Current and voltage as global quantities
Movement treatment
Coupled problems

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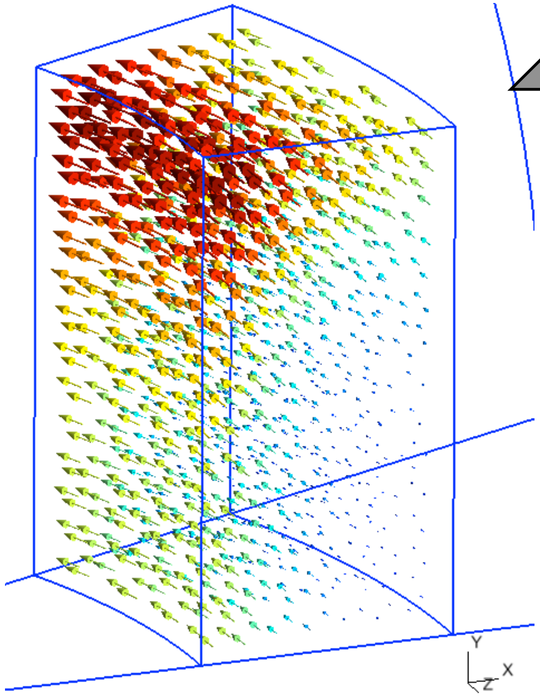
Circuit coupling

- sources of magnetic fields in conductors (two models: massive and stranded)
  - electric current densities (local quantities)
  - current/voltage (global quantities) in a FE computation

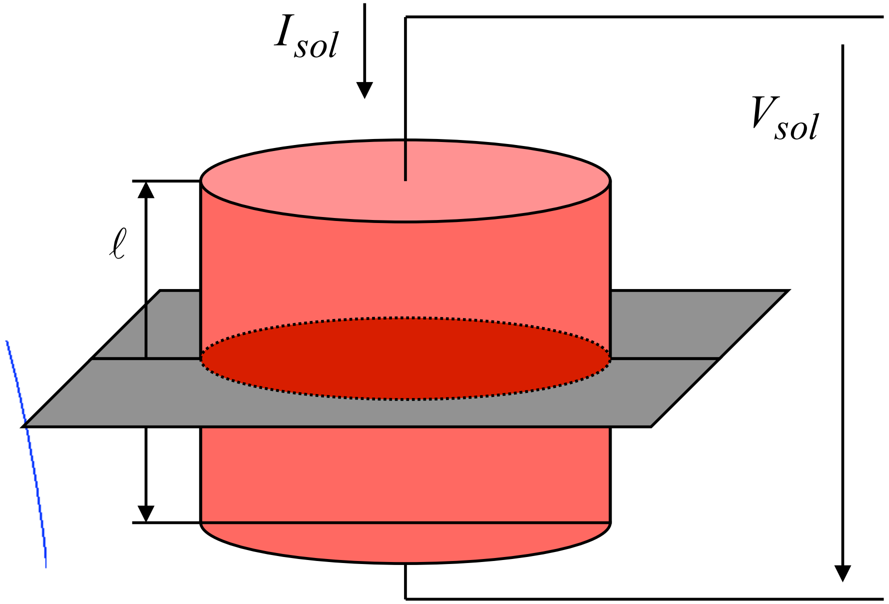
- coupling with **external circuit** source (control/power electronics) is essential for accounting for the operation of the device within the FE simulation
  - sometimes it is enough to perform a set of static or dynamic simulations to characterize the device: tiresome, often unfeasible (saturation, movement, multiple sources...)
  - **strong** coupling: all equations solved simultaneously
  - **weak** coupling: iterative solution with a possibly adaptive time-step

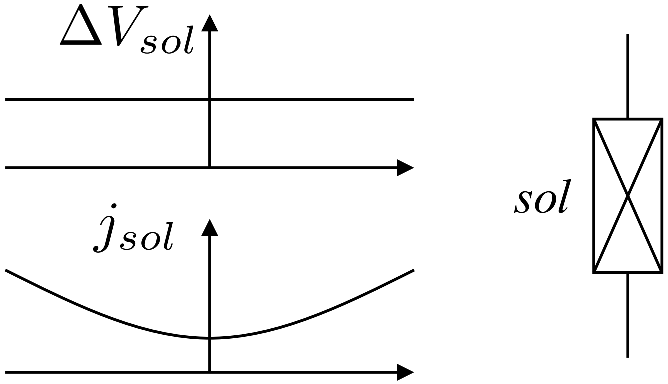
Solid/massive conductor

- allows current redistribution (internal eddy currents causing skin and proximity effect)
- parameters:
  - voltage across conductor
  - length
- equivalent to massive block of conductive material



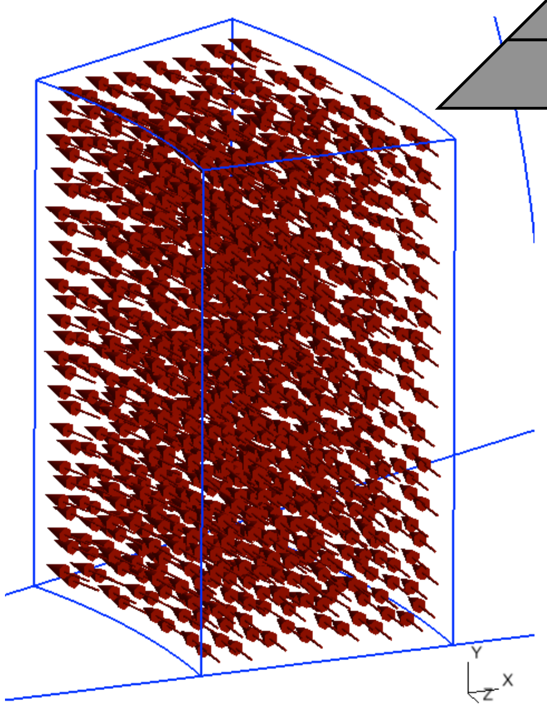
non-homogeneous current distribution

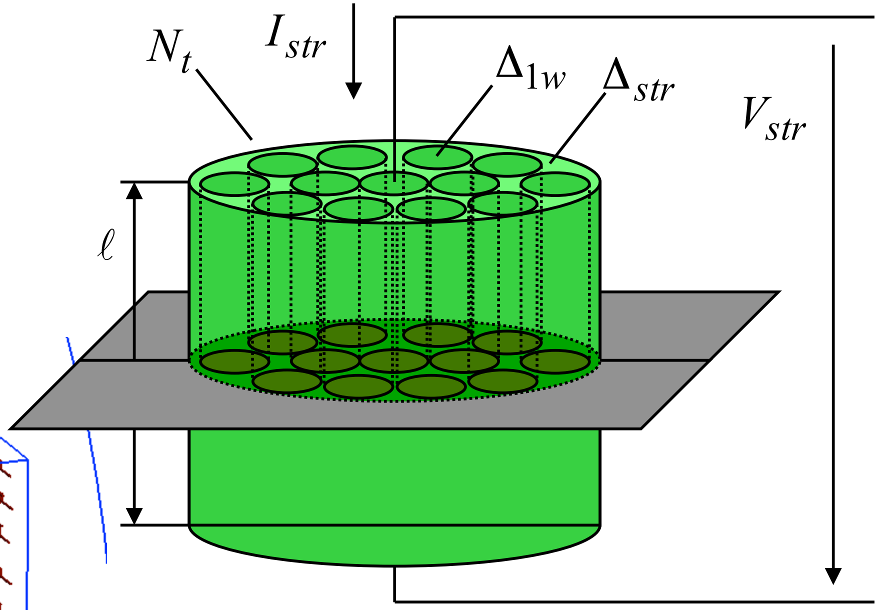


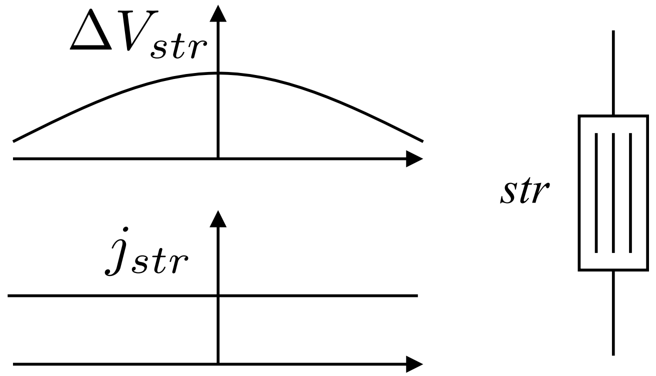


Stranded conductor

- equivalent to bundle of thin (<skin depth) insulated conductors
- eddy currents in wires are disregarded
- parameters: (homogenous) current density, length, number of turns
- electrical conductivity: to correct for cross section => ratio conductor area /total area (fill factor)
- equivalent resistance (lumped element) to couple with external circuit

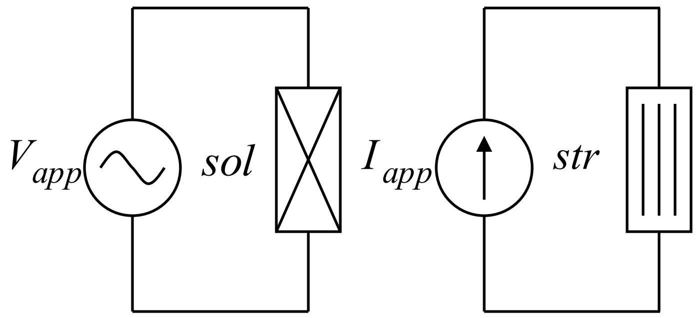






homogeneous current distribution

Circuit equations



- **direct substitution**
  - solid conductor + given voltage source (incl. short-circuit)
  - stranded conductor + given current source (incl. open circuit)
- **extra circuit equations**
  - modified nodal analysis (MNA)
  - signal flow graph
  - block structure => suitable solver
  - off-diagonal: coupling terms
  - can be made symmetric

$$\begin{bmatrix} k_{ij} & k_{ij} & k_{ij} \\ k_{ij} & k_{ij}+j\omega l_{ij} & k_{ij} \\ k_{ij} & k_{ij} & k_{ij} \end{bmatrix}\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{q_i}{\ell}V_{app} \\ -\frac{N_t p_i}{\Delta_{str}} I_{app} \end{bmatrix}$$

$$\left[\begin{array}{ccc|cc} k_{ij} & k_{ij} & k_{ij} & & \\ k_{ij} & k_{ij}+j\omega l_{ij} & k_{ij} & -\frac{q_i}{\ell} & \\ k_{ij} & k_{ij} & k_{ij} & & -\frac{N_t p_i}{\Delta_{str}} \\ \hline & -j\omega q_j & & G_{sol} & \\ & & j\omega\ell\frac{N_t p_i}{\Delta_{str}} & & R_{str} \end{array}\right]\left[\begin{array}{c} A_{zj} \\ \hline V_{sol} \\ I_{str} \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline I_{app} \\ V_{app} \end{array}\right]$$

Circuit equations

- **direct substitution**
  - solid conductor + given voltage source (incl. short-circuit)
  - stranded conductor + given current source (incl. open circuit)
- **extra circuit equations**
  - modified nodal analysis (MNA)
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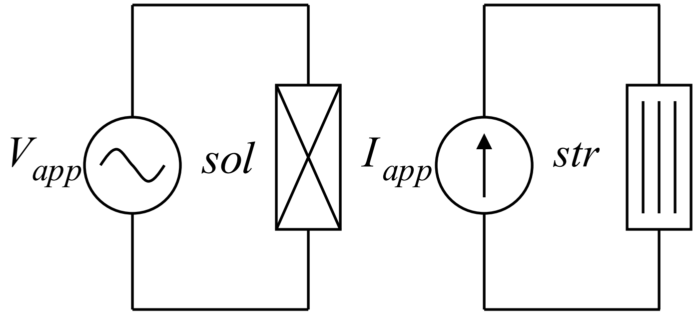


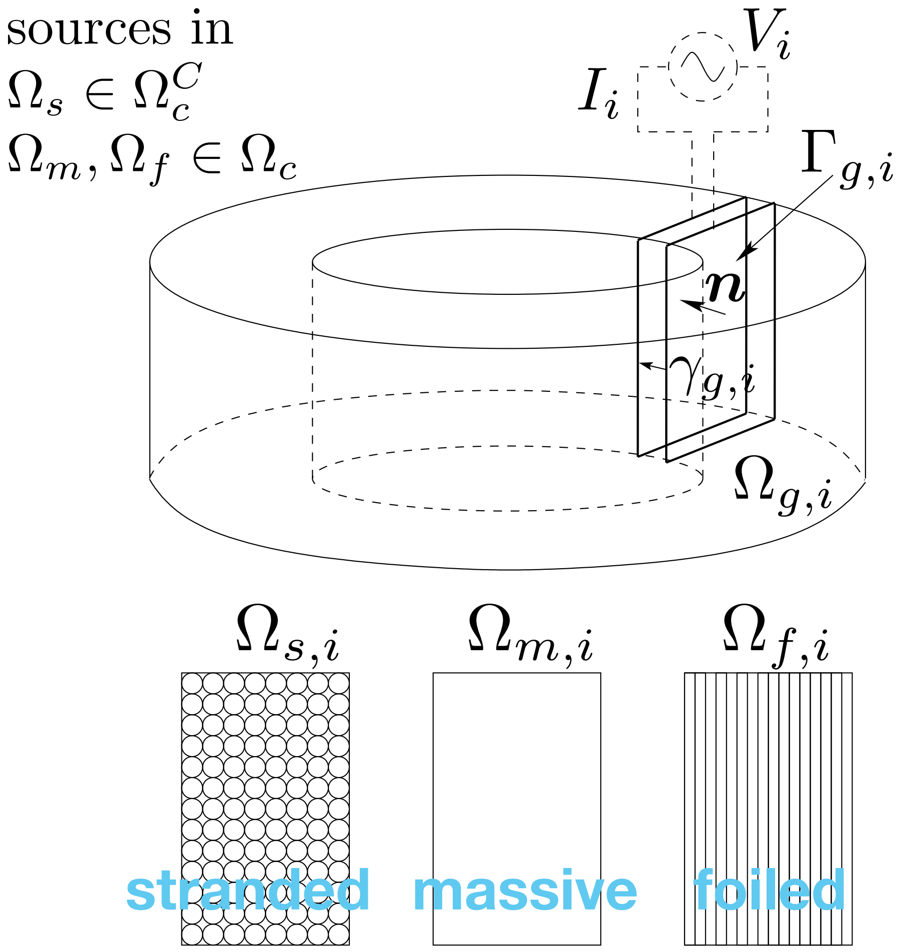
$$\mathbf{S}\,\mathbf{A} + \mathbf{T}_M \frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \mathbf{K}_M^\top \mathbf{R}_M^{-1} \mathbf{V}_M + \mathbf{K}_S^\top \mathbf{I}_S$$

$$\mathbf{V}_M = \mathbf{R}_M \mathbf{I}_M + \mathbf{K}_M \frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}$$

$$\mathbf{V}_S = \mathbf{R}_S \mathbf{I}_S + \mathbf{K}_S \frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}$$

$$\mathbf{D}_{Sl}^\top \mathbf{V}_S + \mathbf{D}_{Ml}^\top \mathbf{V}_M + \mathbf{R}_l \mathbf{I}_l + \mathbf{L}_l \frac{\mathrm{d}\mathbf{I}_l}{\mathrm{d}t} = \mathbf{V}_l$$

Global conditions of currents and voltages



cuts inductors

- $\Omega_{g,i}$ is a source of e.m.f. located between two sections close to each other, i.e. two electrodes (coinciding cuts in practice)
- voltage V_i & current I_i are associated to this source
- current I_i flows through surface $\Gamma_{g,i}$
- $\gamma_{g,i}$ is path connecting the two electrodes

Essential constraints

$$\int_{\gamma_{g,i}} \boldsymbol{e} \cdot \mathrm{d}\boldsymbol{l} = V_i$$

$$\int_{\Gamma_{g,i}} \boldsymbol{n} \cdot \boldsymbol{j} \, \mathrm{d}s = I_i$$

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Differential k-forms
The exterior derivative d applied on a k-form gives a k+1-form

- 0-form, (e.g. potentials φ, v):
  - continuous scalar fields (conform)
  - generated by nodal functions
  - value (point evaluation) at node
  - exterior derivative is grad
- 1-form, e.g. h, e, (potentials a, t):
  - vector fields with continuous tangential trace (curl-conform)
  - generated by edge functions
  - circulation (line integral) along edge
  - exterior derivative is curl
- 2-form, e.g. b, j:
  - vector fields with continuous normal trace (div-conform)
  - generated by facet functions
  - flux (surface integral) through facet
  - exterior derivative is div

$\boldsymbol{a} - v$ formulation
$\boldsymbol{a}$ magnetic vector potential
v electric scalar potential potential

Magnetodynamics

Define *a* in Ω and *v* in Ω_c (discontinuous across electrodes):

- *a* as a 1-form and *v* as a 0-form, $\boldsymbol{b} = \operatorname{curl} a$
- satisfying the local BC $\boldsymbol{e} \times \boldsymbol{n}|_{\Gamma_e} = 0$ $\boldsymbol{e} = -\partial_t \boldsymbol{a} - \operatorname{grad} v$
- and global BC $V_i = \bar{V}_i\,,\ \forall i \in C_V$ (i.e. the circulation of $-\operatorname{grad} v$ around conducting domain Ω_{ci} is equal to V_i).

This strongly satisfies

$$\operatorname{curl} \boldsymbol{e} = -\partial_t \boldsymbol{b}, \quad \operatorname{div} \boldsymbol{b} = 0, \quad \boldsymbol{e} \times \boldsymbol{n}|_{\Gamma_e} = 0, \quad V_i = \bar{V}_i\,,\ \forall i \in C_V$$

What needs to be weakly imposed is

$$\operatorname{curl} \boldsymbol{h} = \boldsymbol{j}, \quad \boldsymbol{j} = \sigma \boldsymbol{e}, \quad \boldsymbol{h} = \nu \boldsymbol{b}, \quad \boldsymbol{h} \times \boldsymbol{n}|_{\Gamma_h} = 0, \quad I_i = \bar{I}_i\ \forall i \in C_I$$

Choosing the potentials

- We still have freedom on the choice of the potentials. Indeed, the fields b and e do not vary for any scalar field ϕ

$$\boldsymbol{a} \to \boldsymbol{a} + \int_0^t \operatorname{grad}\phi \ \mathrm{dt}$$

$$v \to v - \phi$$

- There are different possibilities for gauging *a* and *v*
- **One** global shape function $v_{d,i}$ in each Ω_{c_i} is sufficient for representing a unit voltage, st. we have:

$$\operatorname{grad} v = \sum_{i=1}^{N} V_i \operatorname{grad} v_{d,i}$$

Choosing the potentials
2D case with in-plane magnetic flux b

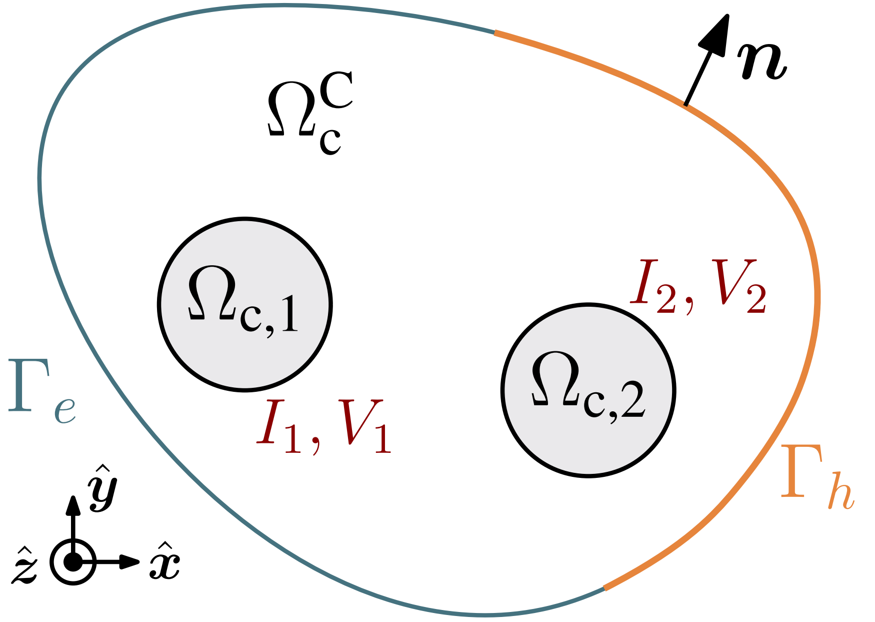
$$\boldsymbol{b} = \operatorname{curl} \boldsymbol{a} \qquad \boldsymbol{e} = -\partial_t \boldsymbol{a} - \operatorname{grad} v \qquad \operatorname{grad} v = \sum_{i=1}^{N} V_i \operatorname{grad} v_{d,i}$$

- MVP along z with nodal basis functions ψ_n

$$\boldsymbol{a} = \sum_n a_n \psi_n \hat{z}$$

NB it is a Coulomb gauge with $\operatorname{div} \boldsymbol{a} = 0$

- $\operatorname{grad} v_{d,i}$ is along $\hat{z}$ and constant $(=1)$ in each Ω_{c_i} (V is a voltage per unit length)
- Remaining constant fixed by BC



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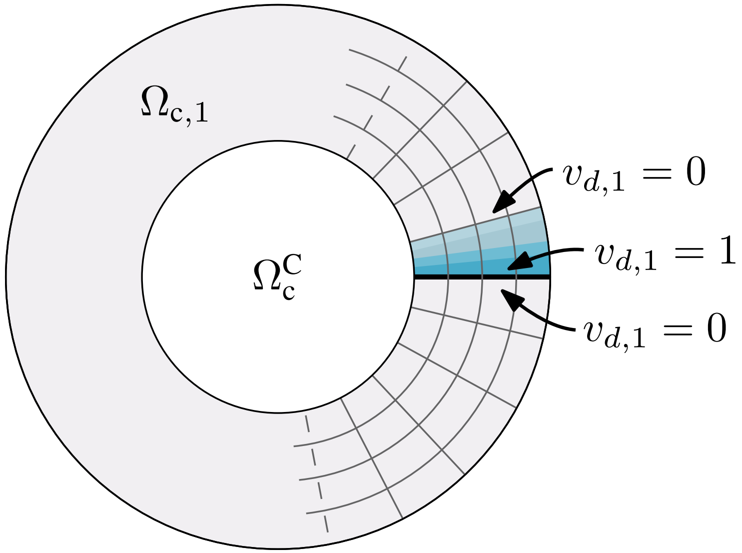
Choosing the potentials
3D case

