

# Introduction to the Finite Element Technique for Electromagnetic Modelling

# Classical and weak formulations

## Partial differential problem

### Classical formulation

$$\begin{aligned} L u &= f & \text{in } \Omega \\ B u &= g & \text{on } \Gamma = \partial\Omega \end{aligned}$$

$u \equiv$  classical solution

### Notations

$$(u, v) = \int_{\Omega} u(\mathbf{x}) v(\mathbf{x}) \, d\mathbf{x} \quad , \quad u, v \in L^2(\Omega)$$
$$(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \mathbf{u}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) \, d\mathbf{x} \quad , \quad \mathbf{u}, \mathbf{v} \in L^2(\Omega)$$

### Weak formulation

$$(u, L^* v) - (f, v) + \int_{\Gamma} Q_g(v) \, ds = 0 \quad , \quad \forall v \in V(\Omega)$$

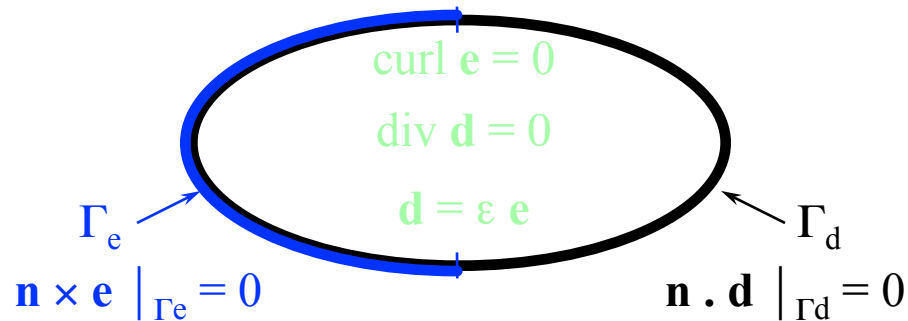
$u \equiv$  weak solution

$v \equiv$  test function

**Continuous level :**  $\infty \times \infty$  system  
**Discrete level :**  $n \times n$  system  
 $\Rightarrow$  **numerical solution**

# Classical and weak formulations

*Application to the electrostatic problem*



**Electrostatic  
classical formulation**

$(\mathbf{d}, \text{grad } v') = 0, \quad \forall v' \in V(\Omega) \quad \text{with } V(\Omega) = \{v \in H^0(\Omega); v|_{\Gamma_e} = 0\}$

$\Rightarrow (\text{div } \mathbf{d}, v') + \langle \mathbf{n} \cdot \mathbf{d}, v' \rangle_{\Gamma} = 0, \quad \forall v' \in V(\Omega)$

$\Downarrow$                        $\Downarrow$   
 $\text{div } \mathbf{d} = 0$        $\mathbf{n} \cdot \mathbf{d} \big|_{\Gamma_d} = 0$

$\mathbf{d} = \varepsilon \mathbf{e} \quad \& \quad \mathbf{e} = -\text{grad } v \Leftrightarrow \text{curl } \mathbf{e} = 0$

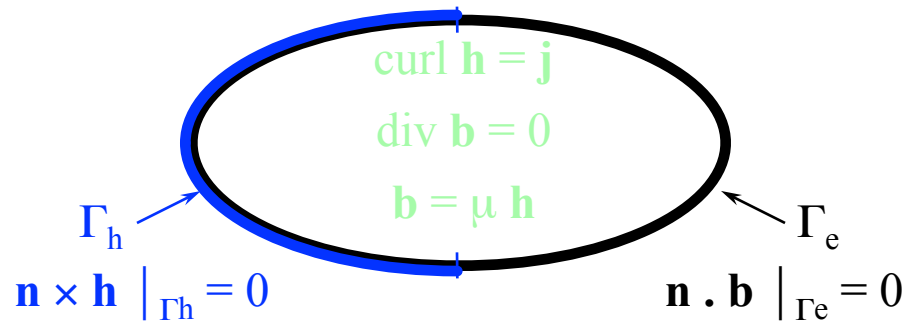
$(-\varepsilon \text{grad } v, \text{grad } v') = 0, \quad \forall v' \in V(\Omega)$

