol{\epsilon} = \epsilon' - \imath\epsilon''$
  - ϵ' = static dielectric contribution in C calculation
  - ϵ'' = energy loss in dielectric medium
  - loss tangent $\tan\delta = \dfrac{\epsilon''}{\epsilon'}$

  (how lossy material is for AC signals)

$$P_{\text{diel}} = \frac{1}{\text{Vol}(\Omega_{\text{diel}})} \int_{\Omega_{\text{diel}}} \omega\epsilon' \tan(\delta)\, e^2 \, \mathrm{d}\Omega_{\text{diel}}$$

$$= \frac{1}{\text{Vol}(\Omega_{\text{diel}})} \int_{\Omega_{\text{diel}}} \omega\epsilon'' \, e^2 \, \mathrm{d}\Omega_{\text{diel}}$$



Losses and power
Frequency domain

Joule losses

$$
\begin{aligned}
I^2(t) &= \hat{I}^2 \cos^2\left(\omega t + \gamma_I\right) \\
&= \frac{1}{2}\hat{I}^2\left(1 + \cos\left(2\omega t + 2\gamma_I\right)\right) \\
&= \frac{1}{2}\hat{I}^2 + \frac{1}{2}\Re\left(\underline{I}\,\underline{I}\,e^{\underline{i}\,2\omega t}\right) \\
&= \frac{1}{2}\hat{I}^2 + \frac{1}{2}\Re\left(\underline{I}\,\underline{I}\right)\cos 2\omega t - \frac{1}{2}\Im\left(\underline{I}\,\underline{I}\right)\sin 2\omega t
\end{aligned}
$$

Instantaneous and average (active) power

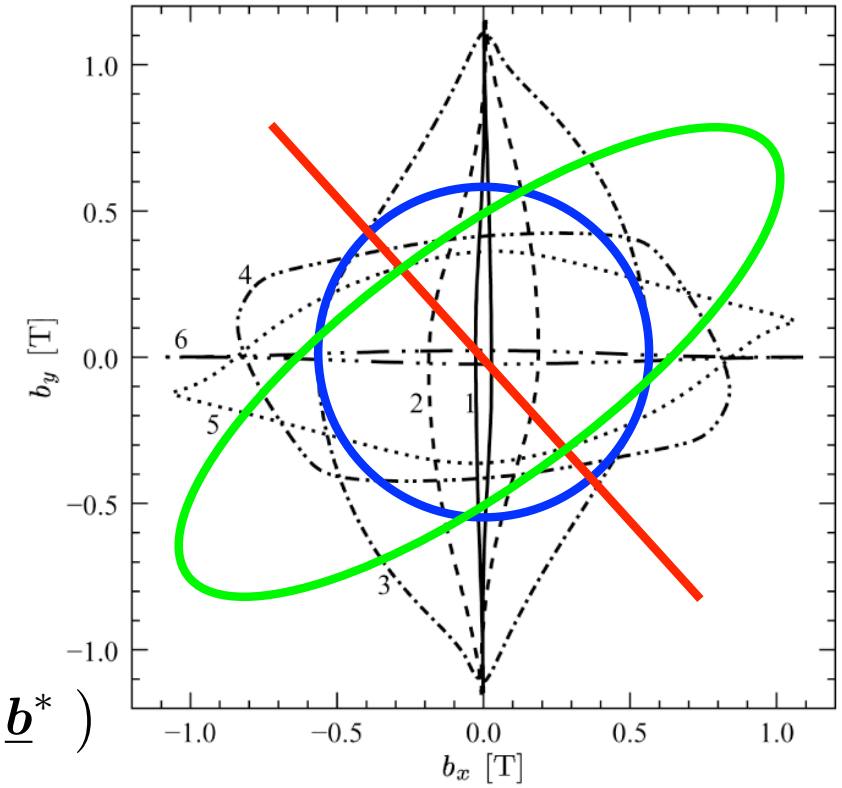
$$
\begin{aligned}
V(t)I(t) &= \hat{V}\hat{I}\cos\left(\omega t + \gamma_V\right)\cos\left(\omega t + \gamma_I\right) \\
&= \frac{1}{2}\hat{V}\hat{I}\left(\cos\left(\gamma_V - \gamma_I\right) + \cos\left(2\omega t + \gamma_V + \gamma_I\right)\right) \\
&= \frac{1}{2}\Re\left(\underline{V}\,\underline{I}^*\right) + \frac{1}{2}\Re\left(\underline{V}\,\underline{I}\,e^{\underline{i}\,2\omega t}\right) \\
&= \frac{1}{2}\Re\left(\underline{V}\,\underline{I}^*\right) + \frac{1}{2}\Re\left(\underline{V}\,\underline{I}\right)\cos 2\omega t - \frac{1}{2}\Im\left(\underline{V}\,\underline{I}\right)\sin 2\omega t
\end{aligned}
$$

Magnetic energy density, Lorentz force density
Frequency domain

Magnetic energy density

$$
\begin{aligned}
w_{\mathrm{mag}}(t) &= \frac{1}{2}\nu\, \boldsymbol{b}(t)\cdot\boldsymbol{b}(t) = \ldots \\
&= \frac{1}{4}\nu\left(\underline{\boldsymbol{b}}\cdot\underline{\boldsymbol{b}}^{*} + \Re\left(\underline{\boldsymbol{b}}\cdot\underline{\boldsymbol{b}}\, e^{\underline{i}\,2\omega t}\right)\right) \\
&= w_{\mathrm{mag,av}} + \Re\left(\underline{w}_{\mathrm{mag},2\omega}\, e^{\underline{i}\,2\omega t}\right) \\
&= w_{\mathrm{mag,av}} + \Re\left(\underline{w}_{\mathrm{mag},2\omega}\right)\cos 2\omega t - \Im\left(\underline{w}_{\mathrm{mag},2\omega}\right)\sin 2\omega t
\end{aligned}
$$

$$
w_{\mathrm{mag,av}} = \frac{1}{4}\nu\,\underline{\boldsymbol{b}}\cdot\underline{\boldsymbol{b}}^{*} \qquad w_{\mathrm{mag},2\omega} = \frac{1}{4}\nu\,\underline{\boldsymbol{b}}\cdot\underline{\boldsymbol{b}} \qquad \left(\ |\underline{\boldsymbol{b}}\cdot\underline{\boldsymbol{b}}| \ \leq\ \underline{\boldsymbol{b}}\cdot\underline{\boldsymbol{b}}^{*}\ \right)
$$



Lorentz force density

$$
\begin{aligned}
\boldsymbol{j}(t)\times\boldsymbol{b}(t) &= \frac{1}{2}\Re\left(\underline{\boldsymbol{j}}\times\underline{\boldsymbol{b}}^{*}\right) + \frac{1}{2}\Re\left(\underline{\boldsymbol{j}}\times\underline{\boldsymbol{b}}\, e^{\underline{i}\,2\omega t}\right) \\
&= \frac{1}{2}\Re\left(\underline{\boldsymbol{j}}\times\underline{\boldsymbol{b}}^{*}\right) + \frac{1}{2}\Re\left(\underline{\boldsymbol{j}}\times\underline{\boldsymbol{b}}\right)\cos 2\omega t - \frac{1}{2}\Im\left(\underline{\boldsymbol{j}}\times\underline{\boldsymbol{b}}\right)\sin 2\omega t
\end{aligned}
$$

average force — 2ω component force