In Ω_c, define $v_{d,i}$ to be zero everywhere except on transition layer in $\Omega_{c,i}$:
layer of one element, on one side of the electrodes, in each $\Omega_{c,i}$
(v has no longer a physical interpretation)

$$\operatorname{grad} v = \sum_{i=1}^{N} V_i \operatorname{grad} v_{d,i}$$

$\boldsymbol{a}$ is generated by edge elements

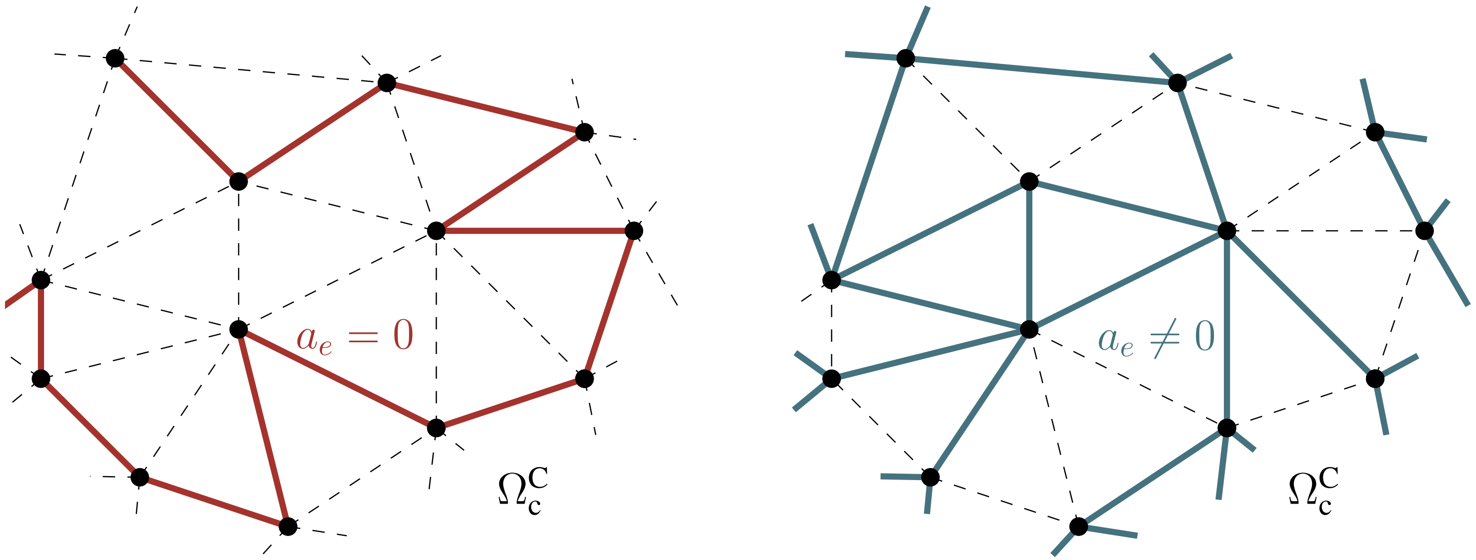


In Ω_c, $\mathbf{a}$ is unique, e.g. outside the transition layer, $\mathbf{e} = -\partial_t \mathbf{a}$
(reduced vector potential)
In Ω_c^C we need a gauge for making $\mathbf{a}$ unique

Co-tree gauge

In Ω_c^C, only $\mathbf{curl}\, \boldsymbol{a} = \boldsymbol{b}$ has a physical meaning. One DOF per facet is sufficient (and necessary), instead of one DOF per edge.
The support entities of the 1-form $\boldsymbol{a}$ are the edges.
To associate a unique edge to each facet: consider only edges in a **co-tree**, i.e. the complementary of a tree:

$$\boldsymbol{a} = \sum_{e \in \Omega_c \cup (\textbf{co-tree} \text{ in } \Omega_c^C)} a_e\, \boldsymbol{\psi}_e.$$



NB: Be careful on the conducting domain boundary $\partial\Omega_c$, no gauge there because $\boldsymbol{a}$ is already unique.

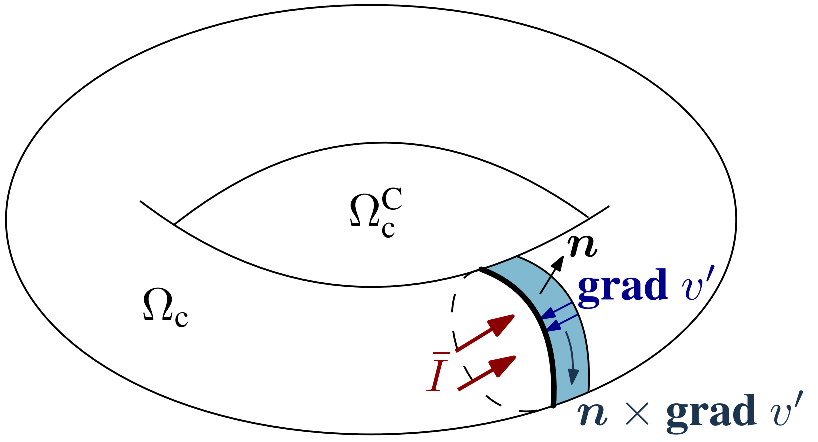
Surface term in MVP formulation

$$\langle \nu \,\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{n} \,, \mathbf{grad}\, v' \rangle_{\partial\Omega_c} = \langle \boldsymbol{h} \times \boldsymbol{n} \,, \mathbf{grad}\, v' \rangle_{\partial\Omega_c}$$

$$= \langle \boldsymbol{h} \,, \boldsymbol{n} \times \mathbf{grad}\, v' \rangle_{\partial\Omega_c}$$

$$= \langle \boldsymbol{h} \,, \boldsymbol{n} \times \mathbf{grad}\, v' \rangle_{\partial(\text{transition layer})}$$

$$= I\, V' = \bar{I}\, V'$$



MVP formulation with V and I

Finally, the a-formulation amounts to find $\boldsymbol{a}$ and v in the choser function spaces such that, $\forall \boldsymbol{a}'$ and v',

$$\left(\nu\, \mathbf{curl}\, \boldsymbol{a}\ ,\mathbf{curl}\, \boldsymbol{a}'\right)_{\Omega} - \left\langle \bar{\boldsymbol{h}} \times \boldsymbol{n}_{\Omega}\ ,\boldsymbol{a}'\right\rangle_{\Gamma_h} + \left(\sigma\, \partial_t \boldsymbol{a}\ ,\boldsymbol{a}'\right)_{\Omega_c} + \left(\sigma\, \mathbf{grad}\ v\ ,\boldsymbol{a}'\right)_{\Omega_c} = 0,$$

$$\left(\sigma\, \partial_t \boldsymbol{a}\ ,\mathbf{grad}\ v'\right)_{\Omega_c} + \left(\sigma\, \mathbf{grad}\ v\ ,\mathbf{grad}\ v'\right)_{\Omega_c} = \sum_{i=1}^{N} I_i \mathcal{V}_i(v'),$$

with $I_i = \bar{I}_i$ for $i \in C_I$,
and $\mathcal{V}_i(v') = V_i'$ (i.e. the DOF associated with the unit voltage function $v_{d,i}$).

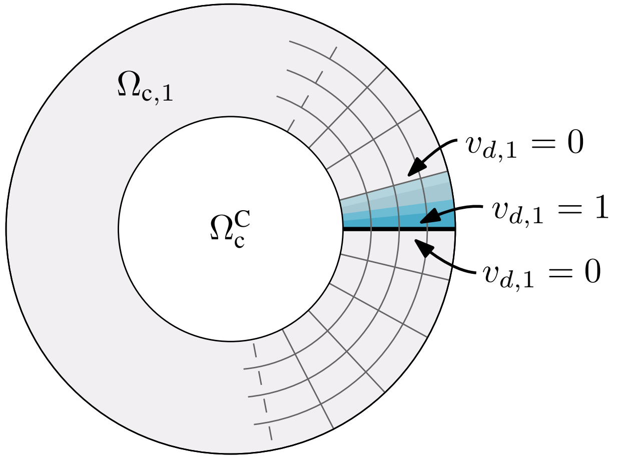
MVP formulation interpretation

When the test function $v' = v_{d,i}$ is chosen ($\mathcal{V}_i(v_{d,i}) = 1$), the second equation reads

$$\left(\sigma\left(\partial_t \boldsymbol{a} + \mathbf{grad}\ v\right), \mathbf{grad}\ v_{d,i}\right)_{\Omega_c} = I_i$$

$$\Rightarrow \qquad \left(\sigma\, \boldsymbol{e}\ , -\mathbf{grad}\ v_{d,i}\right)_{\Omega_c} = I_i.$$

”Flux of $\sigma\boldsymbol{e}$ $(= \boldsymbol{j})$ averaged over a transition layer = total current”



NB: The flux of $\sigma\boldsymbol{e}$ depends on the chosen cross-section as $\sigma\boldsymbol{e}$ is not a 2-form (as $\boldsymbol{j}$ should be). Conservation of current is weakly satisfied.

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Current as a strong global quantity

$$\boldsymbol{h} = \sum_{e \in \mathcal{E}} h_e \boldsymbol{s}_e$$

$\mathcal{E}$ edges in Ω

Edge FEs to interpolate curl-conform fields (h)
s_e edge BF associated with edge e
h_e = circulation of h along edge e = difference of scalar potentials at the two ends of the edge in the non-conducting domain
=> Coupling of edge FEs (field) and nodal FEs (scalar potential)

Explicit constraints for circulations and zero curl, i.e. currents I_i

$$\boldsymbol{h} = \sum_{k \in \mathcal{E}_c} h_k \boldsymbol{s}_k + \sum_{n \in \mathcal{N}_c^C} \phi_n \operatorname{grad} \psi_n + \sum_{i \in \mathcal{C}} I_i \boldsymbol{c}_i$$

$\mathcal{E}_c$ inner edges in Ω_c
$\mathcal{N}_c^C$ nodes in Ω_c^C and $\partial\Omega_c^C$
$\mathcal{C}$ cuts

$\operatorname{curl} \boldsymbol{h} = \boldsymbol{j}$ in Ω_c
$\operatorname{curl} \boldsymbol{h} = 0$ in Ω_c^C

Basis functions

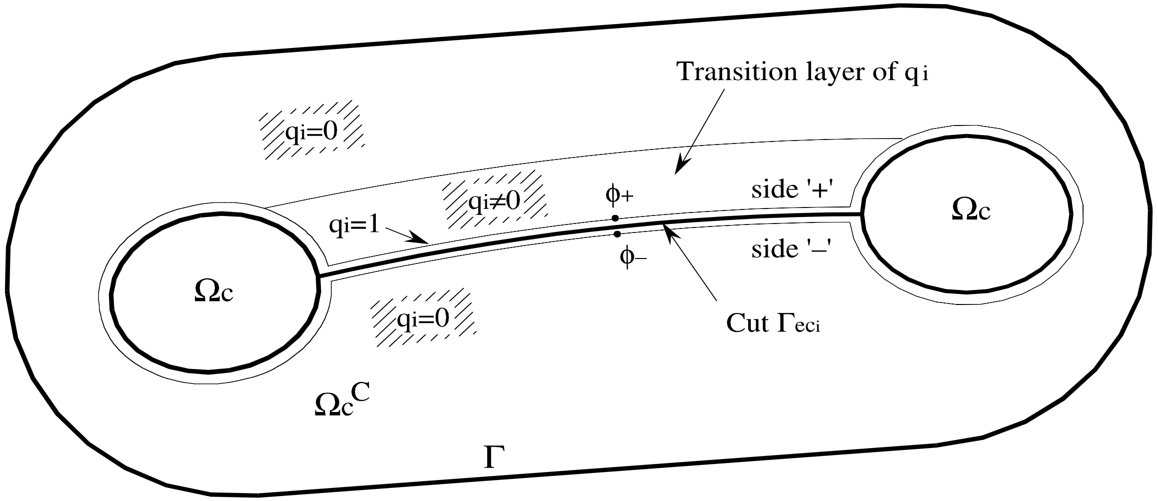
$$\phi = \phi_{\text{cont}} + \phi_{\text{discont}} \quad \in \Omega_c^C$$

Current as a strong global quantity

$$\boldsymbol{h} = \sum_{k \in \mathcal{E}_c} h_k \boldsymbol{s}_k + \sum_{n \in \mathcal{N}_c^C} \phi_n \operatorname{grad} \psi_n + \sum_{i \in \mathcal{C}} I_i \boldsymbol{c}_i$$

$$\phi = \phi_{\text{cont}} + \phi_{\text{discont}} \quad \in \Omega_c^C$$

Introducing cuts for making Ω_c^C simply connected



$$\boldsymbol{c}_i = -\operatorname{grad} q_i \quad \text{in } \Omega_c^C$$
$$\boldsymbol{n} \times \boldsymbol{c}_i = -\boldsymbol{n} \times \operatorname{grad} q_i \quad \text{on } \partial\Omega_c$$

Treatment of multivalued potentials

- coefficients I_i are circulations along well-defined paths
- c_i are vector ‘circulation’ BFs linked to a group of edges from a cut, accounting for discontinuity

=> its circulation equals 1 along a closed path around Ω_c

Elementary geometrical entities (nodes, edges) and global ones (groups of edges)

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Voltage as a weak global quantity

$$\boldsymbol{h} = \sum_{k\in\mathcal{E}_c} h_k \boldsymbol{s}_k + \sum_{n\in\mathcal{N}_c^C} \phi_n \boldsymbol{v}_n + \sum_{i\in\mathcal{C}} I_i \boldsymbol{c}_i$$

Strong formulation

$$\operatorname{curl}\rho\operatorname{curl}\boldsymbol{h} + \partial_t(\mu\boldsymbol{h}) = 0 \quad \text{in } \Omega_c\,, \qquad \operatorname{div}\Big(\mu(\boldsymbol{h}_s - \operatorname{grad}\phi)\Big) = 0 \quad \text{in } \Omega_c^C$$

Weak formulation of Faraday equation

find $\boldsymbol{h}$ such that

$$\int_\Omega \mu\,\partial_t\boldsymbol{h}\cdot\boldsymbol{w}\,\mathrm{d}\Omega + \int_{\Omega_c} \rho\operatorname{curl}\boldsymbol{h}\cdot\operatorname{curl}\boldsymbol{w}\,\mathrm{d}\Omega + \int_\Gamma \boldsymbol{n}\times\boldsymbol{e}_s\cdot\boldsymbol{w}\,\mathrm{d}\Gamma = 0\,,\ \forall\boldsymbol{w}$$

$$\boldsymbol{h} = \sum_{k\in\mathcal{E}_c} h_k \boldsymbol{s}_k + \sum_{n\in\mathcal{N}_c^C} \phi_n \operatorname{grad}\psi_n + \sum_{i\in\mathcal{C}} I_i \boldsymbol{c}_i$$

$$\operatorname{grad}\psi_n = \boldsymbol{v}_n$$

system of equations (symmetrical matrix)

test function $\boldsymbol{w} = \boldsymbol{s}_k,\ \boldsymbol{v}_n$ → no additional contribution to surface integral

test function $\boldsymbol{w} = \boldsymbol{c}_i$ → **contribution to surface integral**

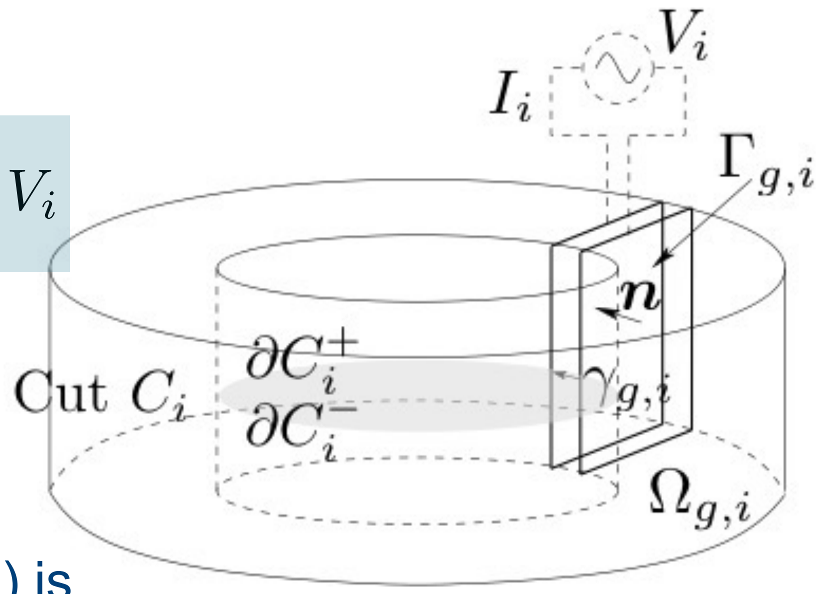
electromotive force = weak global quantity

$$\int_\Gamma \boldsymbol{n}\times\boldsymbol{e}_s\cdot\boldsymbol{w}\,\mathrm{d}\Gamma = \int_\Gamma \boldsymbol{n}\times\boldsymbol{e}_s\cdot\boldsymbol{c}_i\,\mathrm{d}\Gamma = \int_\Gamma \boldsymbol{n}\times\boldsymbol{e}_s\cdot(-\operatorname{grad} q_i)\,\mathrm{d}\Gamma = \oint_{\gamma_i} \boldsymbol{e}_s\times\mathrm{d}\boldsymbol{l} = V_i$$

Voltage as a weak global quantity and circuit relations

electromotive force

$$\int_{\Gamma} \boldsymbol{n} \times \boldsymbol{e}_s \cdot \boldsymbol{c}_i \, \mathrm{d}\Gamma = \oint_{\gamma_i} \boldsymbol{e}_s \times \mathrm{d}\boldsymbol{l} = V_i$$



Natural way to compute a weak voltage !
Better than an explicit non-unique line integration

When the test function c_i ($I_i(c_i) = 1$) is chosen, we get the equation:

$$\int_{\Omega} \mu \, \partial_t \boldsymbol{h} \cdot \boldsymbol{c}_i \, \mathrm{d}\Omega + \int_{\Omega_c} \sigma^{-1} \operatorname{curl} \boldsymbol{h} \cdot \operatorname{curl} \boldsymbol{c_i} \, \mathrm{d}\Omega = -V_i,$$

Weak circuit relation between voltage & current

Flux change μ*h* (= *b*) + circulation of ρ *j* (= *e*),
both averaged over a transition layer = total voltage".

$$\text{“}\partial_t(\text{Magnetic flux}) + \text{Resistance} \times \text{Current} = \text{Voltage”}$$

GetDP implementation

$$\boldsymbol{h} = \sum_{k\in\mathcal{E}_c} h_k \boldsymbol{s}_k + \sum_{n\in\mathcal{N}_c^C} \phi_n \,\mathrm{grad}\,\psi_n + \sum_{i\in\mathcal{C}} I_i \boldsymbol{c}_i$$

```
FunctionSpace{
    { Name h_space; Type Form1;
        BasisFunction {
            // Nodal functions
            { Name gradpsin; NameOfCoef phin; Function BF_GradNode;
                Support Omega_h_OmegaCC_AndBnd; Entity NodesOf[OmegaCC]; }
            { Name gradpsin; NameOfCoef phin2; Function BF_GroupOfEdges;
                Support Omega_h_OmegaC; Entity GroupsOfEdgesOnNodesOf[BndOmegaC]; }
            // Edge functions
            { Name psie; NameOfCoef he; Function BF_Edge;
                Support Omega_h_OmegaC_AndBnd; Entity EdgesOf[All, Not BndOmegaC]; }
            // Cut functions
            { Name ci; NameOfCoef Ii; Function BF_GradGroupOfNodes;
                Support ElementsOf[Omega_h_OmegaCC, OnPositiveSideOf Cuts];
                Entity GroupsOfNodesOf[Cuts]; }
            { Name ci; NameOfCoef Ii2; Function BF_GroupOfEdges;
                Support Omega_h_OmegaC_AndBnd;
                Entity GroupsOfEdgesOf[Cuts, InSupport TransitionLayerAndBndOmegaC]; }
        }
        GlobalQuantity {
            { Name I ; Type AliasOf        ; NameOfCoef Ii ; }
            { Name V ; Type AssociatedWith ; NameOfCoef Ii ; }
        }
        Constraint {
            { [...] }
            { [...] }
}}}
```

Massive and stranded inductors

- Massive inductor => direct application
- Stranded inductor (uniform current density) => additional treatment
- Reaction field + Source field due to a magnetomotive force N_j (= turns) (one basis function for each stranded inductor)

$$\boldsymbol{h} = \boldsymbol{h}_r + \sum_{j \in \Omega_s} I_{s,j} \boldsymbol{h}_{s,j}$$

find $\boldsymbol{h}$ such that

$$\int_{\Omega} \mu\, \partial_t \boldsymbol{h} \cdot \boldsymbol{w}\, \mathrm{d}\Omega + \int_{\Omega_c} \sigma^{-1} \operatorname{curl} \boldsymbol{h} \cdot \operatorname{curl} \boldsymbol{w}\, \mathrm{d}\Omega + \int_{\Omega_s} \sigma^{-1} \boldsymbol{j}_s \cdot \operatorname{curl} \boldsymbol{w}\, \mathrm{d}\Omega = -\int_{\Gamma} \boldsymbol{n} \times \boldsymbol{e}_s \cdot \boldsymbol{w}\, \mathrm{d}\Gamma$$

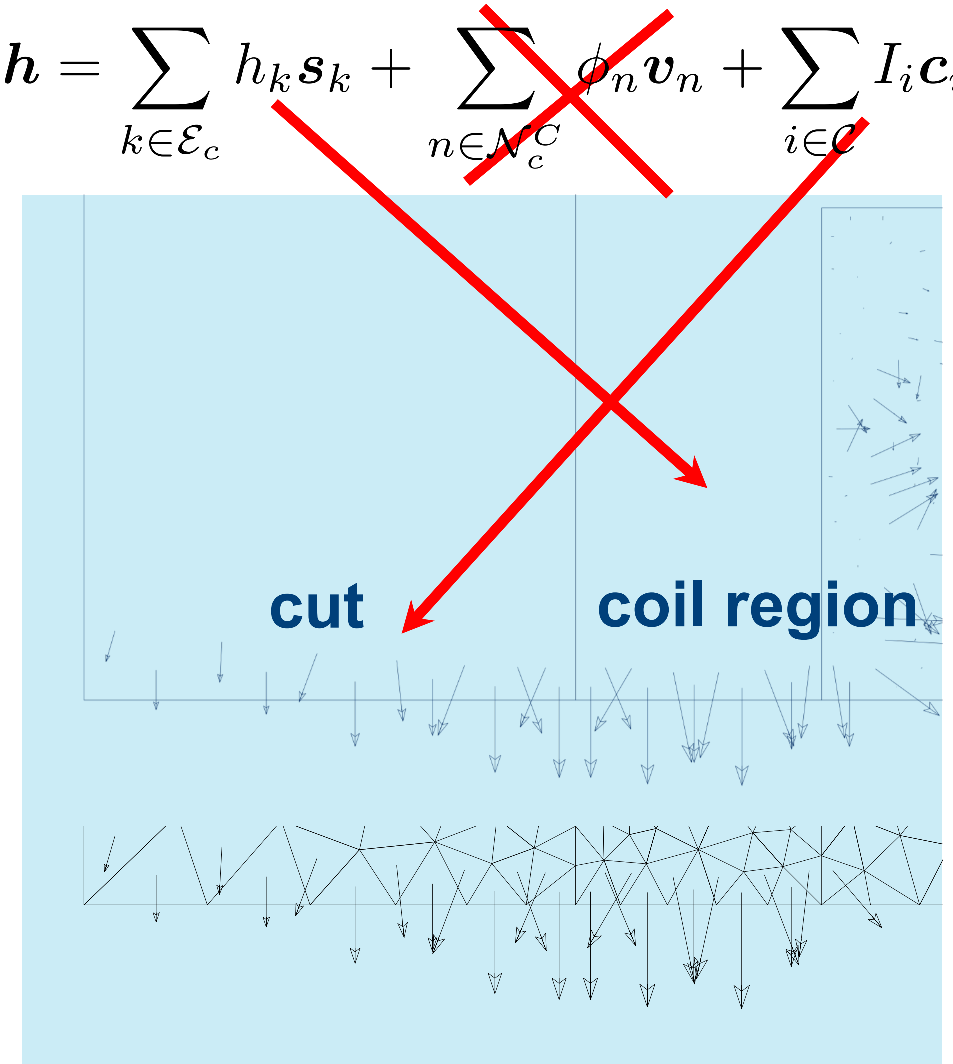
holds $\forall \boldsymbol{w}$

$$\int_{\Omega} \mu\, \partial_t \boldsymbol{h} \cdot \boldsymbol{w}_{s,j}\, \mathrm{d}\Omega + I_{s,j} \int_{\Omega_s} \sigma^{-1} \boldsymbol{j}_{s,j} \cdot \operatorname{curl} \boldsymbol{w}_{s,j}\, \mathrm{d}\Omega_s = -V_j \,, \forall \boldsymbol{w}_{s,j}$$

Weak circuit relation between *Vj* and *Ij* for stranded inductor *j*
Natural way to compute the magnetic flux through all the wires !