**Weak formulation  
of  $\text{div } \mathbf{d} = 0$   
(+ boundary condition)**

**Electrostatic  
weak formulation with  $v$**

# Classical and weak formulations

*Application to the magnetostatic problem*



**Magnetostatic  
classical formulation**

$(\mathbf{b}, \text{grad } \phi') = 0, \quad \forall \phi' \in \Phi(\Omega) \quad \text{with } \Phi(\Omega) = \{ \phi \in H^0(\Omega); \phi \big|_{\Gamma_h} = 0 \}$

$\Rightarrow (\text{div } \mathbf{b}, \phi') + \langle \mathbf{n} \cdot \mathbf{b}, \phi' \rangle_{\Gamma} = 0, \quad \forall \phi' \in \Phi(\Omega)$

$\Downarrow$                        $\Downarrow$   
 $\text{div } \mathbf{b} = 0$        $\mathbf{n} \cdot \mathbf{b} \big|_{\Gamma_e} = 0$

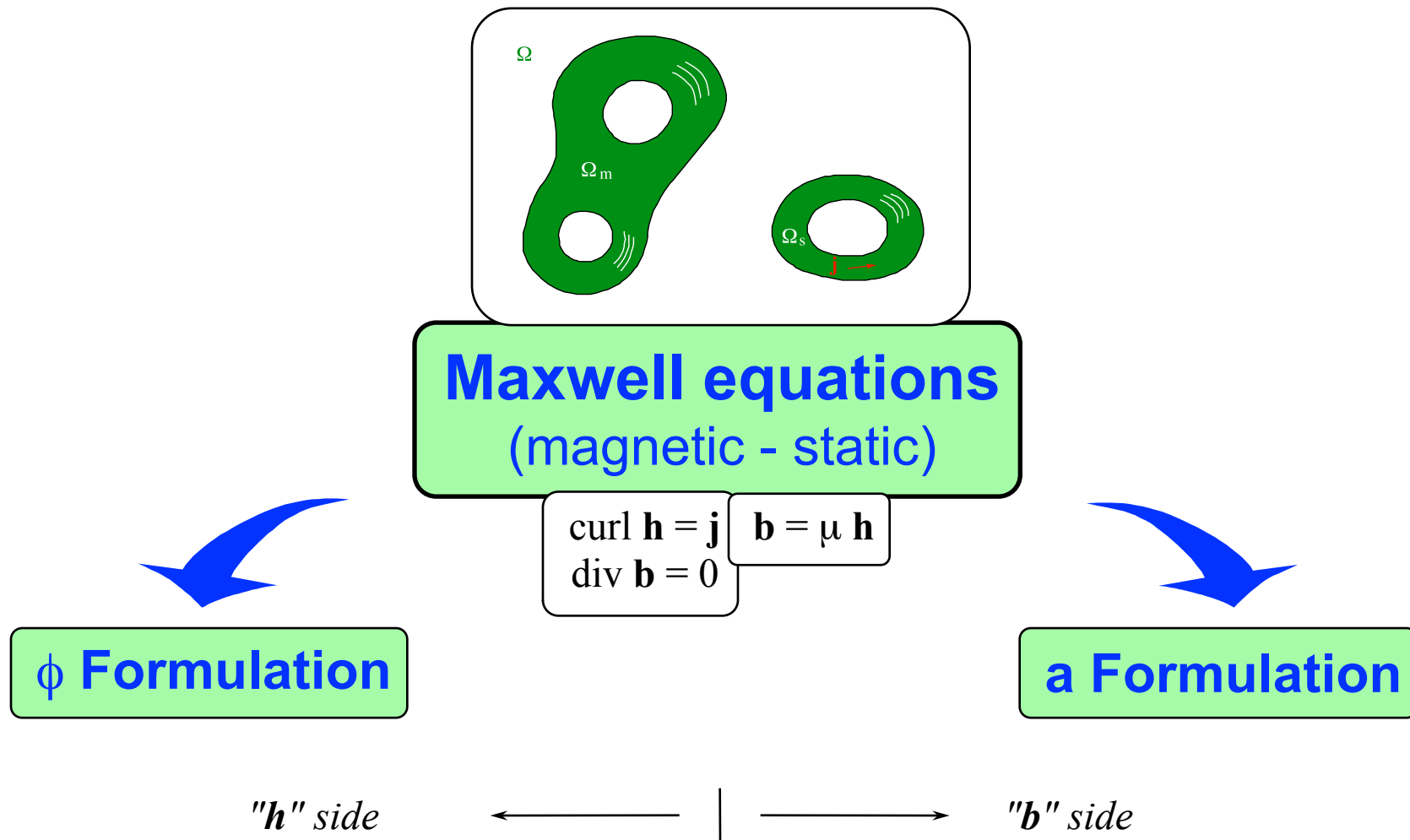
**Weak formulation  
of  $\text{div } \mathbf{b} = 0$   
(+ boundary condition)**