Stranded inductor — source field

$$\boldsymbol{h} = \sum_{k \in \mathcal{E}_c} h_k \boldsymbol{s}_k + \sum_{n \in \mathcal{N}_c^C} \phi_n \boldsymbol{v}_n + \sum_{i \in \mathcal{C}} I_i \boldsymbol{c}_i$$



Simplified source computation:

- projection method
- electrokinetic problem

$$\int_{\Omega_{s,j}} \operatorname{curl} \boldsymbol{h}_{s,j} \cdot \operatorname{curl} \boldsymbol{w} \, \mathrm{d}\Omega_{s,j} = \int_{\Omega_{s,j}} \boldsymbol{j}_{s,j} \cdot \operatorname{curl} \boldsymbol{w} \, \mathrm{d}\Omega_{s,j}$$

$$\int_{\Omega_{s,j}} \sigma^{-1} \operatorname{curl} \boldsymbol{h}_{s,j} \cdot \operatorname{curl} \boldsymbol{w} \, \mathrm{d}\Omega_{s,j} = 0$$

Conclusions FE formulations with massive and stranded inductors

h-ϕ magnetodynamic formulation — use of edge & nodal FEs for *h* and ϕ

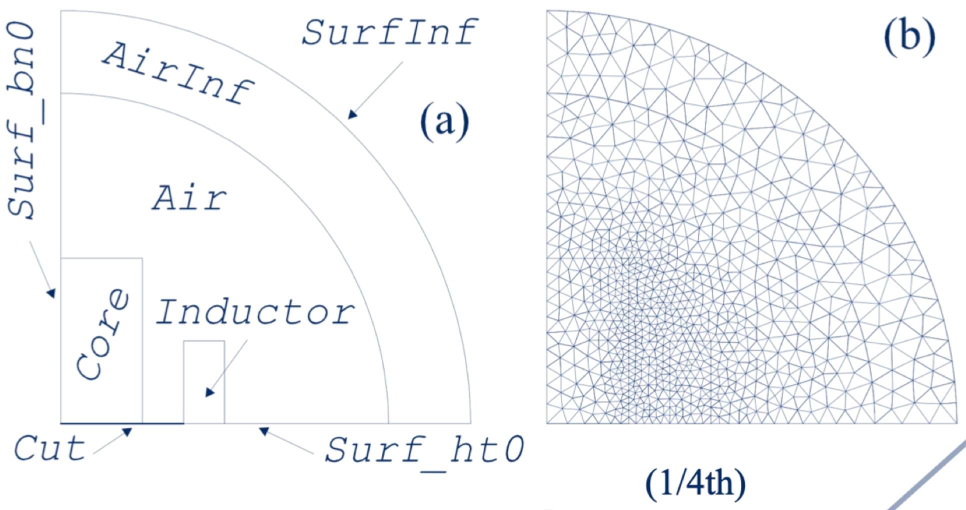
- Natural coupling between *h* and ϕ with BFs for either massive or stranded inductors
- Definition of current in a strong sense/voltage in a weak sense
- Efficient definition of a source magnetic field: limited support
- Natural coupling between fields, currents and voltages

a-v_0 magnetodynamic formulation — use of edge & nodal FEs for a and v_0

- Natural coupling between a and v_0 for massive inductors, adaptation for stranded inductors
- Definition of voltage in a strong sense/current in a weak sense
- Efficient definition of a source electric scalar potential v_0 in massive inductors: limited support
- Natural coupling between fields and currents and voltage

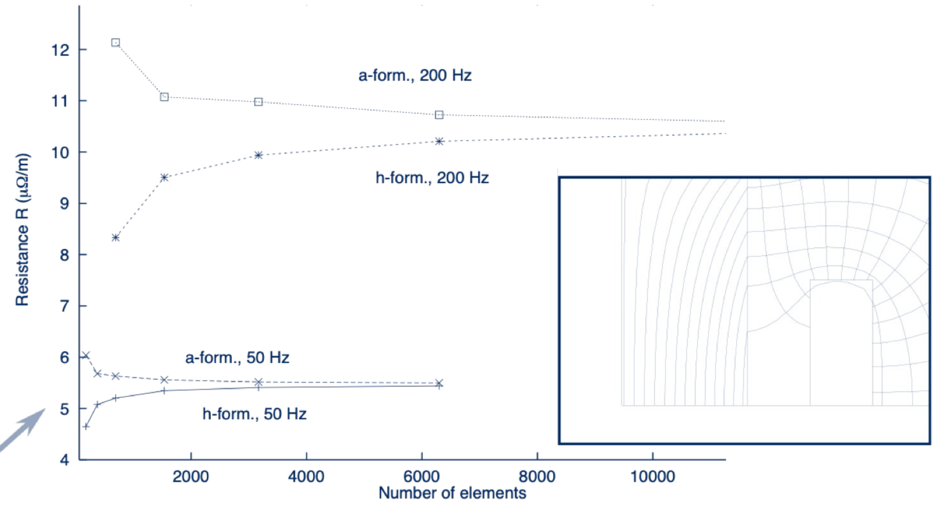
Application — Massive inductor

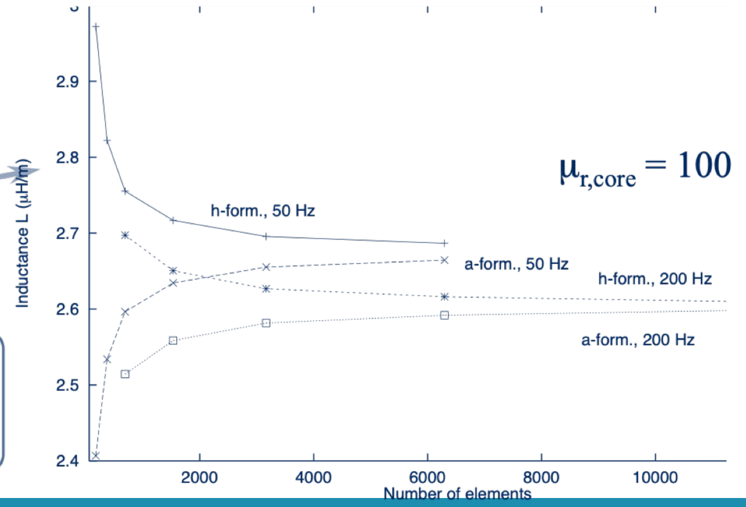
Inductor-Core system in air



$\mu_{r,core} = 1, 10, 100$, $\sigma = 5.9\ 10^7$m S/m
Frequency f = 50, 200 Hz

Computation of resistance and inductance





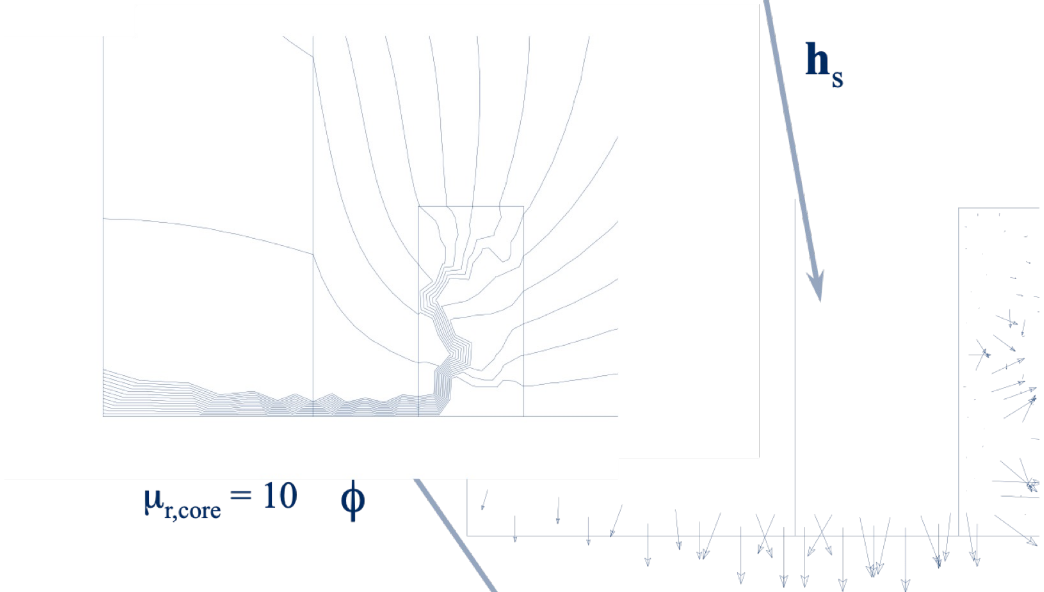
Complementarity between a-v and h-ϕ formulations → validation at global level

Application — Stranded inductor

Inductor-Core system in air

Computation of a source field

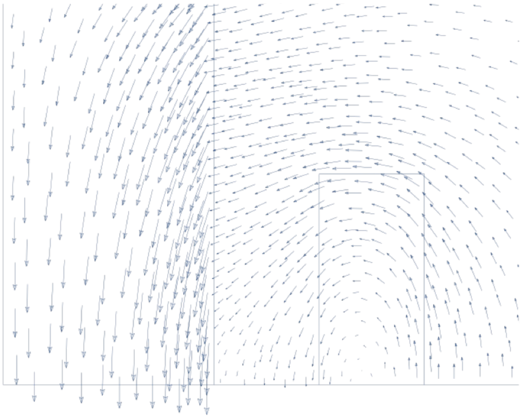
$\mathbf{h}_s$

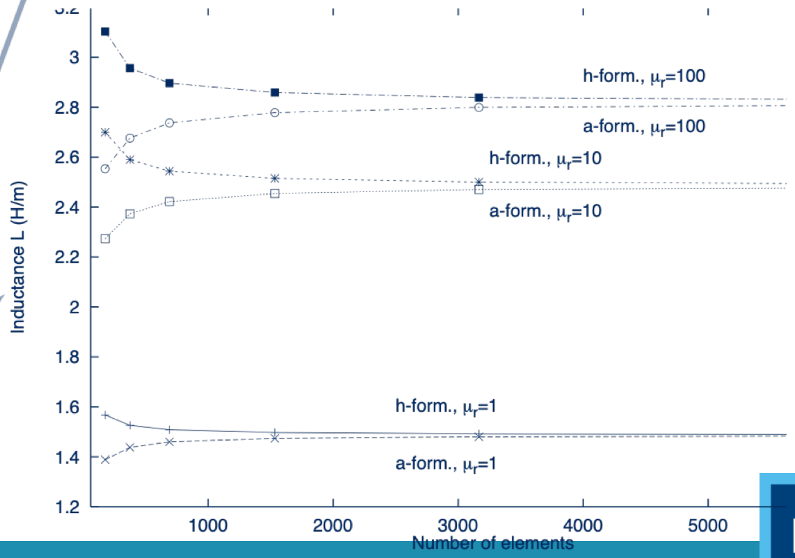


$\mu_{r,core} = 10$ ϕ

$\mu_{r,core} = 10$

$\mathbf{h}$





Computation of reaction field, total field and inductance

Complementarity between h-ϕ and a-v formulations → validation at global level

Coupled problems involving electromagnetism

EM + thermal problem
EM + mechanical problem
...

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Why studying coupled-field problems?

- civil engineering systems involve integration of structural/solid mechanics, fluid/gas/air dynamics, acoustics, propagation, magnetodynamics, electrodynamics, thermal fields, material processing, control, etc.
- an interdisciplinary approach is more effective than single-discipline approach
- tackle coupled-field problems taking advantage of all the know-how (design, modeling and analysis software) for the individual disciplines
- Examples involving electromagnetism:
  - **structure-electromagnetic interaction**: electrical machines, magnetically levitated vehicle, electromagnetic break, microelectromechanical systems MEMs (resonators, gyroscope, sensors of all categories, accelerometers)...
  - **thermal-electromagnetic interaction**: induction heating, electromagnetic stirring and melting, hyperthermia in bio-electromagnetism...

Coupled problem

- Definition:

  coupled system or formulation, defined on **multiple domains**, possibly overlapping, involving **dependent variables** that cannot be eliminated at the equation level
- Strong and weak coupling:

  strength of the interaction between the sub-problems involved. The degree of mutual influence is often unknown/poorly known due to nonlinear coupling mechanisms

  (strong => weak; weak => strong)
- Strong and weak **nonlinear** numerical solution algorithm

Coupled problem — Classification

- Extent:

  fields interact on **partially or entirely overlapping domains** (e.g. thermal-magnetic), or interact **through an interface** (e.g. a massive body and its cooling flow)
- Discretization method:

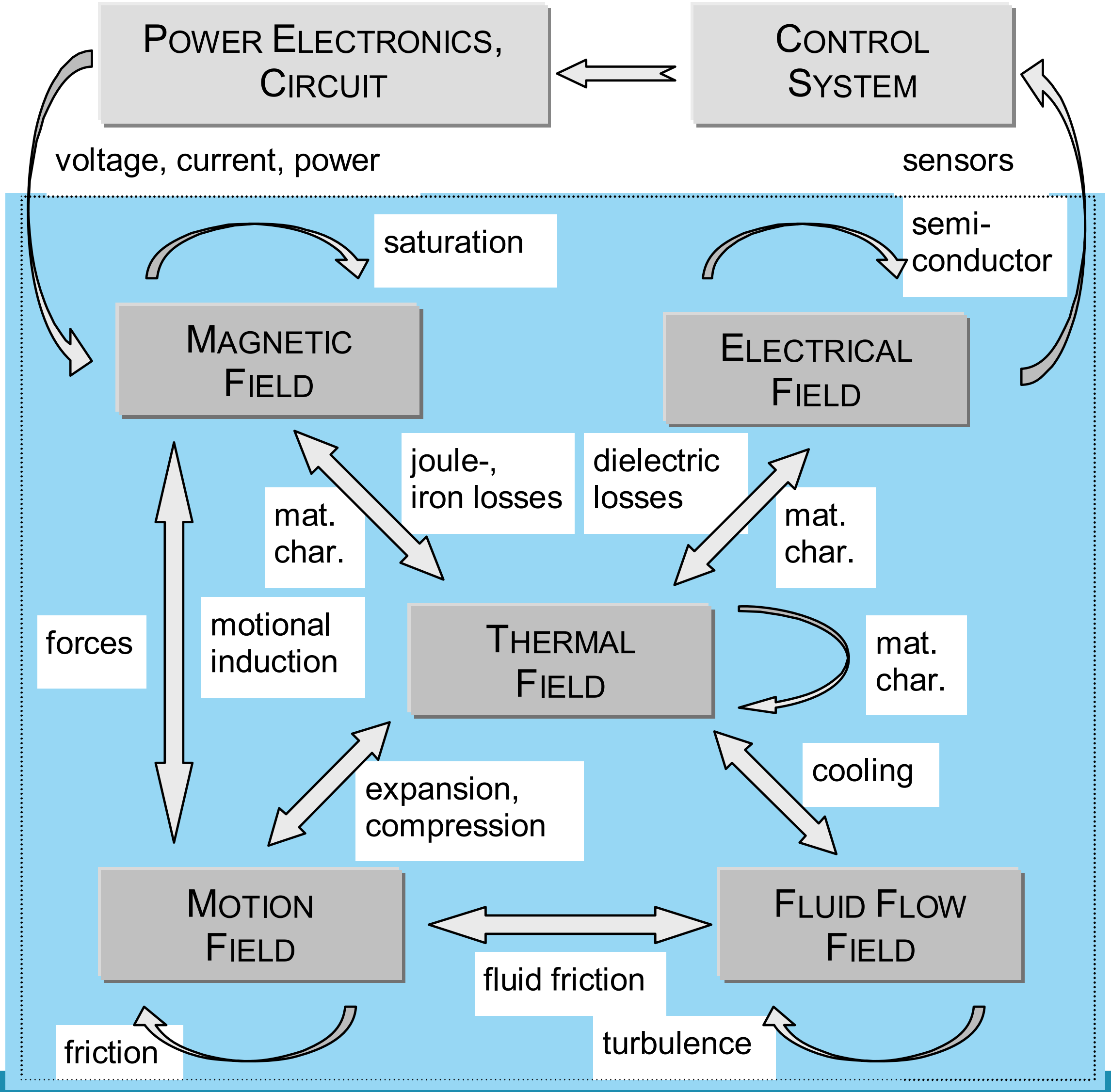
  Continuous sub-problems are discretized to obtain a mathematical model with finite unknowns => **homogeneous or hybrid discretization**
- **Global** non-linear numerical solution algorithm:

  When equations are combined and solved simultaneously in one large matrix system, a fully coupled algorithm is applied

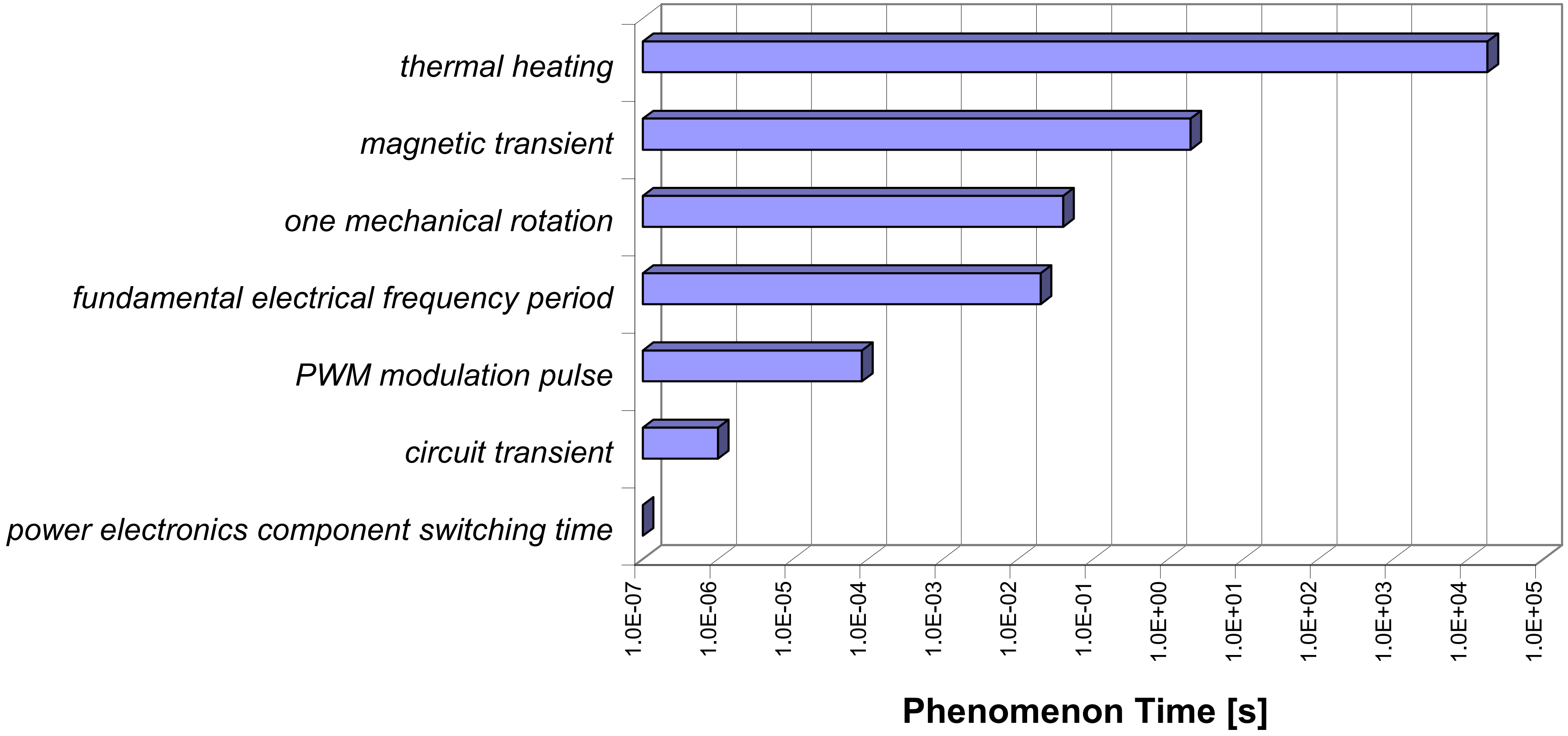
Physical couplings — Electromagnetic fields &