$\mathbf{b} = \mu \mathbf{h} \quad \& \quad \mathbf{h} = \mathbf{h}_s - \text{grad } \phi \quad (\text{with } \text{curl } \mathbf{h}_s = \mathbf{j}) \Leftrightarrow \text{curl } \mathbf{h} = \mathbf{j}$

$(\mu (\mathbf{h}_s - \text{grad } \phi), \text{grad } \phi') = 0, \quad \forall \phi' \in \Phi(\Omega)$

**Magnetostatic  
weak formulation with  $\phi$**

# Magnetostatic formulations



# Magnetostatic formulations

## Basis equations

$$\text{curl } \mathbf{h} = \mathbf{j} \quad (\text{h})$$

$$\mathbf{b} = \mu \mathbf{h} \quad (\text{m})$$

$$\text{div } \mathbf{b} = 0 \quad (\text{b})$$

### $\phi$ Formulation

#### Magnetic scalar potential $\phi$

$$\mathbf{h} = \mathbf{h}_s - \text{grad } \phi \Rightarrow (\text{h}) \text{ OK}$$

$$\mathbf{h}_s \text{ given such as } \text{curl } \mathbf{h}_s = \mathbf{j} \quad (\text{non-unique})$$

$$\text{div} ( \mu ( \mathbf{h}_s - \text{grad } \phi ) ) = 0 \quad \leftarrow (\text{b}) \ \& \ (\text{m})$$

#### Multivalued potential

Cuts

### $\mathbf{a}$ Formulation

#### Magnetic vector potential $\mathbf{a}$

$$(\text{b}) \text{ OK} \leftarrow \mathbf{b} = \text{curl } \mathbf{a}$$

$$(\text{h}) \ \& \ (\text{m}) \rightarrow \text{curl} ( \mu^{-1} \text{curl } \mathbf{a} ) = \mathbf{j}$$

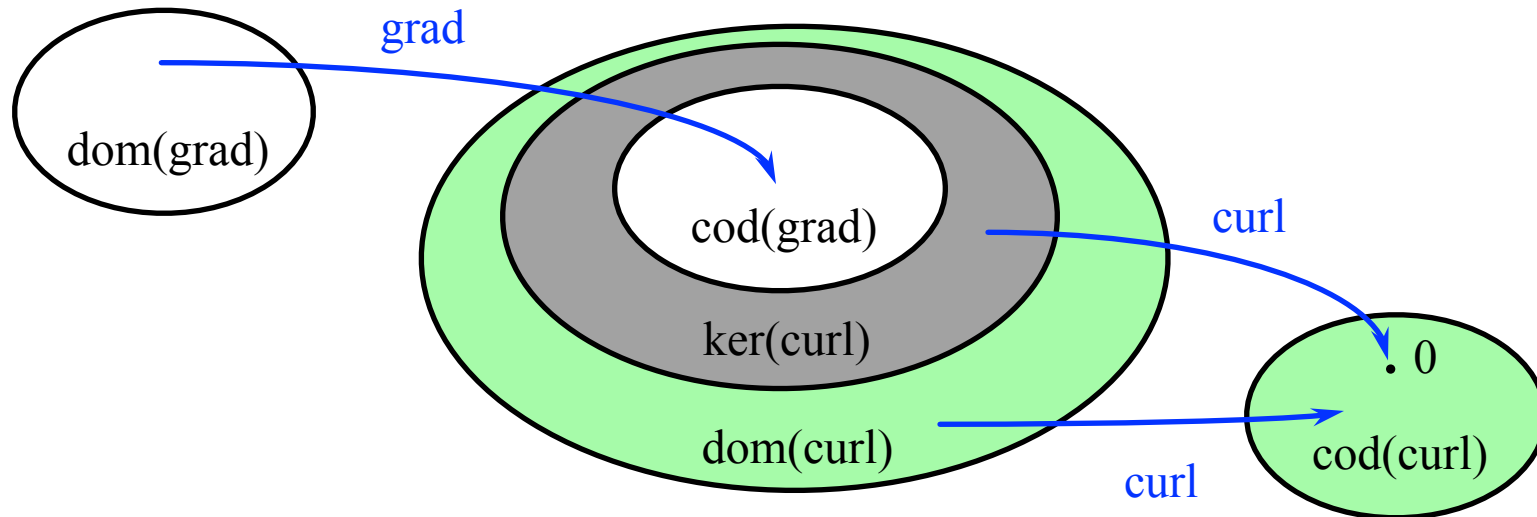
#### Non-unique potential

Gauge condition

# Multivalued scalar potential

## Kernel of the curl (in a domain $\Omega$ )

$$\ker(\text{curl}) = \{ \mathbf{v} : \text{curl } \mathbf{v} = 0 \}$$



$$\text{cod}(\text{grad}) \subset \text{ker}(\text{curl})$$

# Multivalued scalar potential - Cut

$$\text{curl } \mathbf{h} = 0 \quad \text{in } \Omega$$

OK



Scalar potential  $\phi$

$$\mathbf{h} = -\text{grad } \phi \quad \text{in } \Omega$$



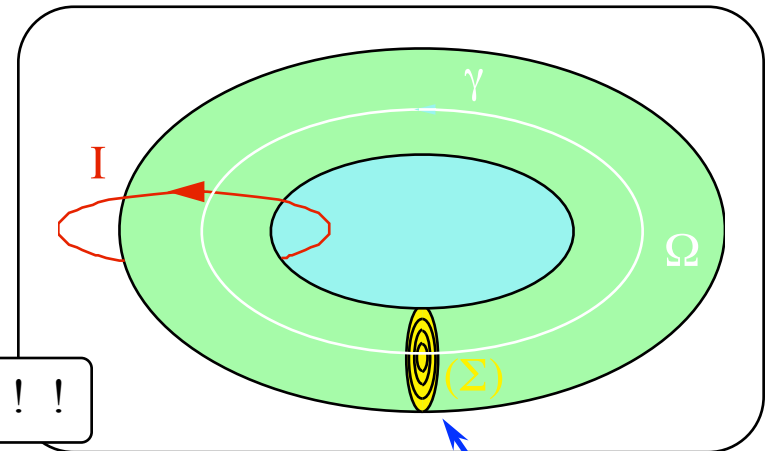
?

Circulation of  $\mathbf{h}$  along path  $\gamma_{AB}$  in  $\Omega$

$$\int_{\gamma_{AB}} \mathbf{h} \cdot d\mathbf{l} = \int_{\gamma_{AB}} -\text{grad } \phi \cdot d\mathbf{l} = \phi_A - \phi_B$$

Closed path  $\gamma_{AB}$  ( $A \equiv B$ )  
surrounding a conductor (with current  $I$ )

$$\Rightarrow \phi_A - \phi_B = 0 \neq I \quad !!!$$



$\phi$  must be discontinuous ... through a **cut**

$$\Delta \phi = I$$

# Vector potential - gauge condition

$$\text{div } \mathbf{b} = 0 \quad \text{in } \Omega$$

OK



Vector potential  $\mathbf{a}$

$$\mathbf{b} = \text{curl } \mathbf{a} \quad \text{in } \Omega$$



?