- supply **circuits** (lumped parameters), control system
- **thermal**/temperature fields
- **Solid/rigid body** movement: rotation/translation
- (elastic) **deformation** & vibrations
- **fluid flows**: cooling, molten metals, …
- “material fields” (material behaviour depends on other physics)
- multi-scale problems (e.g., scale of the circuit, scale of the lamination, scale of the machine, scale of the power grid)
  - multi-scale in space
  - multi-scale in time

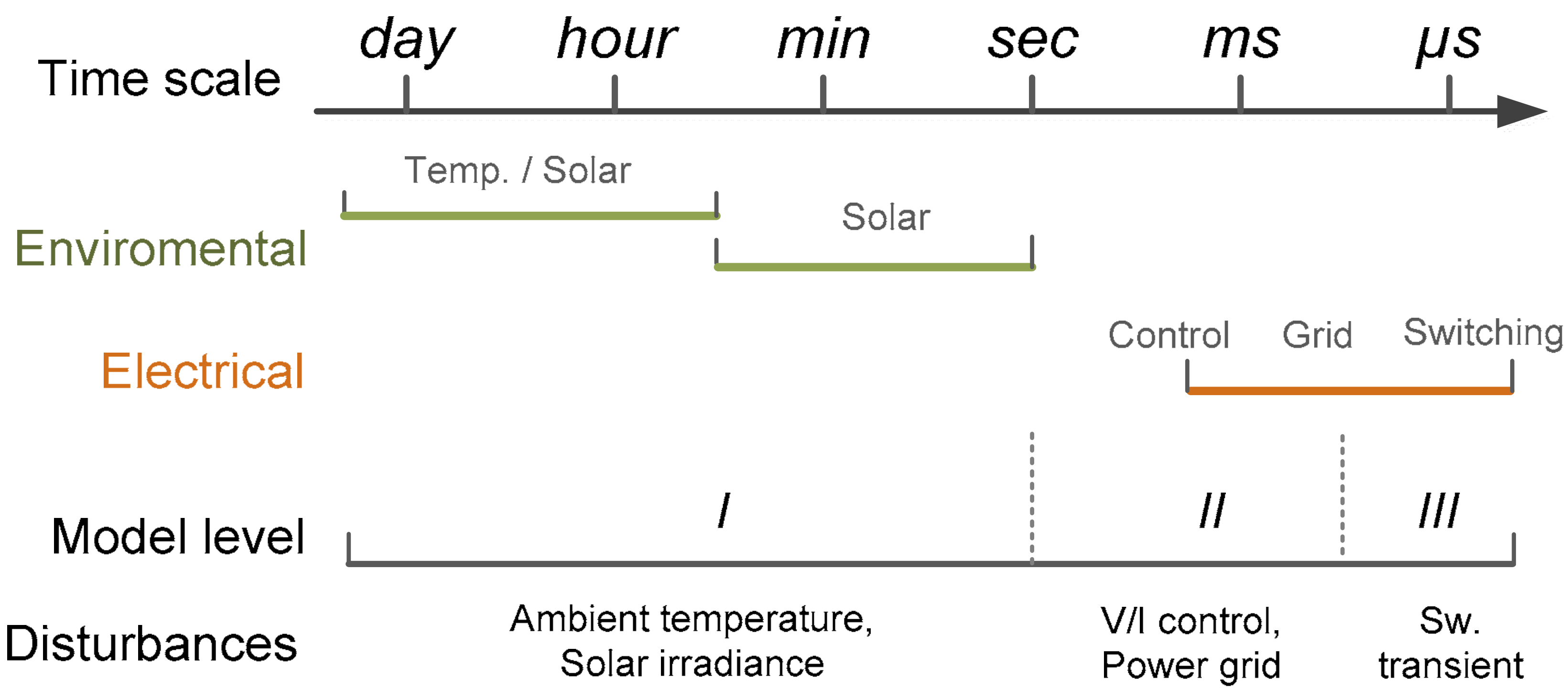
Electrical energy transfer — fields



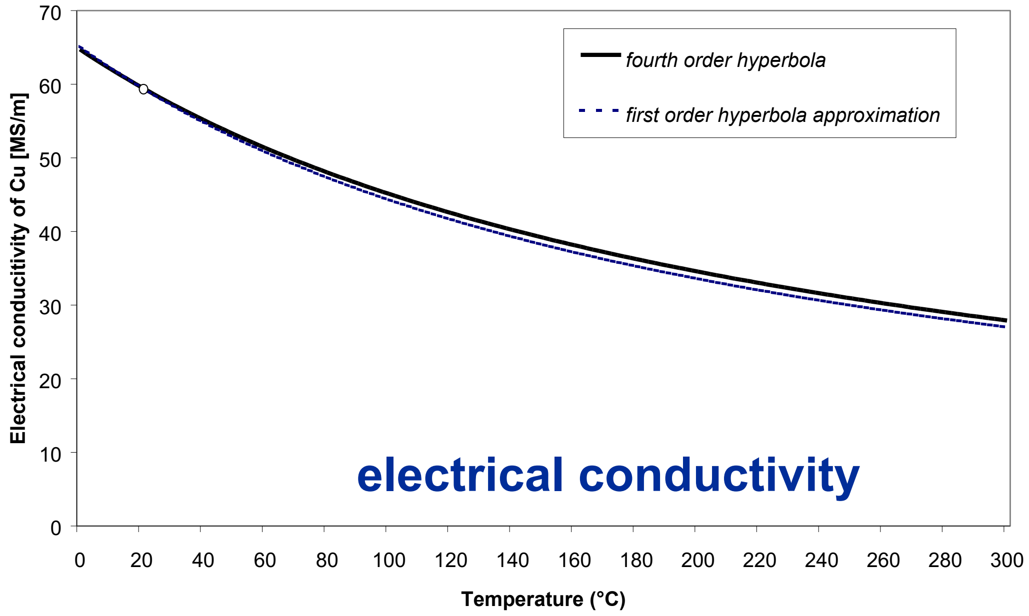
Electrical energy transfer — time scales

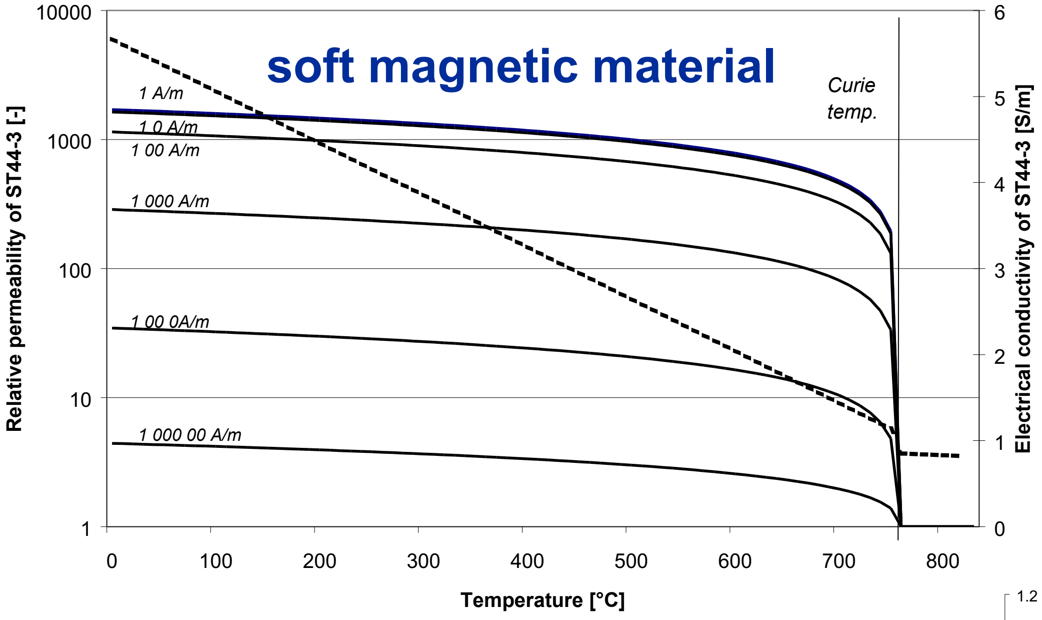


Typical disturbances to the loading of power devices in a PV system

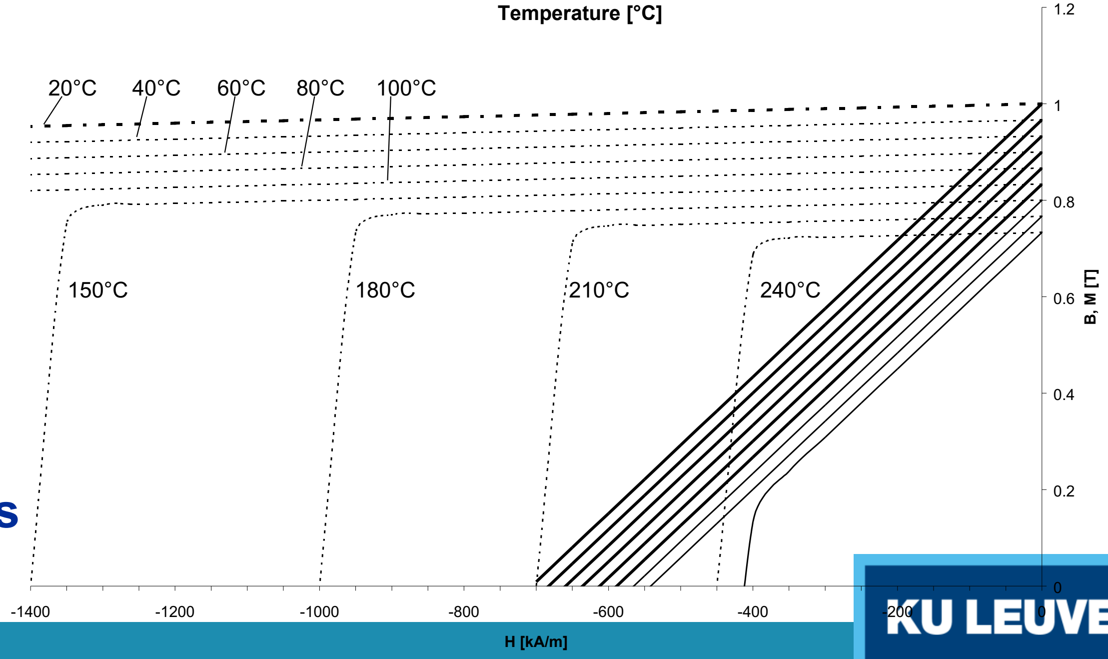


Temperature dependence





permanent magnets

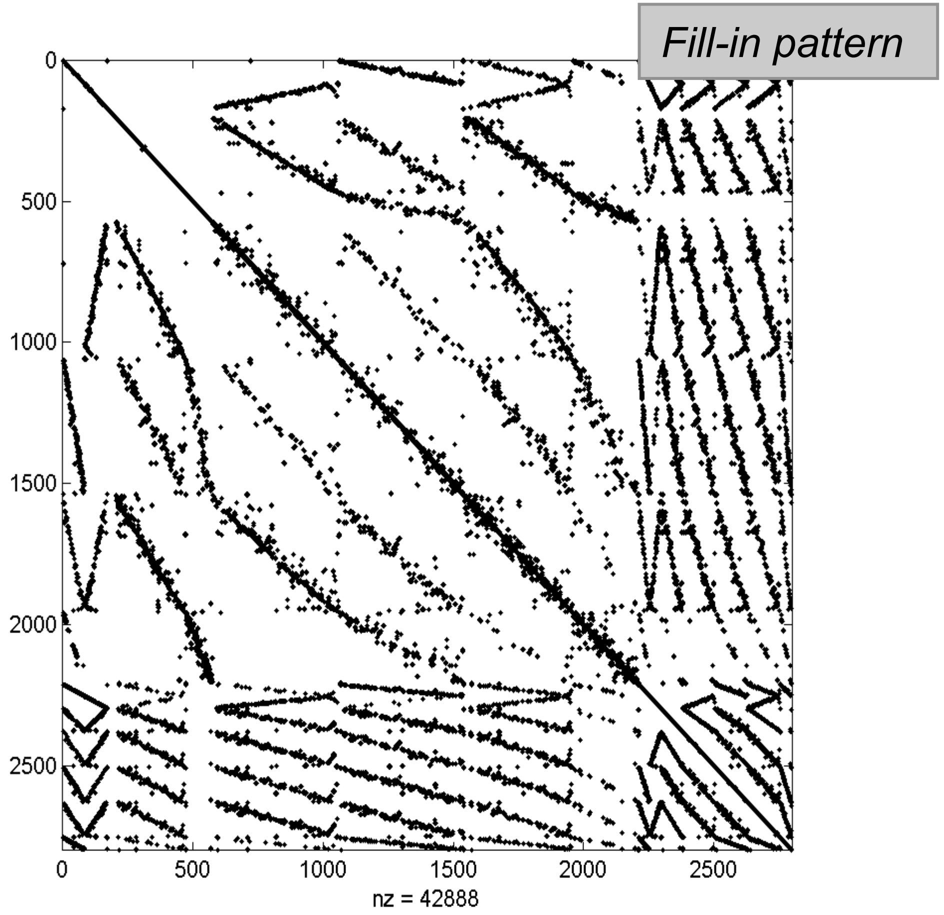


Coupled problem issues

- How to tackle using FEM ?
  - multiple meshes (non-conforming) => in principle projection required!
  - extra non-linear iteration
- weak solution approach
  - outer non-linear loop iterating between subproblems
- strong solution approach
  - all linearized FEM subproblems in one matrix
  - ill-conditioned system and high numerical stiffness

Strong coupled approach
Properties of the Jacobian (Newton algorithm)

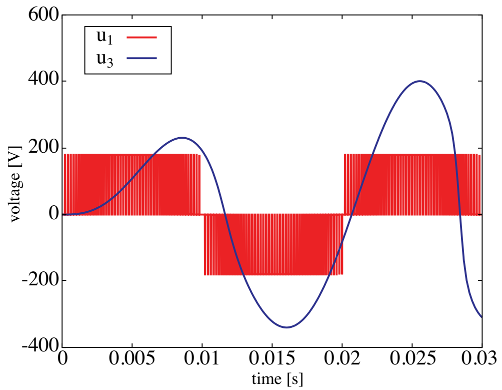
- complicated procedure for computation:
  - dependencies, projection, ...
  - not always possible
  - ill-conditioned
  - asymmetric
  - requires expensive solvers



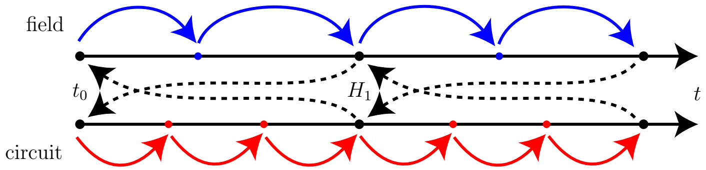
Global convergence issues

- problem:
  - excellent convergence behavior when close to the solution but getting in the vicinity of the solution is a major issue
  - global convergence requirement
- solution: additional measures to ensure convergence
  - relaxation — evolution in the desired direction
  - estimate optimal damping parameter(s) — decreasing residual
- stability — choice of time step
  - smoother pseudo-transient continuation method leading to the desired steady-state solution

Coupled (pseudo-)transient problem approaches e.g. Magneto-thermal



- different time steps for the different sub-problems
  - simulate at the time step of e.g. the magnetic problem
  - simulate at the time step of e.g. the thermal problem
  - simulate using both time scales and extrapolate



- assume ‘a continuously changing steady-state’
- envelope approach:

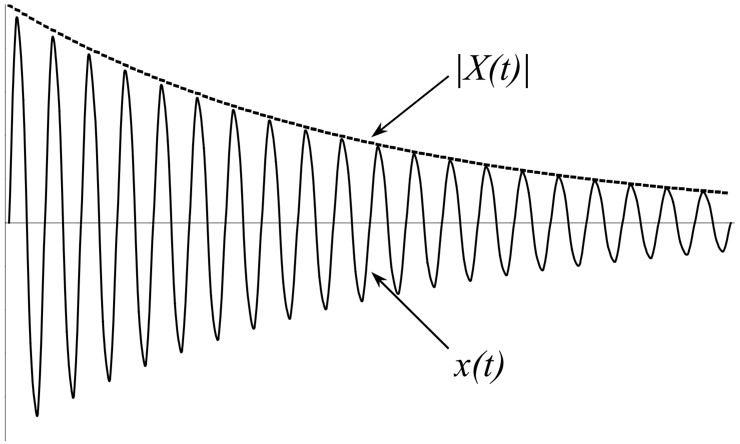
  X = envelope function

  with slow time scale & parameter ω

  linked to the fast time scale

  dealt with in the frequency domain

$$x(t) = X(t, \omega) e^{\imath \omega t}$$



Coupled problems involving electromagnetism
Examples

Electro/Magneto—mechanical coupling
Rigid body

- Translational or rotational motion: all points of body with same (angular) speed

$$\boldsymbol{F}_{\text{ext}} = m\frac{d\boldsymbol{v}}{dt} - \lambda_1 \boldsymbol{v} - \lambda_2 \boldsymbol{v}^2 \qquad \boldsymbol{T}_{\text{ext}} = I\frac{d\boldsymbol{\Omega}}{dt} - \lambda_1 \boldsymbol{\Omega} - \lambda_2 \boldsymbol{\Omega}^2$$

- Magnetic/electric source for mechanics

$$\boldsymbol{F}_{\text{mag/ele}} = \int_V \operatorname{div} \boldsymbol{T}\, \mathrm{d}V = \oint_{\partial V} \boldsymbol{T}\cdot\boldsymbol{n}\, \mathrm{d}S \Longrightarrow \boldsymbol{F}_{\text{ext}} \quad (\text{or } \boldsymbol{T}_{\text{mag/ele}} \Longrightarrow \boldsymbol{T}_{\text{ext}})$$

- Electromagnetic formulation
  - e.g. magnetostatics, magnetodynamics

$$\boldsymbol{F}_{\text{mag}} = \oint_{\partial V} \left[(\boldsymbol{n}\cdot\boldsymbol{b})\boldsymbol{h} - \frac{1}{2}(\boldsymbol{h}\cdot\boldsymbol{b})\boldsymbol{n} \right] \mathrm{d}S = \oint_{\partial V} \nu \left[(\boldsymbol{n}\cdot\boldsymbol{b})\boldsymbol{b} - \frac{1}{2}|\boldsymbol{b}|^2\boldsymbol{n} \right] \mathrm{d}S$$

  - e.g. electrostatics

$$\boldsymbol{F}_{\text{ele}} = \oint_{\partial V} \left[(\boldsymbol{n}\cdot\boldsymbol{d})\boldsymbol{e} - \frac{1}{2}(\boldsymbol{e}\cdot\boldsymbol{d})\boldsymbol{n} \right] \mathrm{d}S = \oint_{\partial V} \epsilon \left[(\boldsymbol{n}\cdot\boldsymbol{e})\boldsymbol{e} - \frac{1}{2}|\boldsymbol{e}|^2\boldsymbol{n} \right] \mathrm{d}S$$

Electro/Magneto—mechanical coupling
Elastic body

- Use of EM formulation for computing the local mechanical source
  - e.g. electrostatics => $\boldsymbol{f}_{\text{ele}} = \operatorname{div}\left[(\boldsymbol{d}\cdot\boldsymbol{e}) - \frac{1}{2}(\boldsymbol{d}\cdot\boldsymbol{e})\underline{\underline{\mathbf{1}}}\right]$
  - e.g. magnetostatics => $\boldsymbol{f}_{\text{mag}} = \operatorname{div}\left[(\boldsymbol{h}\cdot\boldsymbol{b}) - \frac{1}{2}(\boldsymbol{h}\cdot\boldsymbol{b})\underline{\underline{\mathbf{1}}}\right]$
- Mechanical formulation
  - e.g. elastic deformation, local displacement $\underline{\underline{D}}^T\,\underline{\underline{E}}\,\underline{\underline{D}}\,\boldsymbol{u} + \boldsymbol{f} = 0$

$$\boldsymbol{D} = \begin{bmatrix} \partial_x & 0 & 0 \\ 0 & \partial_y & 0 \\ 0 & 0 & \partial_z \\ \partial_y & \partial_x & 0 \\ \partial_z & 0 & \partial_x \\ 0 & \partial_z & \partial_y \end{bmatrix}$$

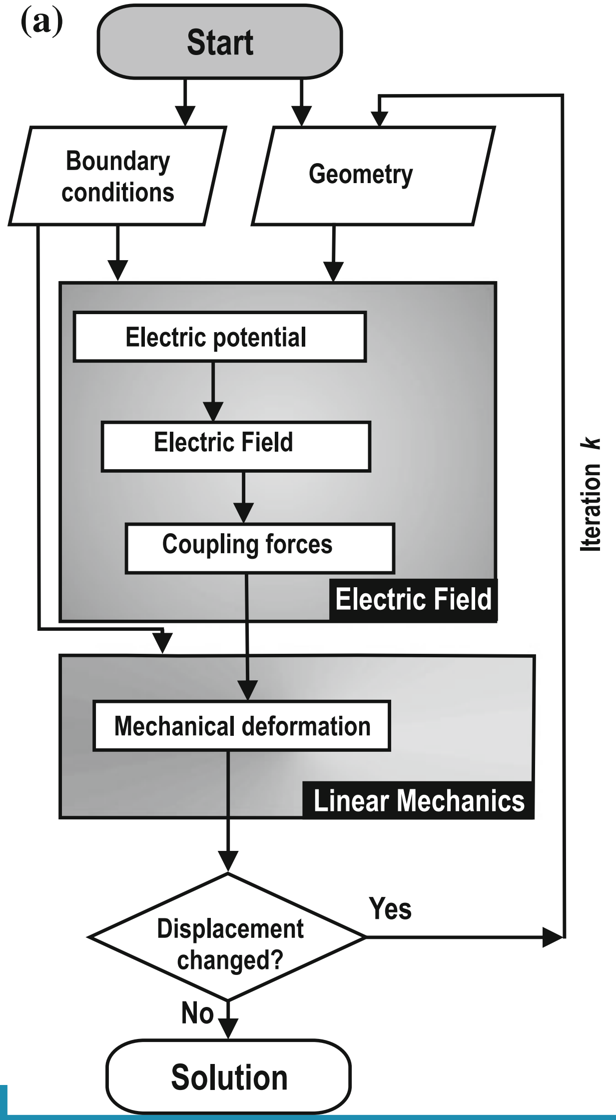
elasticity tensor

$$\underline{\underline{E}} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & & & \\ \nu & 1-\nu & \nu & & \mathbf{0} & \\ \nu & \nu & 1-\nu & & & \\ & & & \frac{1-2\nu}{2} & 0 & 0 \\ & \mathbf{0} & & 0 & \frac{1-2\nu}{2} & 0 \\ & & & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}.$$

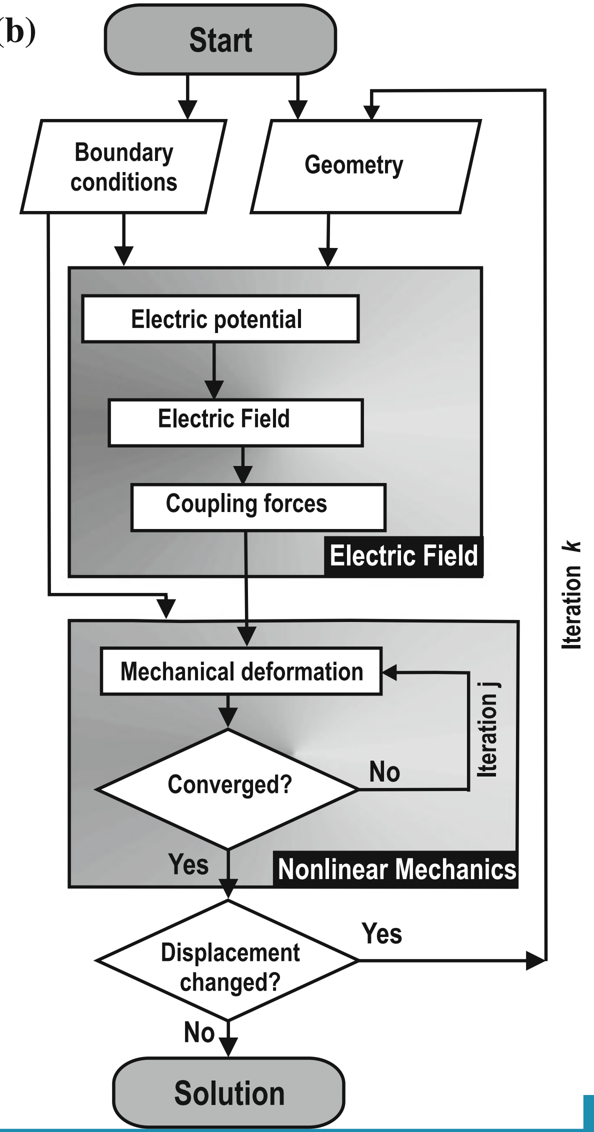
Young modulus E, Poisson ratio ν

Coupled electrostatic-mechanical simulation schemes

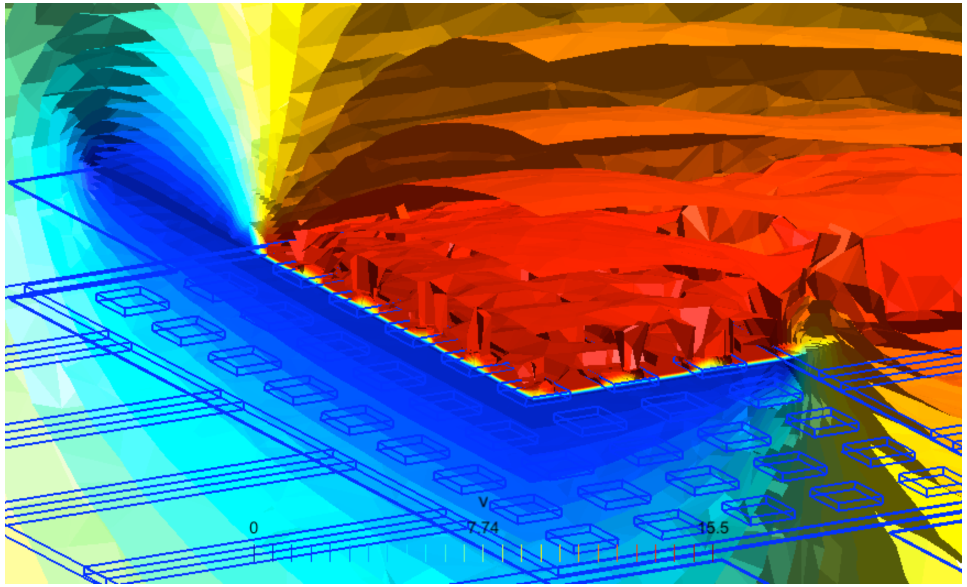
(a)

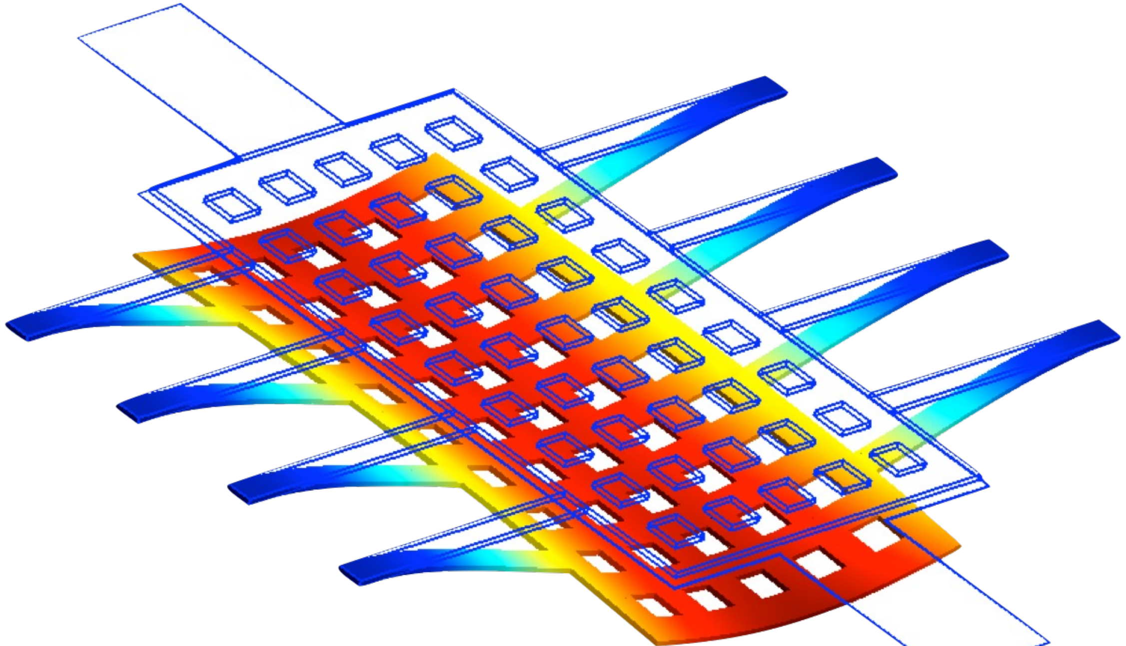


(b)



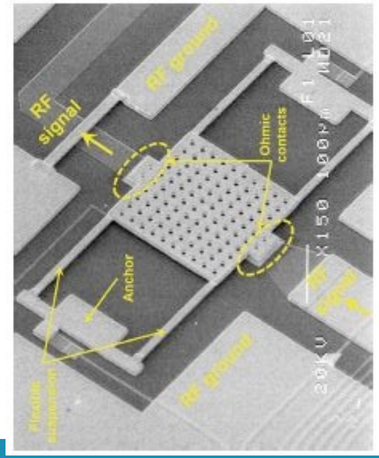
Electro-mechanical

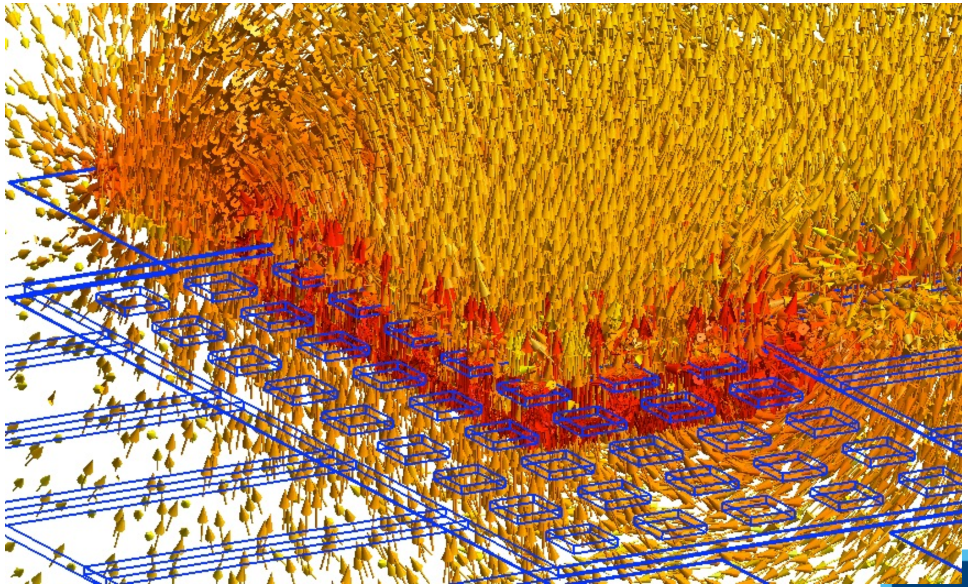




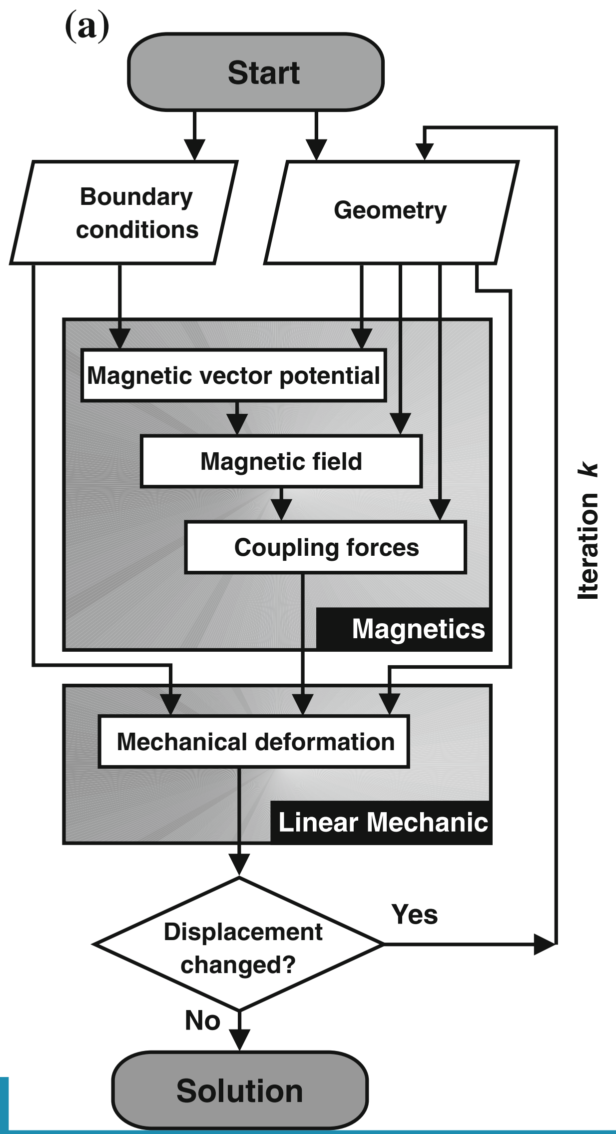
shunt capacitive MEM switch

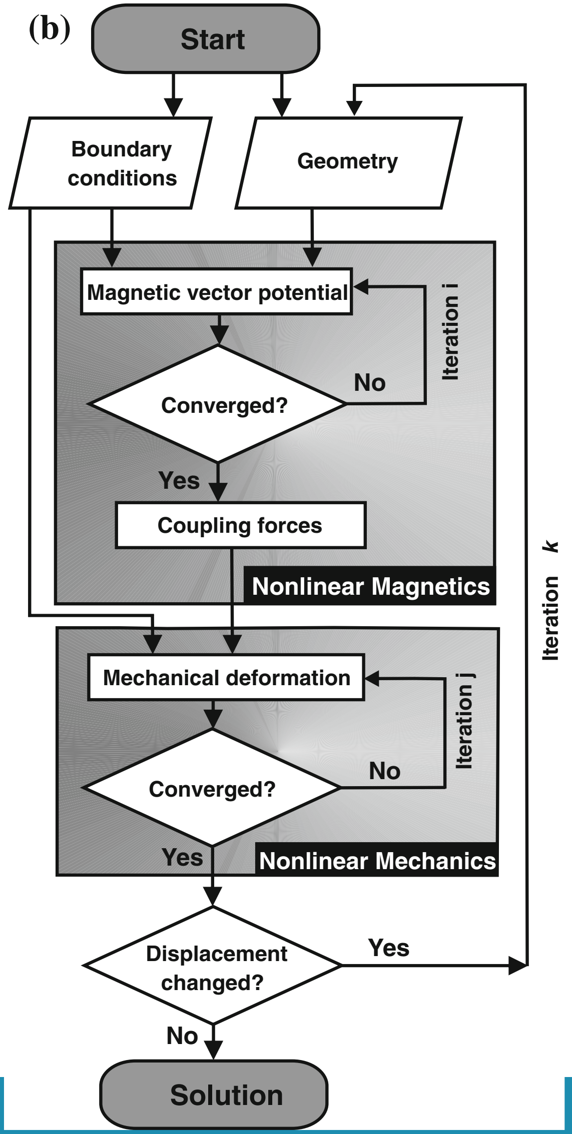
computation of
* electrostatic force
* elastic deformation



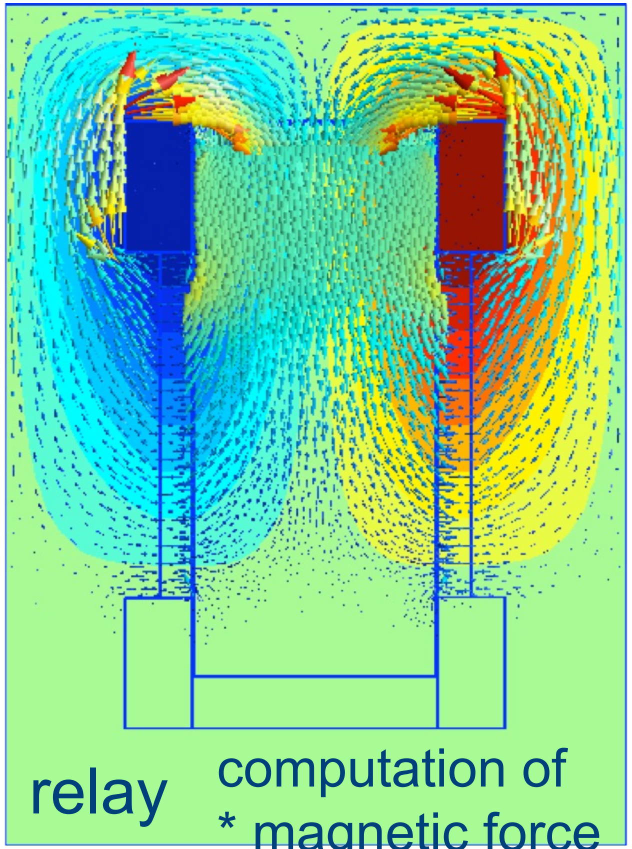


Coupled magneto-mechanical simulation schemes





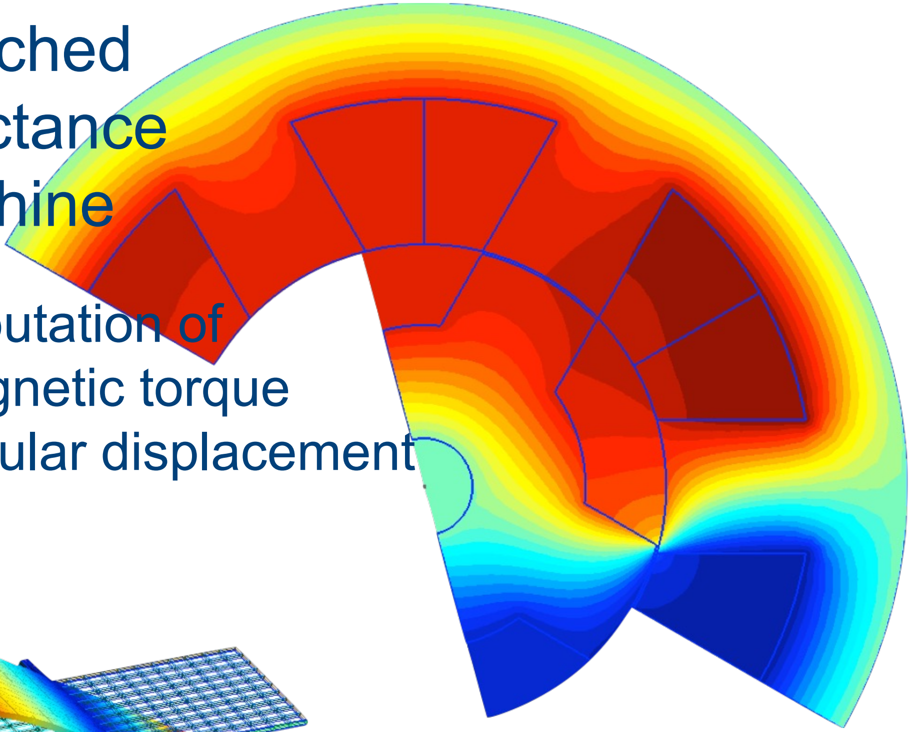
Magneto-mechanical



relay

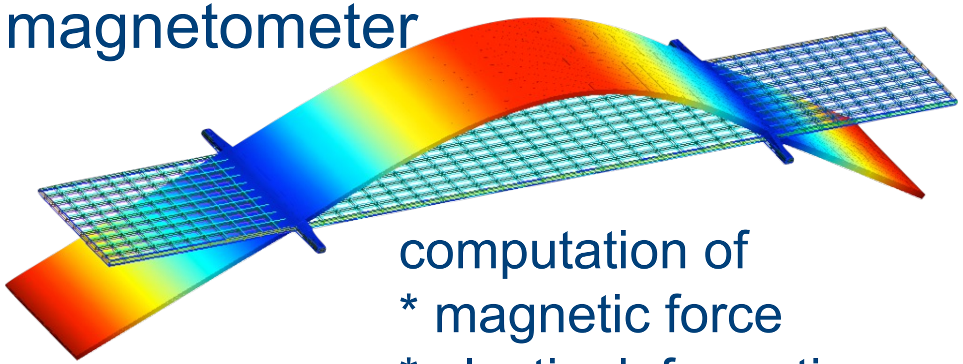
computation of
* magnetic force
* trans. displacement

Switched reluctance machine



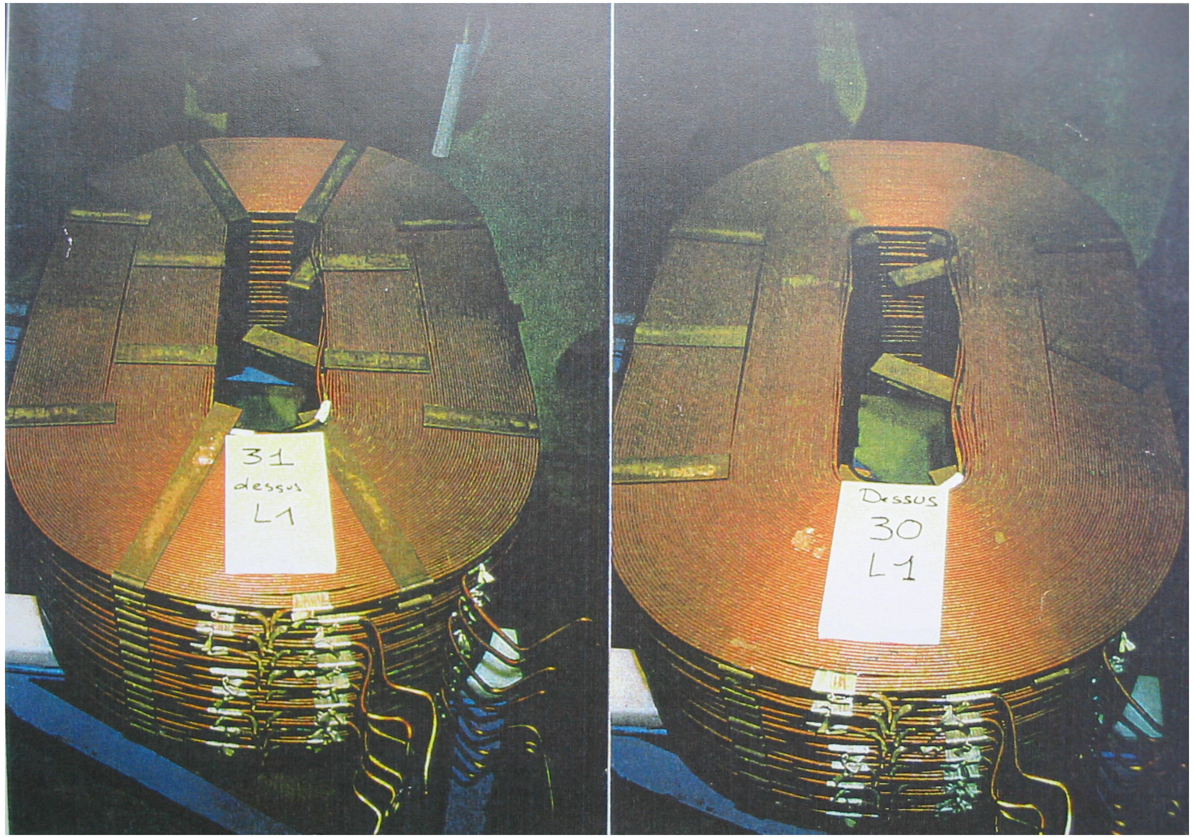
computation of
* magnetic torque
* angular displacement