Non-uniqueness of vector potential  $\mathbf{a}$

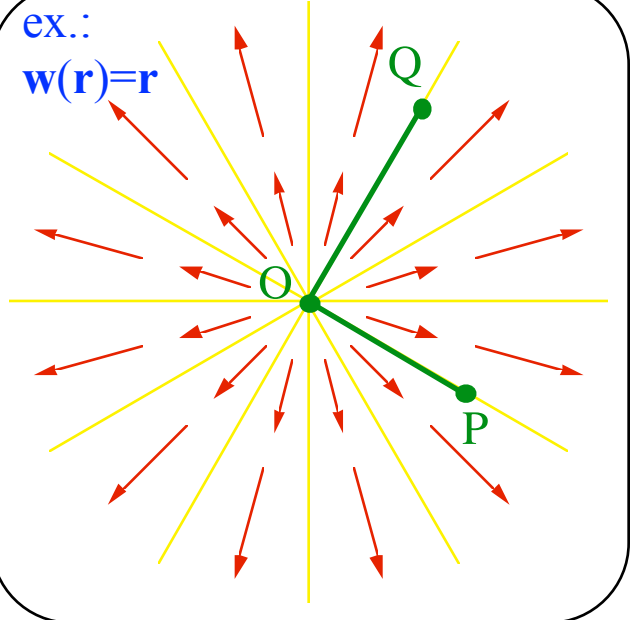
$$\mathbf{b} = \text{curl } \mathbf{a} = \text{curl} (\mathbf{a} + \text{grad } \eta)$$

Gauge condition

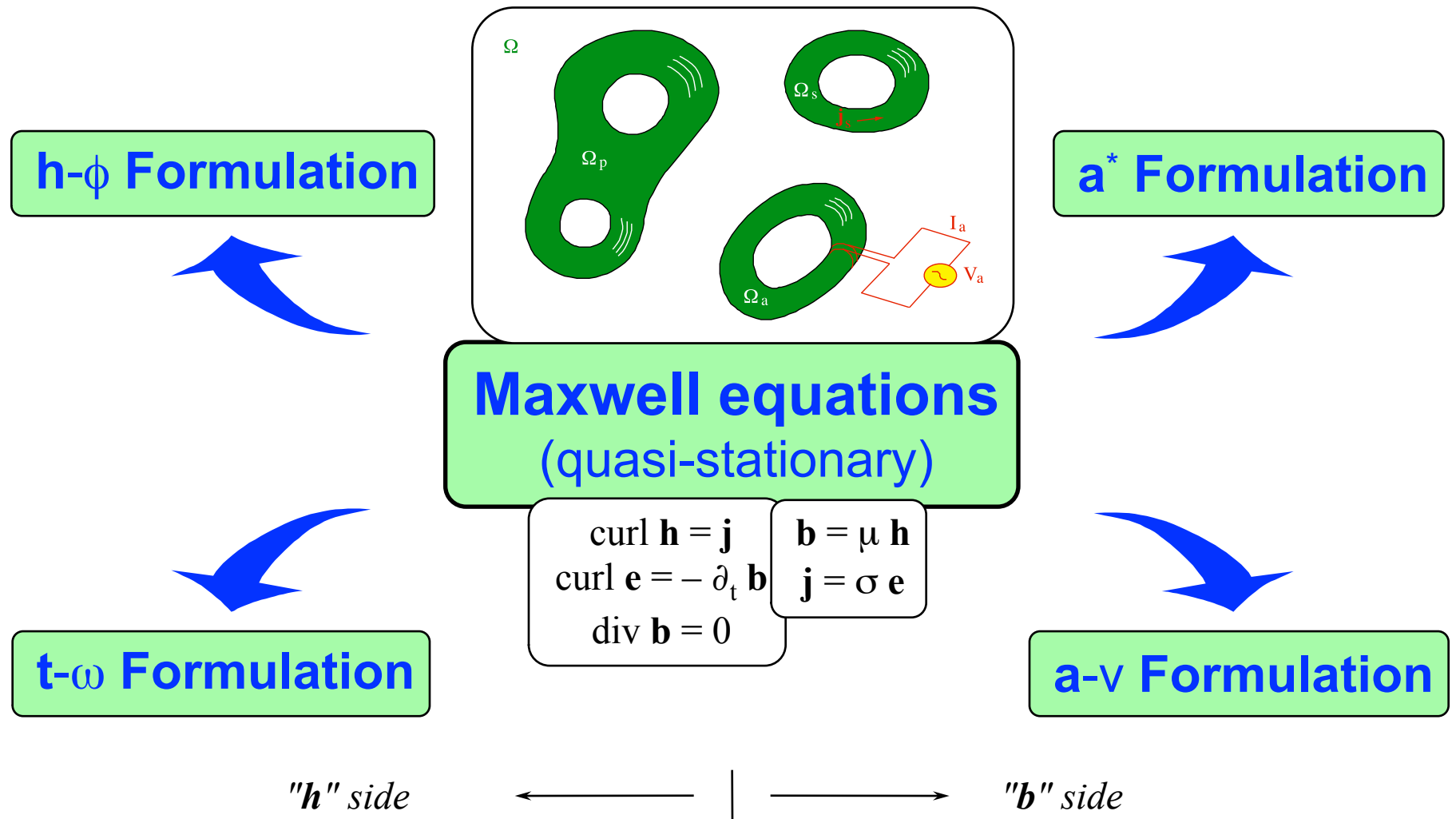
Coulomb gauge  $\text{div } \mathbf{a} = 0$