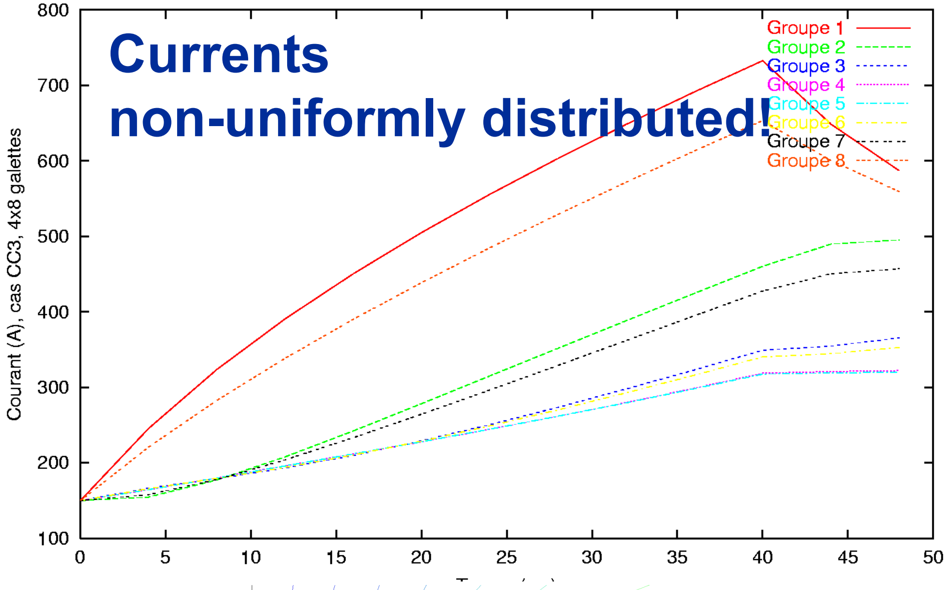
MEMS magnetometer

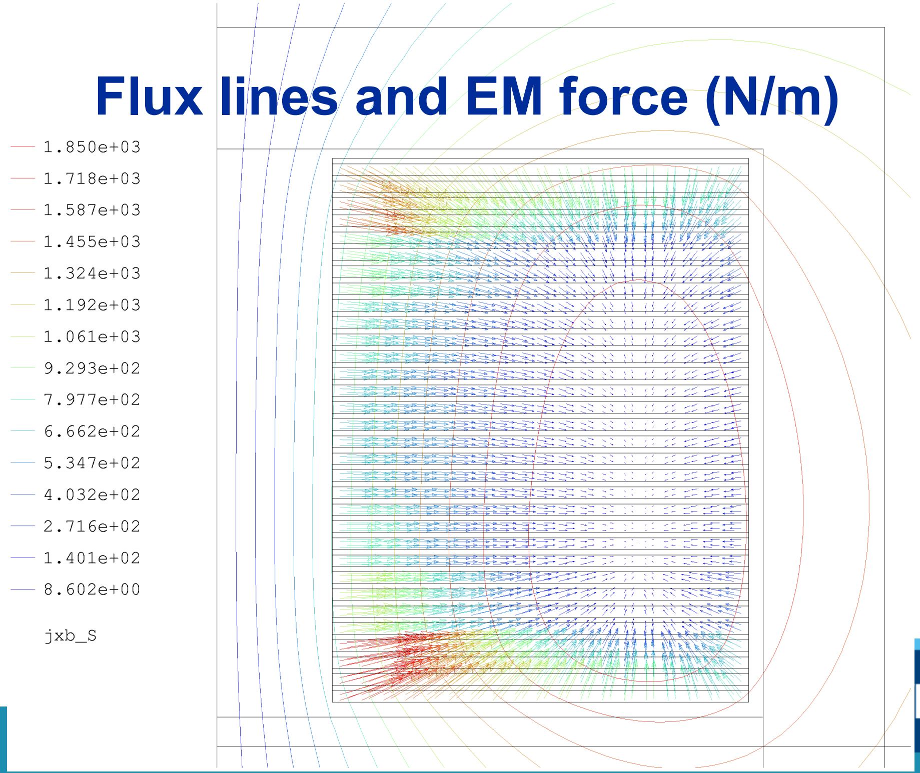


computation of
* magnetic force
* elastic deformation

Magneto-mechanical







Total current (3200 A)
distributed in 8 groups in parallel

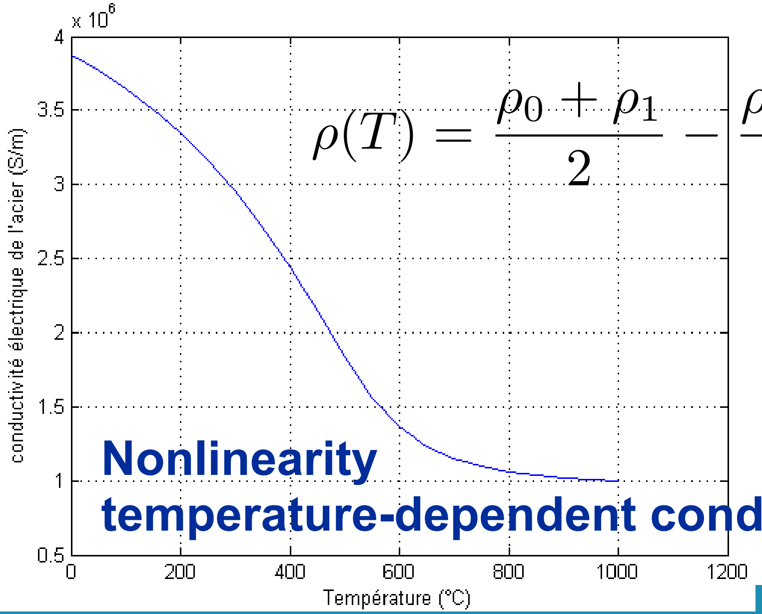
Magneto-thermal coupling

- Magnetodynamic formulation

$$\operatorname{curl} \nu \operatorname{curl} \boldsymbol{a} + \sigma\,(\partial_t \boldsymbol{a} + \operatorname{grad} v) = \boldsymbol{j}_s\,, \quad \operatorname{div}\Big(-\sigma(\partial_t \boldsymbol{a} + \operatorname{grad} v)\Big) = 0 \quad \text{in } \Omega^C$$

$$\text{with } \boldsymbol{b} = \operatorname{curl} \boldsymbol{a}\,, \quad \boldsymbol{e} = -\operatorname{grad} v - \partial_t \boldsymbol{a}$$

- Electromagnetic energy as thermal source => Joule losses $p = \boldsymbol{e} \cdot \boldsymbol{j}$
- Thermal formulation, e.g. heat conduction problem $-\operatorname{div}(\kappa \operatorname{grad} T) = p$



$$\rho(T) = \frac{\rho_0 + \rho_1}{2} - \frac{\rho_0 - \rho_1}{\pi} \arctan \frac{T - 273.15 - T_c}{T_r}$$

Nonlinearity
temperature-dependent conductivity

$\kappa =$ thermal conductivity
$T =$ temperature
$p =$ thermal power $\Rightarrow$ Joule losses
$\rho = \sigma^{-1} =$ resistivity

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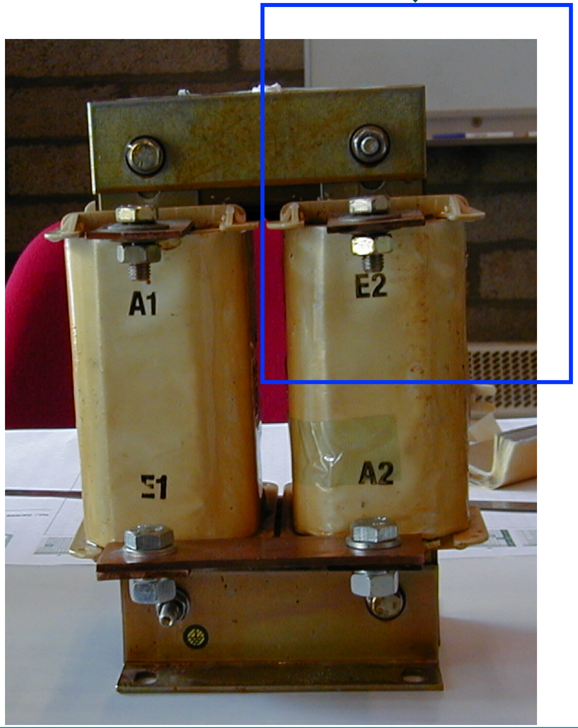
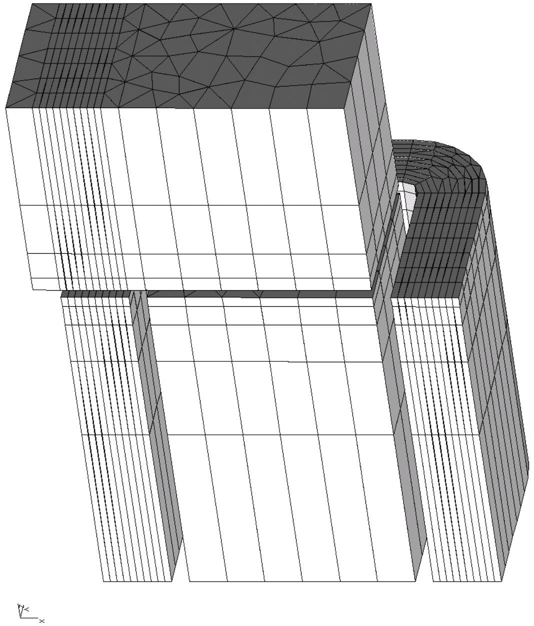
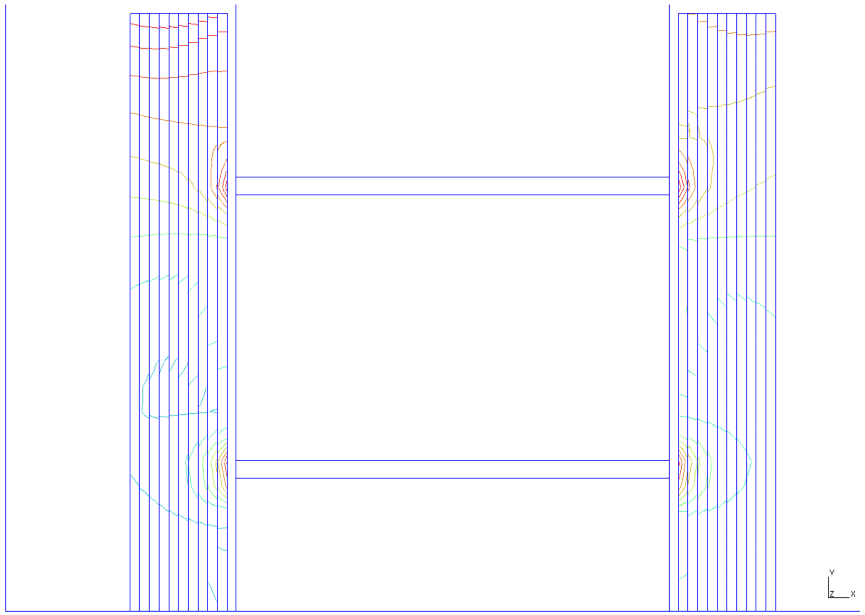
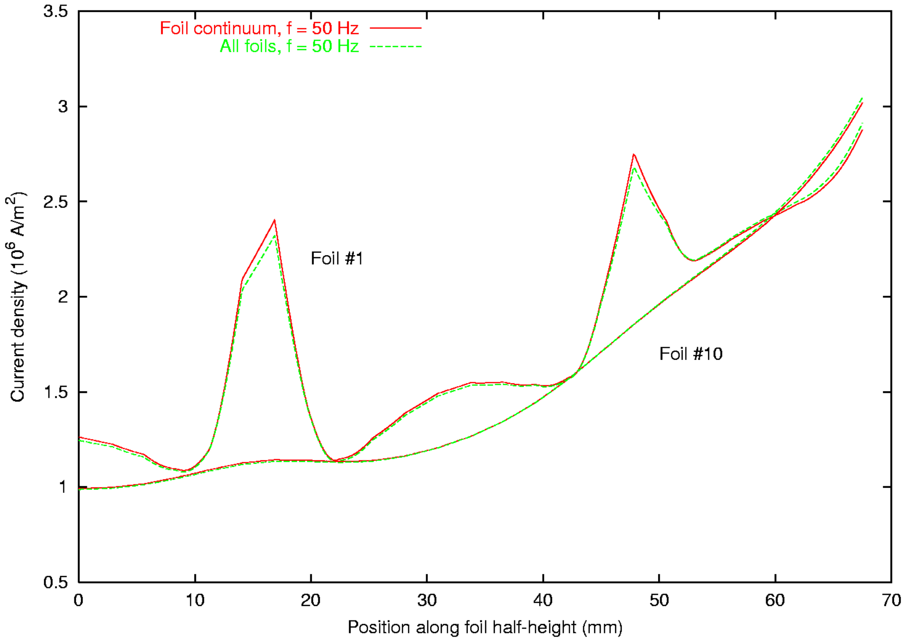
Magneto-thermal coupling

Foil winding inductor



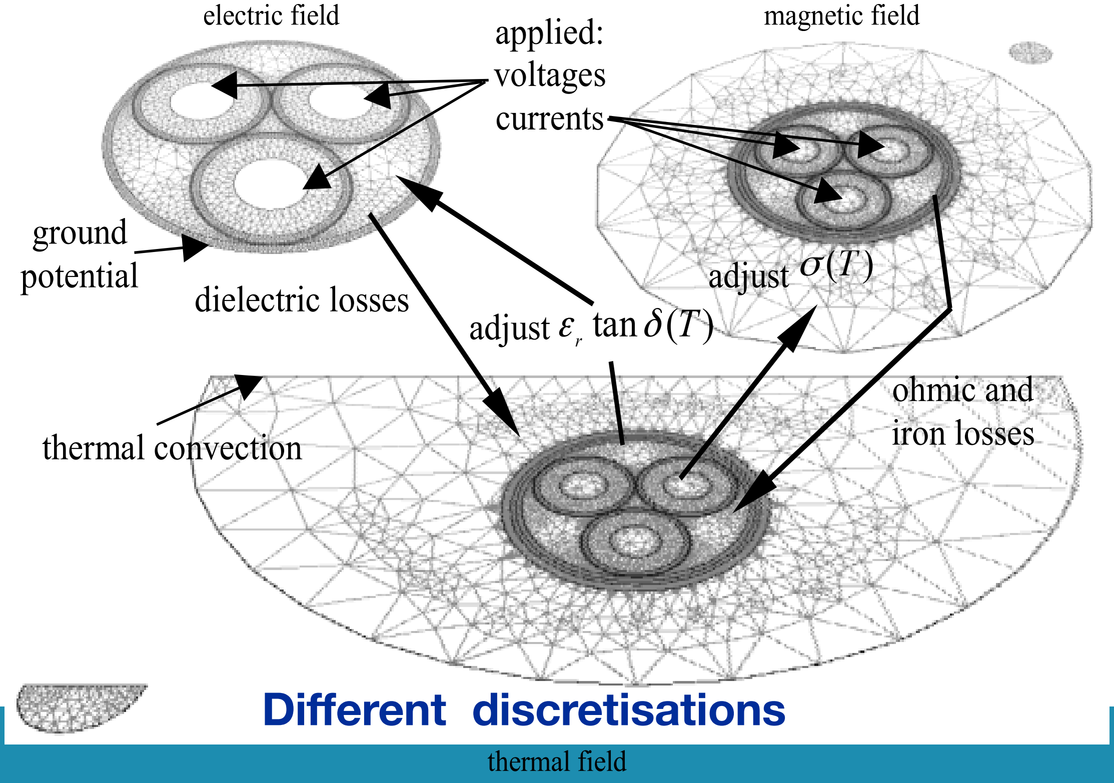




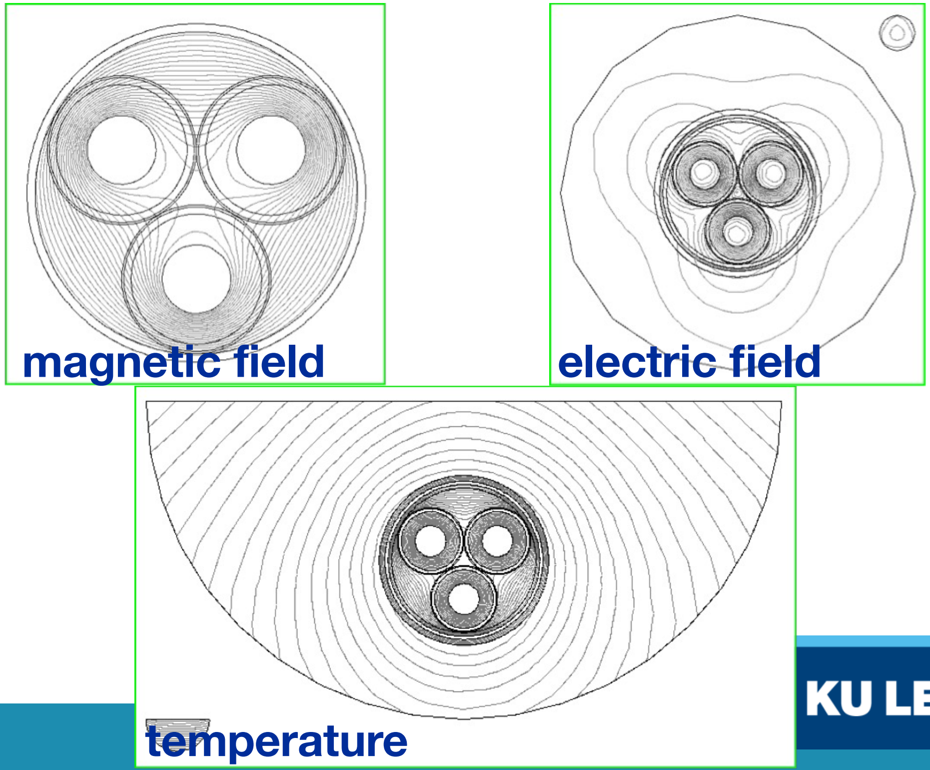


Magnetic/electrostatic/thermal coupling

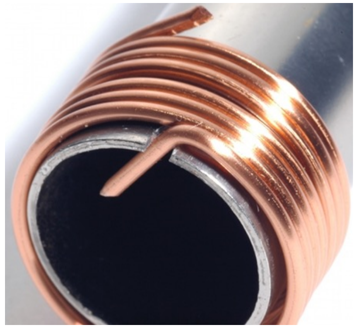
- High voltage power cable
  - ohmic losses in conductors
  - dielectric losses in insulation
  - magnetic losses —hysteresis & eddy currents in shield

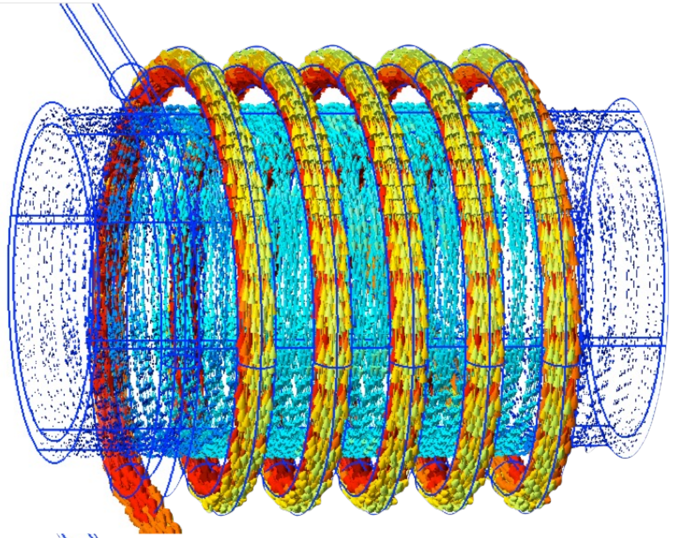


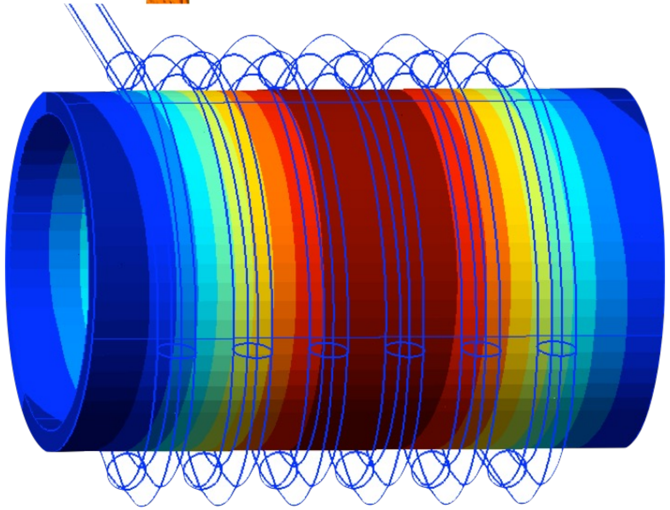
Different discretisations



Magneto-thermal Induction heating

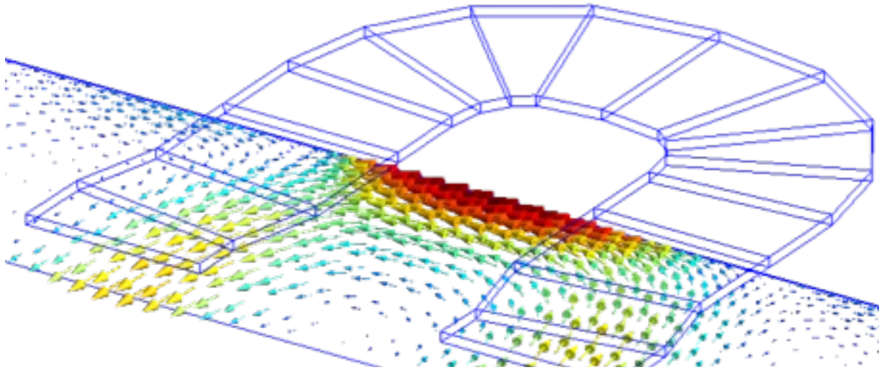




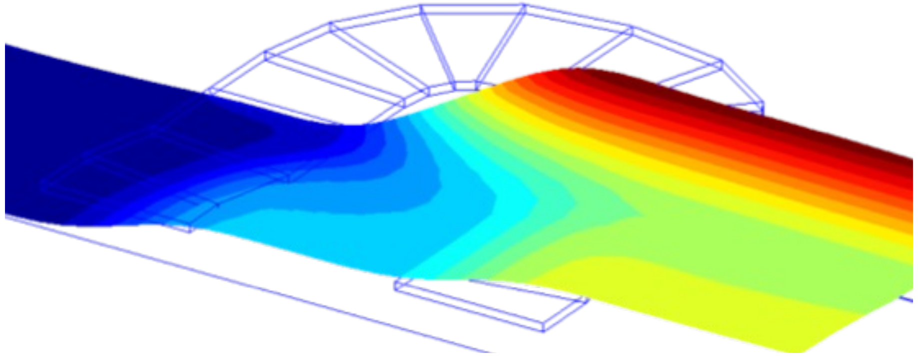


eddy current distribution

computation of
* Joule losses
* temperature
* additional non-linearity σ(T)



temperature distribution



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Movement treatment

Eulerian vs Lagrangian approach
Moving-band techniques
Mortar method

Movement with one degree of freedom
Solid body motion

- In most electromechanical systems, one degree of freedom suffices to describe the movement
- contacts and frictions appear in the equation of motion
- Translational motion: all the points of the body have the same velocity

$$\boldsymbol{F}_{\text{ext}} = m\frac{d\boldsymbol{v}}{dt} - \lambda_1 \boldsymbol{v} - \lambda_2 \boldsymbol{v}^2$$

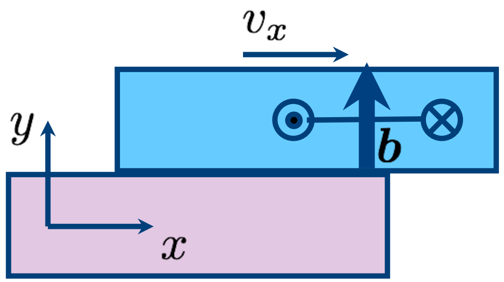
- Rotational motion: all the points of the body rotate at an angular velocity

$$\boldsymbol{T}_{\text{ext}} = I\frac{d\boldsymbol{\Omega}}{dt} - \lambda_1 \boldsymbol{\Omega} - \lambda_2 \boldsymbol{\Omega}^2$$

$$\begin{aligned} \boldsymbol{F}_{\text{ext}} &= \text{ external force} \\ m &= \text{ masse} \\ v &= \text{ speed} \\ \lambda_1 &= \text{ dry friction coefficient} \\ \lambda_2 &= \text{ viscous friction coefficient} \end{aligned}$$

$$\begin{aligned} \boldsymbol{T}_{\text{ext}} &= \text{ external torque} \\ I &= \text{ Inertia} \\ \Omega &= \text{ angular speed} \end{aligned}$$

Eulerian approach



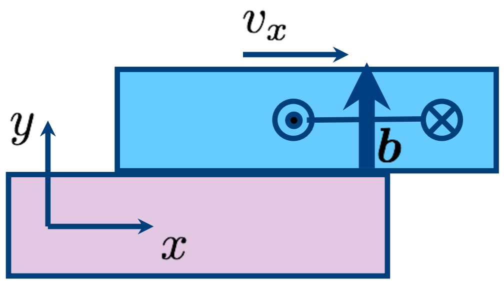
- all phenomena described from same reference (standstill/laboratory) frame
- modified Ohm's law $\boldsymbol{j} = \sigma \boldsymbol{e} + \sigma \boldsymbol{v} \times \boldsymbol{b}$
- terms associated with speed are transformed $v =$ scalar potential, $\boldsymbol{v} =$ speed

$$\mathrm{curl}\,(\nu\,\mathrm{curl}\,\boldsymbol{a}) + \sigma\,(\partial_t \boldsymbol{a} + \boldsymbol{a} \times \Omega + \mathrm{grad}\,(v - \boldsymbol{a} \cdot \boldsymbol{v})) = \boldsymbol{j}_s$$

$$\mathrm{curl}\,(\rho\,\mathrm{curl}\,\boldsymbol{t}) + \mu\,(\partial_t(\boldsymbol{t} - \mathrm{grad}\,\phi) + \Omega \times (\boldsymbol{t} - \mathrm{grad}\,\phi)) = 0$$

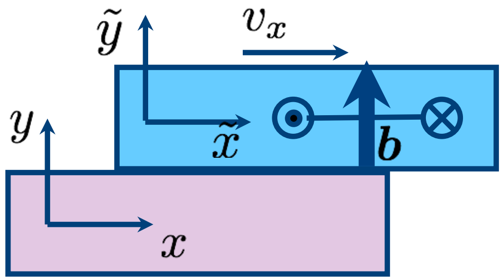
- two types of terms
  - with a total **derivative**: it measures changes in a material point of the body in motion. This can be evaluated by finite differences. In general, the simplest way to calculate it is to move the mesh with the moving part
  - with a (rotational) speed: source of numerical instabilities

Eulerian approach



- discretization with classical interpolation functions and Galerkin approach leads to stability problems
- transport term gives rise to non-physical oscillations that grow with the speed
- with nodal basis functions (first degree Lagrange polynomials) and Galerkin approach, there is a number that measures the unstable nature of the numerical solution $P_e = \mu\sigma v l_c$
- oscillations appear for $P_e \geq 2$, as l_c is the characteristic length of the mesh elements, a good mesh allows for a non-oscillating solution (rest of the parameters are fixed by the problem)
- quality of the solution can be improved by using a Petrov-Galerkin method, with a weighting function better adapted to the physical solution
- Eulerian approach not used in practice, Lagrangian approach is preferred in EM

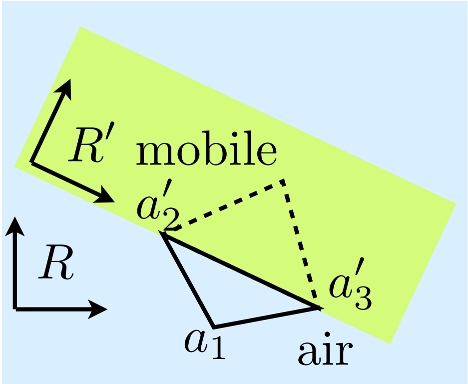
Lagrangian approach



- phenomena observed from each material point: a reference frame for each part of moving part (n moving parts => n reference frames)
- Maxwell's equations with same form in fixed and moving reference frame

$$\operatorname{curl}(\nu \operatorname{curl} \boldsymbol{a}) + \sigma (\partial_t \boldsymbol{a} + \operatorname{grad} v) = \boldsymbol{j}_s$$

$$\operatorname{curl}(\rho \operatorname{curl} \boldsymbol{t}) + \mu \, \partial_t (\boldsymbol{t} - \operatorname{grad} \phi) = 0$$



- each part of the problem is 'at rest' in the associated frame
- mesh attached to each moving part
- any speed term is discarded

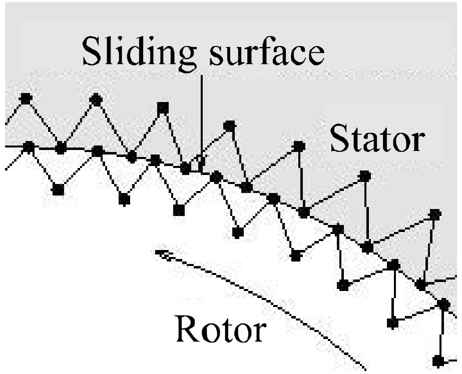
$$a = \alpha_1 a_1 + \alpha_2 P a_2' + \alpha_3 P a_3'$$

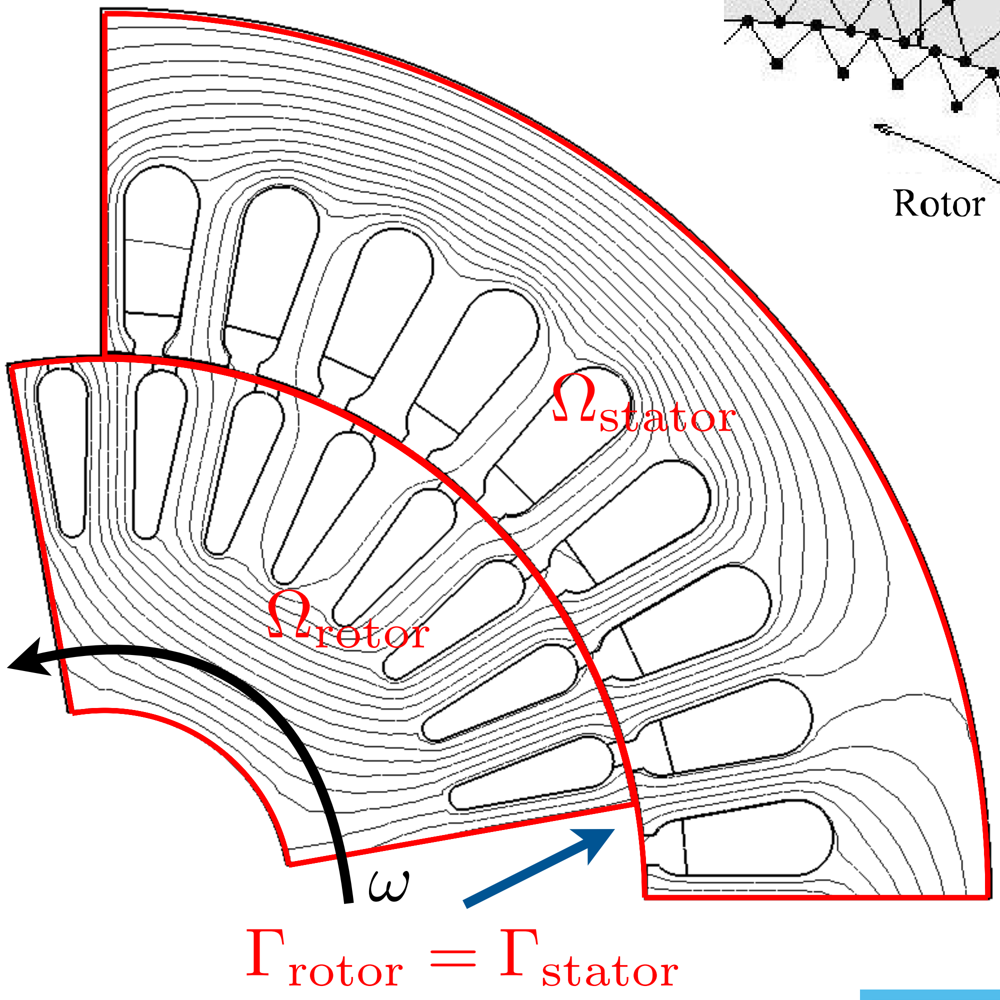
- relation between Euler and Lagrangian approach, e.g., for the electric field

$$\boldsymbol{e}_{\text{Euler}} = \boldsymbol{e}_{\text{Lagrange}} + \boldsymbol{v} \times \boldsymbol{b}$$

Explicitly considering motion (Lagrange)

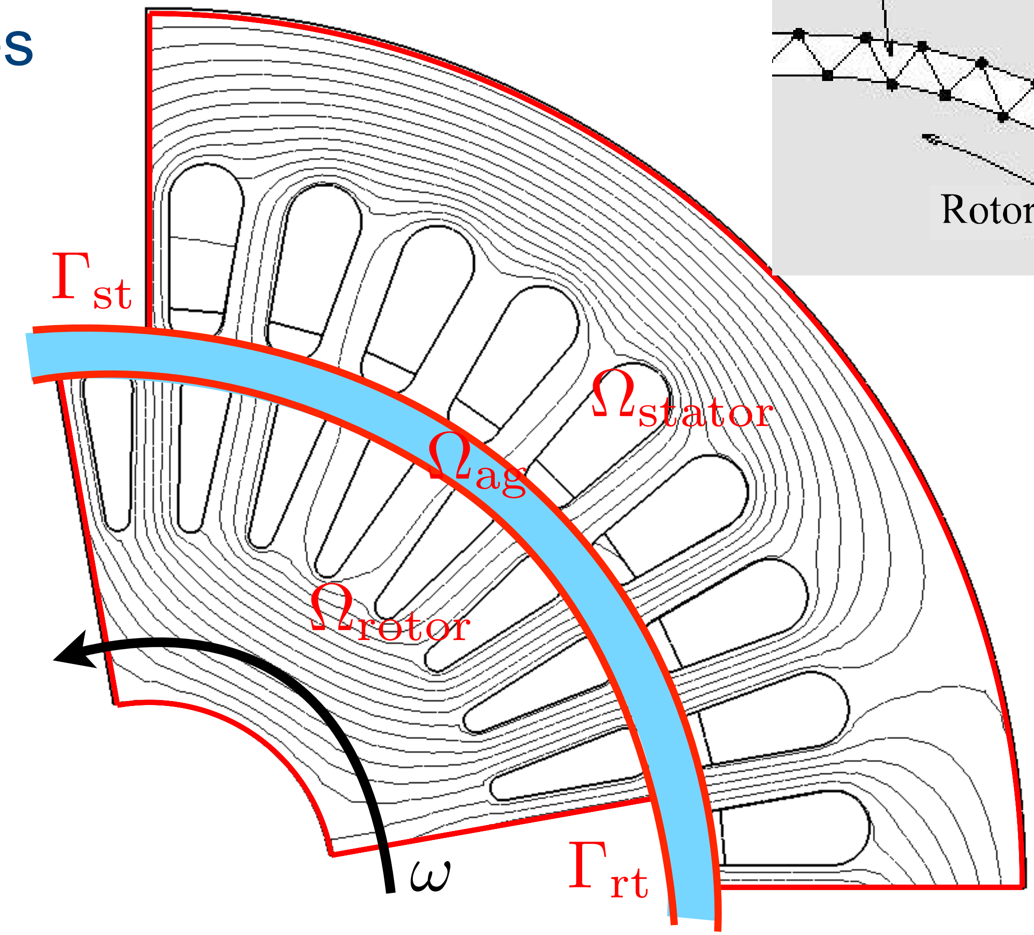
- Modeling rotation in electrical machines
  - sliding-surface techniques
    - locked step approach
    - linear/quadratic interpolation
    - trigonometric interpolation

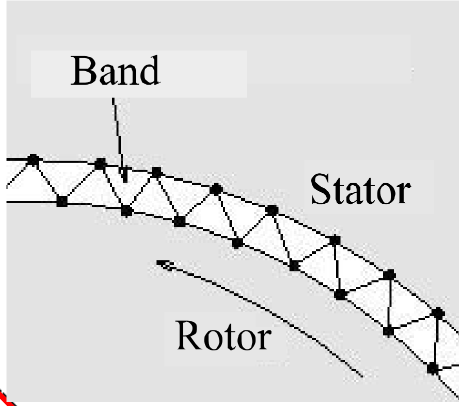




Explicitly considering motion (Lagrange)

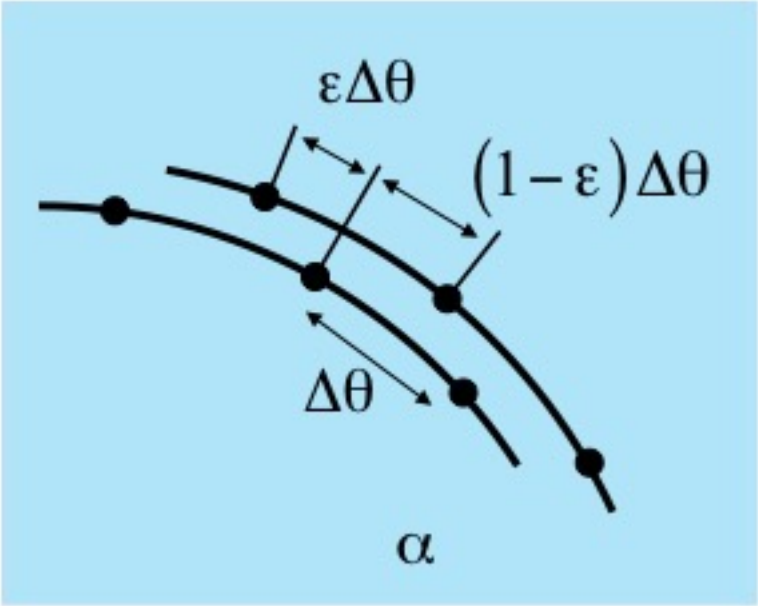
- Modeling rotation in electrical machines
  - sliding-surface techniques
    - locked step approach
    - linear/quadratic interpolation
    - trigonometric interpolation
  - air-gap models
    - moving-band techniques
    - boundary elements
    - discontinuous Galerkin technique
    - air-gap element (spectral elements)





Sliding-surface technique

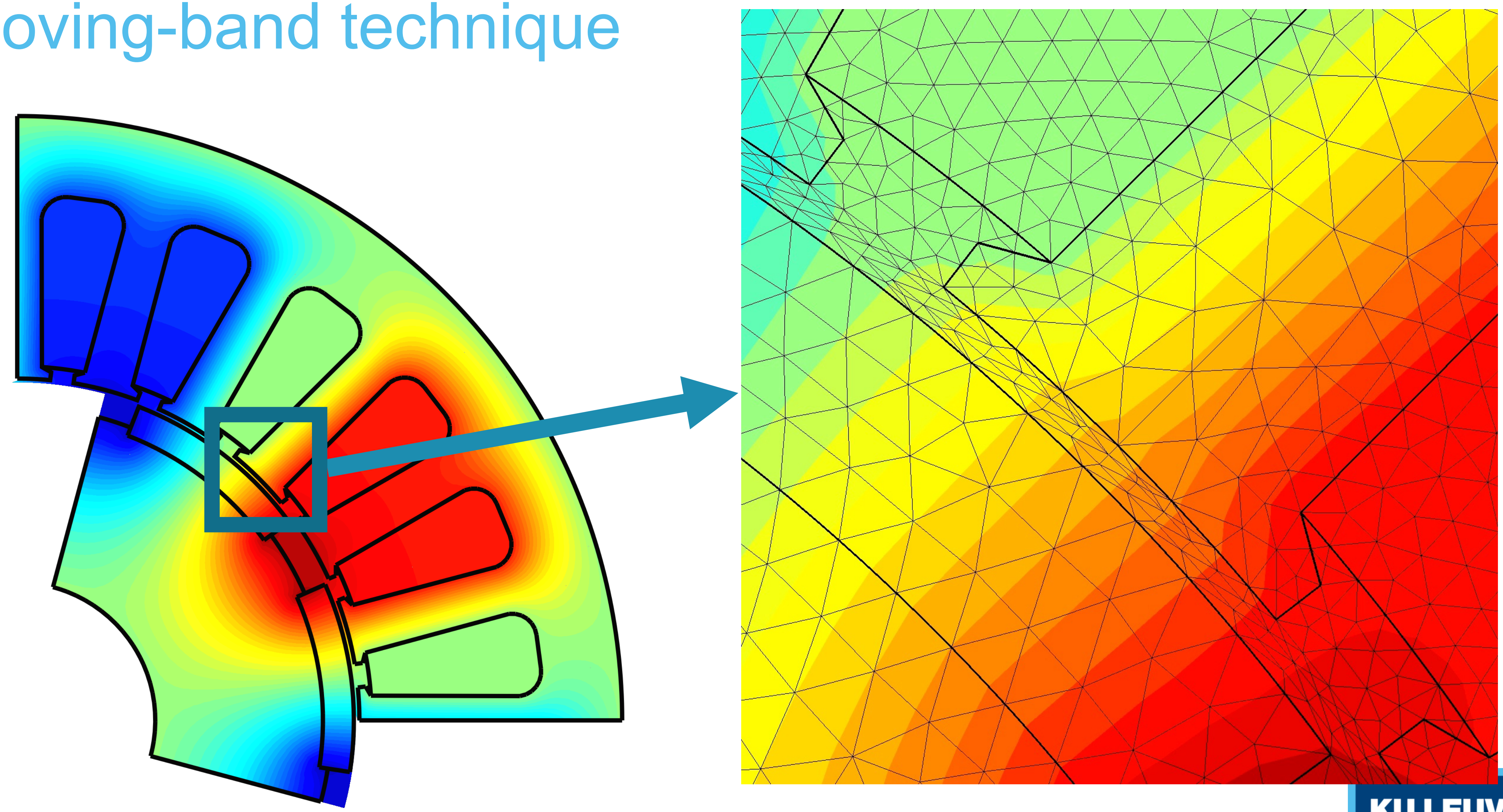
- non-matching grids at interface
  - linear interpolation/extrapolation
  - mortar approach (transmission conditions between elements via Lagrange multipliers)



↓

☹ eccentricity
☹ consistency error
☹ torque ripple

Moving-band technique



Interface conditions

- decoupled FE system $[\ \boldsymbol{K}\ ][\ \boldsymbol{A}\ ]+[\ \boldsymbol{G}\ ]=[\ \boldsymbol{J}\ ]$

$$\begin{bmatrix}\boldsymbol{K}_{\mathrm{st}} & \\ & \boldsymbol{K}_{\mathrm{rt}}\end{bmatrix}\begin{bmatrix}\boldsymbol{a}_{\mathrm{st}}\\ \boldsymbol{a}_{\mathrm{rt}}\end{bmatrix}+\begin{bmatrix}\boldsymbol{g}_{\mathrm{st}}\\ \boldsymbol{g}_{\mathrm{rt}}\end{bmatrix}=\begin{bmatrix}\boldsymbol{j}_{\mathrm{st}}\\ \boldsymbol{j}_{\mathrm{rt}}\end{bmatrix}$$

- continuity of normal component of magnetic flux density (induction)

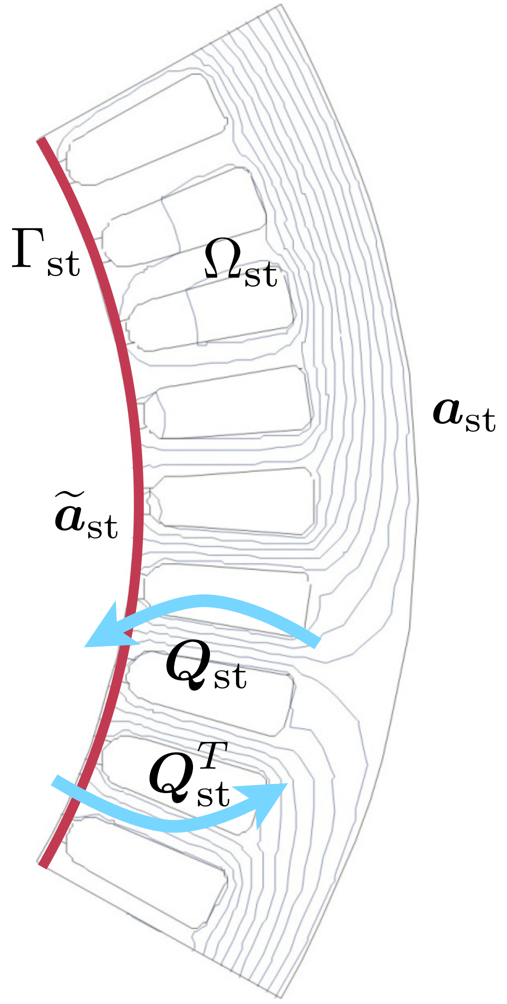
  ⟷ magnetic vector potential continuous

$$\widetilde{\boldsymbol{a}}_{\mathrm{st}}=\widetilde{\boldsymbol{a}}_{\mathrm{rt}}$$

- continuity of tangential component of magnetic field

  ⟷ fictitious source currents vanish

$$\widetilde{\boldsymbol{g}}_{\mathrm{st}}+\widetilde{\boldsymbol{g}}_{\mathrm{rt}}=0$$



Select comps. at interface

$$\widetilde{\boldsymbol{a}}_{\mathrm{st}}=\boldsymbol{Q}_{\mathrm{st}}\boldsymbol{a}_{\mathrm{st}}$$

Prolongate interface comps.

$$\boldsymbol{g}_{\mathrm{st}}=\boldsymbol{Q}_{\mathrm{st}}^T\,\widetilde{\boldsymbol{g}}_{\mathrm{st}}$$

Interface conditions (cont'd)
Reminder matrix system 2D case

$$\int_\Omega \left(\nu \, \mathrm{grad}\, a \cdot \mathrm{grad}\, w_i + \sigma \partial_t a \cdot w_i\right) \mathrm{d}\Omega + \int_\Gamma \nu \, \partial_n a \, w_i \, \mathrm{d}\Gamma = \int_\Omega j_s \cdot w_i \, \mathrm{d}\Omega$$

$$[\ \boldsymbol{K}\ ][\ \boldsymbol{A}\ ] + [\ \boldsymbol{M}\ ]\,\partial_t [\ \boldsymbol{A}\ ] + [\ \boldsymbol{G}\ ] = [\ \boldsymbol{J}\ ]$$

$$k_{i,j} = \int_\Omega \nu \ \mathrm{grad}\, w_i \cdot \mathrm{grad}\, w_j \, \mathrm{d}\Omega$$

$$m_{i,j} = \int_{\Omega_c} \sigma \, w_i \, w_j \, \mathrm{d}\Omega$$

$$g_i = \int_\Gamma \nu \, \partial_n a \, w_i \, \mathrm{d}\Omega$$

$$\boldsymbol{A} = \begin{bmatrix} a_1 & \dots & a_{N_a} \end{bmatrix}^T$$

$$j_i = \int_{\Omega_s} j_s \, w_i \, \mathrm{d}\Omega$$

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_{st} & \boldsymbol{a}_{rt} \end{bmatrix}^T$$

Locked-Step Approach (Sliding-surface techniques)

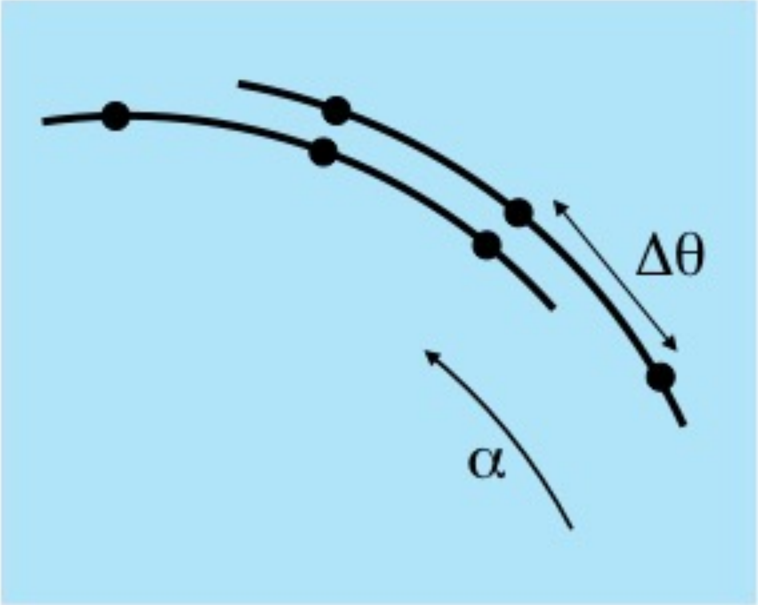
- rotation over an integer number of mesh steps (very simple to use)
- FE with same properties, no additional unknowns
- but: mesh equidistant at the interface
  restriction on the time step

$$\widetilde{\boldsymbol{a}}_{\mathrm{rt}} = \boldsymbol{K}_{\mathrm{shift}}^{q} \widetilde{\boldsymbol{a}}_{\mathrm{st}}$$

$$\alpha = q\Delta\theta$$

$$\boldsymbol{K}_{\mathrm{shift}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

cyclic permutation



Linear interpolation (Sliding-surface techniques)

- mesh of lines on rotor and stator side performed independently

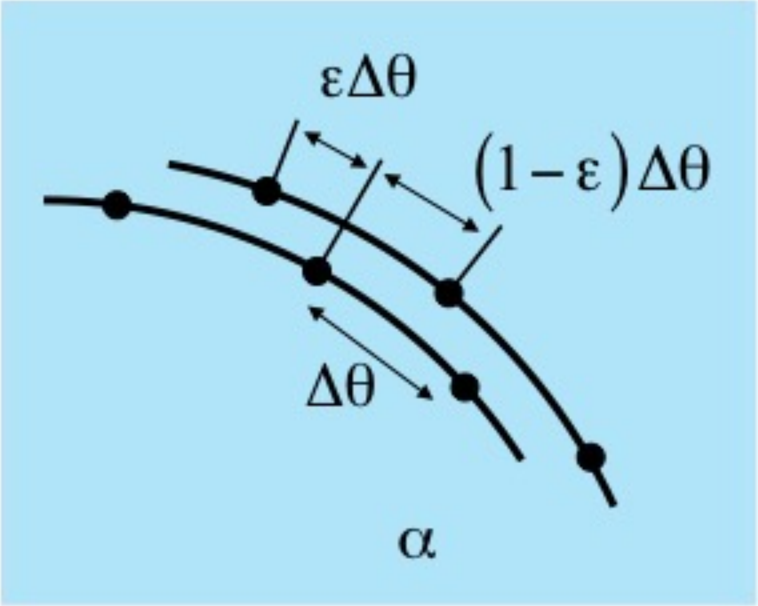
$$\widetilde{\boldsymbol{a}}_{\mathrm{rt}} = \boldsymbol{K}^{q}_{\mathrm{shift}} \boldsymbol{K}_{\varepsilon} \widetilde{\boldsymbol{a}}_{\mathrm{st}}$$

$$\alpha = q\Delta\theta + \varepsilon\Delta\theta$$

$$\boldsymbol{K}_{\mathrm{shift}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{K}_{\varepsilon} = \begin{bmatrix} 1-\varepsilon & \varepsilon & 0 & 0 \\ 0 & 1-\varepsilon & \varepsilon & 0 \\ 0 & 0 & 1-\varepsilon & \varepsilon \\ \varepsilon & 0 & 0 & 1-\varepsilon \end{bmatrix}$$

reduces to locked-step approach when $\varepsilon = 0$



Coupled formulation

- rotation operators
  - forward rotation operator $\widetilde{\boldsymbol{H}}_\alpha = \boldsymbol{K}^q_{\text{shift}} \boldsymbol{K}_\varepsilon$
  - backward rotation operator $\widetilde{\boldsymbol{H}}^H_\alpha = \boldsymbol{K}^T_\varepsilon \boldsymbol{K}^{qT}_{\text{shift}} = \widetilde{\boldsymbol{H}}_{-\alpha}$

- saddle-point formulation — we look for the stationary point

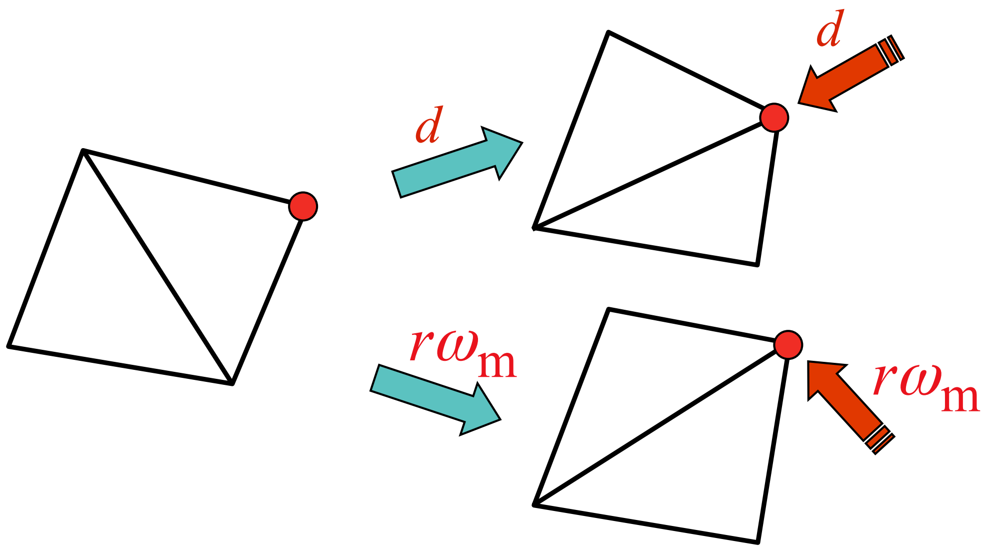
$$\begin{bmatrix} \boldsymbol{M}_{\text{st}} & 0 & 0 \\ 0 & \boldsymbol{M}_{\text{rt}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \partial_t \begin{bmatrix} \boldsymbol{a}_{\text{st}} \\ \boldsymbol{a}_{\text{rt}} \\ \widetilde{\boldsymbol{g}}_{\text{rt}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{K}_{\text{st}} & 0 & -\boldsymbol{Q}^T_{\text{st}} \widetilde{\boldsymbol{H}}^H_\alpha \\ 0 & \boldsymbol{K}_{\text{rt}} & \boldsymbol{Q}^T_{\text{rt}} \\ -\widetilde{\boldsymbol{H}}_\alpha \boldsymbol{Q}_{\text{st}} & 0 & \boldsymbol{Q}_{\text{rt}} \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_{\text{st}} \\ \boldsymbol{a}_{\text{rt}} \\ \widetilde{\boldsymbol{g}}_{\text{rt}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{j}_{\text{st}} \\ \boldsymbol{j}_{\text{rt}} \\ 0 \end{bmatrix}$$

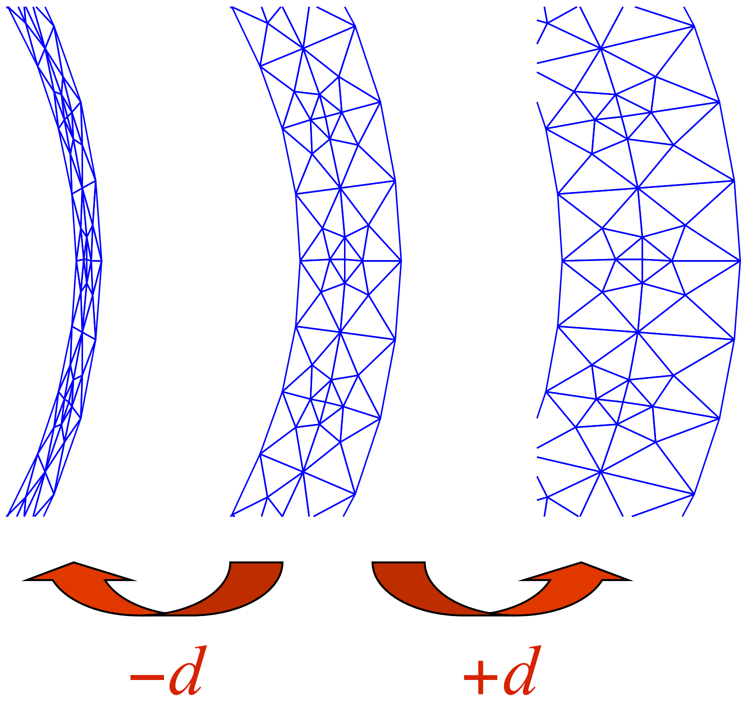
- projected system (eliminate $\widetilde{\boldsymbol{a}}_{\text{rt}}$) with projector

$$\boldsymbol{P}^H \left(\boldsymbol{M}\partial_t + \boldsymbol{K} \right) \boldsymbol{P}\boldsymbol{a} = \boldsymbol{P}^H \boldsymbol{j}$$

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{I} & 0 \\ \boldsymbol{Q}^T_{\text{rt}} \widetilde{\boldsymbol{H}}_\alpha \boldsymbol{Q}_{\text{st}} & \boldsymbol{I} - \boldsymbol{Q}^T_{\text{rt}} \boldsymbol{Q}_{\text{rt}} \end{bmatrix}$$

Moving band technique





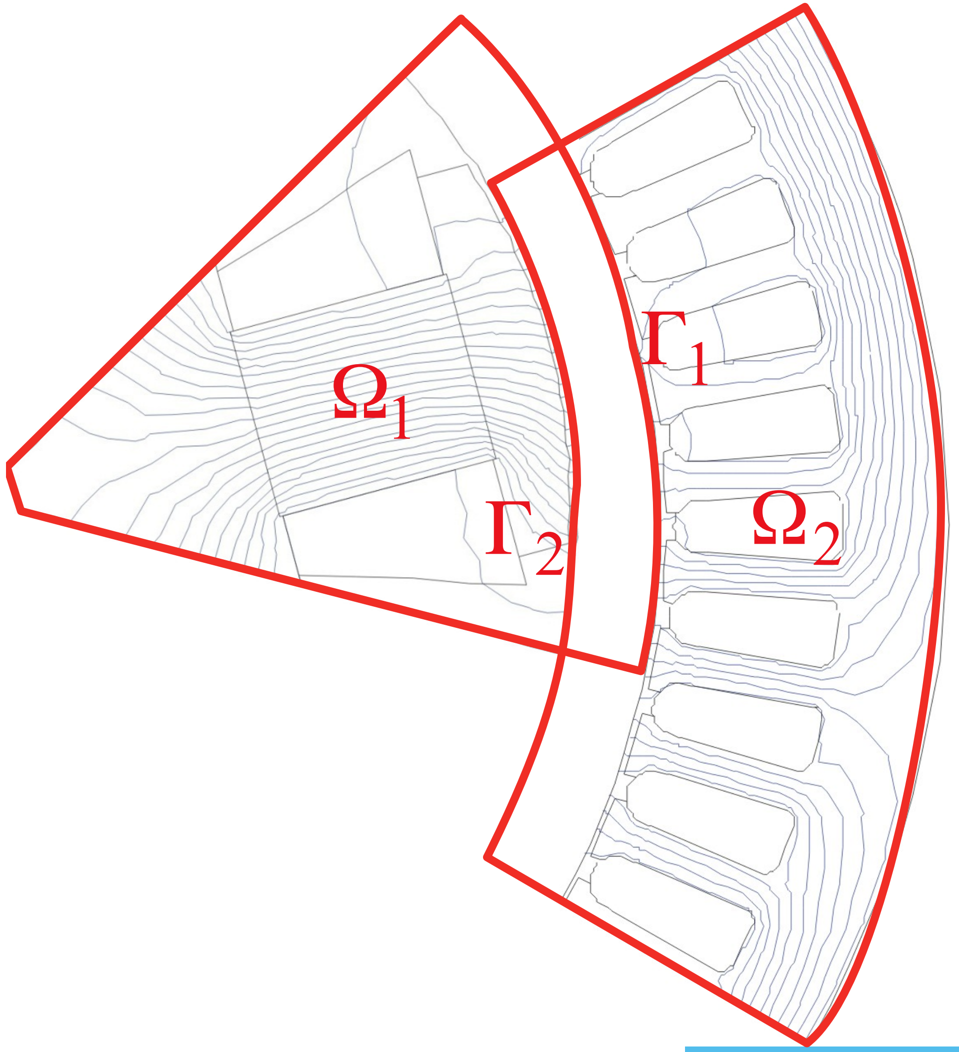
small rotation ω_m
small displacement d → change of the mesh topology

consistent change of mesh according to eccentricity

☹ possibly bad meshes
☹ difficulties when rotation has to be considered

Mortar element method

- variational non-conforming domain decomposition: moving and fixed domain
- movement without re-meshing the whole computational domain
- transmission conditions weakly imposed via e.g. Lagrange multipliers
- overlapping non-matching grid
  = two overlapping FE models
- additive/multiplicative Schwarz
- flexible and accurate



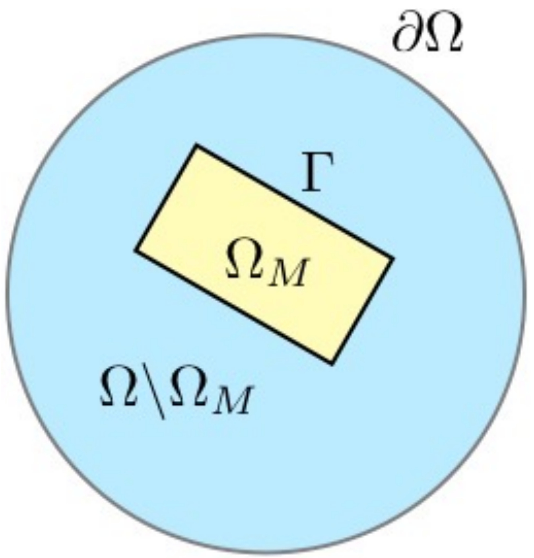
Mortar FE method — Lagrange multipliers

find $\boldsymbol{a}$ and $\boldsymbol{a}_M$ so that

$$\int_{\Omega\backslash\Omega_M} \sigma\, \partial_t \boldsymbol{a} \cdot \boldsymbol{a}' \,\mathrm{d}\Omega + \int_{\Omega\backslash\Omega_M} \nu \operatorname{curl} \boldsymbol{a} \cdot \operatorname{curl} \boldsymbol{a}' \,\mathrm{d}\Omega = 0$$

$$\int_{\Omega_M} \sigma\, \partial_t \boldsymbol{a}_M \cdot \boldsymbol{a}'_M \,\mathrm{d}\Omega + \int_{\Omega_M} \nu \operatorname{curl} \boldsymbol{a}_M \cdot \operatorname{curl} \boldsymbol{a}'_M \,\mathrm{d}\Omega = 0$$

holds $\forall \boldsymbol{a}'$ and $\boldsymbol{a}'_M$ in suitable function spaces



- Two spatial discretizations, one per subdomain
  - non-matching grids
  - completely independent in overlapping region
- Transfer of information from Ω to Ω_M : $\boldsymbol{C}\boldsymbol{A}_M|_\Gamma = \boldsymbol{D}\boldsymbol{A}$
- Transfer of information from Ω_M to Ω : $\boldsymbol{E}\boldsymbol{A}|_\gamma = \boldsymbol{H}\boldsymbol{A}_M$

http://en.wikipedia.org/wiki/Lagrange_multipliers

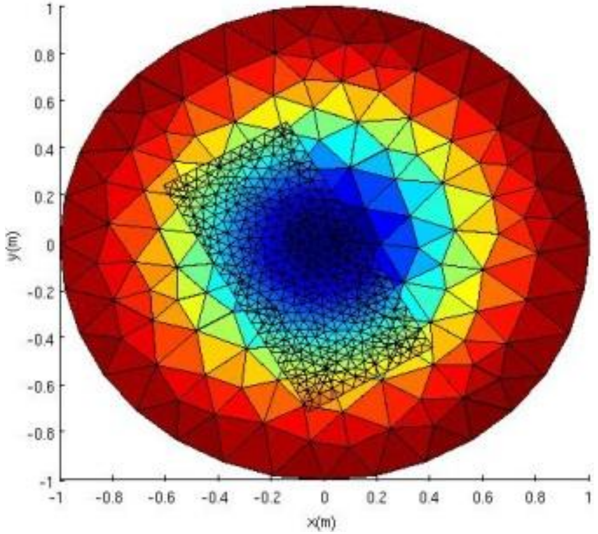
Mortar FE method — Lagrange multipliers

- The coupling matrices are described on the edges of the concern interface

$$c_{ij} = \int_{e\in\Gamma} \left(\boldsymbol{w}_i^M \lambda_e\right)\left(\boldsymbol{w}_j^M \lambda_e\right), \quad d_{ij} = \int_{e\in\Gamma} \left(\boldsymbol{w}_i^M \lambda_e\right)\left(\boldsymbol{w}_k^F \lambda_e\right)$$

$$e_{ij} = \int_{e\in\gamma} \left(\boldsymbol{w}_i^F \lambda_e\right)\left(\boldsymbol{w}_j^F \lambda_e\right), \quad h_{ij} = \int_{e\in\gamma} \left(\boldsymbol{w}_i^F \lambda_e\right)\left(\boldsymbol{w}_k^M \lambda_e\right)$$

λ_e = Lagrange multipliers
$\boldsymbol{w}_i^F$ = BFs in Ω
$\boldsymbol{w}_i^M$ = BFs in Ω_M



- C and E defined on same mesh: not difficult
- D and H defined on different meshes:
- division of elements of the triangulations
- overlapped by the edge e

KU LEUVEN

Hybrid FE-BE model

- FE model
  - domains with nonlinear media and/or induced currents
  - explicit mechanisms for truncating the domain required
  - sparse system matrix
- BE model
  - movement ⇒ no re-meshing, no moving band,...
  - rigorous treatment for open problems
  - dense system matrix
- Hybrid FE-BE model + acceleration method
  - FE model for device at hand
  - BE model for surrounding air and air gaps
  - acceleration method:
    - fast multipole method
    - hierarchical matrix technique