Gauge  $\mathbf{a} \cdot \boldsymbol{\omega} = 0$

$\boldsymbol{\omega}$  vector field with non-closed lines linking any 2 points in  $\Omega$



# Magnetodynamic formulations



# Magnetodynamic formulations

## Basis equations

$$\text{curl } \mathbf{h} = \mathbf{j}$$

(h)

$$\mathbf{b} = \mu \mathbf{h}$$

$$\mathbf{j} = \sigma \mathbf{e}$$

$$\text{curl } \mathbf{e} = -\partial_t \mathbf{b}$$

$$\text{div } \mathbf{b} = 0$$

(b)

## h- $\phi$ Formulation

Magnetic field  $\mathbf{h}$

Magnetic scalar potential  $\phi$

$$\mathbf{h} = \mathbf{h}_s - \text{grad } \phi$$

$ds \Omega_c$

$ds \Omega_c^C$

$\Rightarrow$  (h) OK

$$\text{curl } \mathbf{h}_s = \mathbf{j}_s$$

$$\text{curl } (\sigma^{-1} \text{curl } \mathbf{h}) + \partial_t (\mu \mathbf{h}) = 0$$

$$\text{div } (\mu (\mathbf{h}_s - \text{grad } \phi)) = 0$$

$\leftarrow$  (b)  $\rightarrow$

$\leftarrow$  in  $\Omega_c \rightarrow$

$\leftarrow$  in  $\Omega_c^C \rightarrow$

## t- $\omega$ Formulation

Electric vector potential  $\mathbf{t}$

Magnetic scalar potential  $\omega$

$$\mathbf{j} = \text{curl } \mathbf{t}$$

(h) OK  $\Leftarrow$

$$\mathbf{h} = \mathbf{t} - \text{grad } \omega$$

$$\text{curl } (\sigma^{-1} \text{curl } \mathbf{t}) + \partial_t (\mu (\mathbf{t} - \text{grad } \omega)) = 0$$

$$\text{div } (\mu (\mathbf{t} - \text{grad } \omega)) = 0$$

+ Gauge

# Magnetodynamic formulations

## Basis equations

$$\text{curl } \mathbf{h} = \mathbf{j}$$

(h)

$$\mathbf{b} = \mu \mathbf{h}$$

$$\mathbf{j} = \sigma \mathbf{e}$$

$$\text{curl } \mathbf{e} = -\partial_t \mathbf{b}$$

$$\text{div } \mathbf{b} = 0$$

(b)

## $\mathbf{a}^*$ Formulation

### Magnetic vector potential $\mathbf{a}^*$

$$\mathbf{b} = \text{curl } \mathbf{a}^*$$

$\Rightarrow$  (b) OK

$$\mathbf{e} = -\partial_t \mathbf{a}^*$$

$$\text{curl} (\mu^{-1} \text{curl } \mathbf{a}^*) + \sigma \partial_t \mathbf{a}^* = \mathbf{j}_s$$

+ Gauge in  $\Omega_c^C$

## $\mathbf{a}$ - $\mathbf{v}$ Formulation

### Magnetic vector potential $\mathbf{a}$ Electric scalar potential $v$

$$\mathbf{b} = \text{curl } \mathbf{a}$$

(b) OK  $\Leftarrow$

$$\mathbf{e} = -\partial_t \mathbf{a} - \text{grad } v$$

$$\text{curl} (\mu^{-1} \text{curl } \mathbf{a}) + \sigma (\partial_t \mathbf{a} + \text{grad } v) = \mathbf{j}_s$$

+ Gauge in  $\Omega$

$\leftarrow$  (h)  $\rightarrow$

# Electrostatic problem

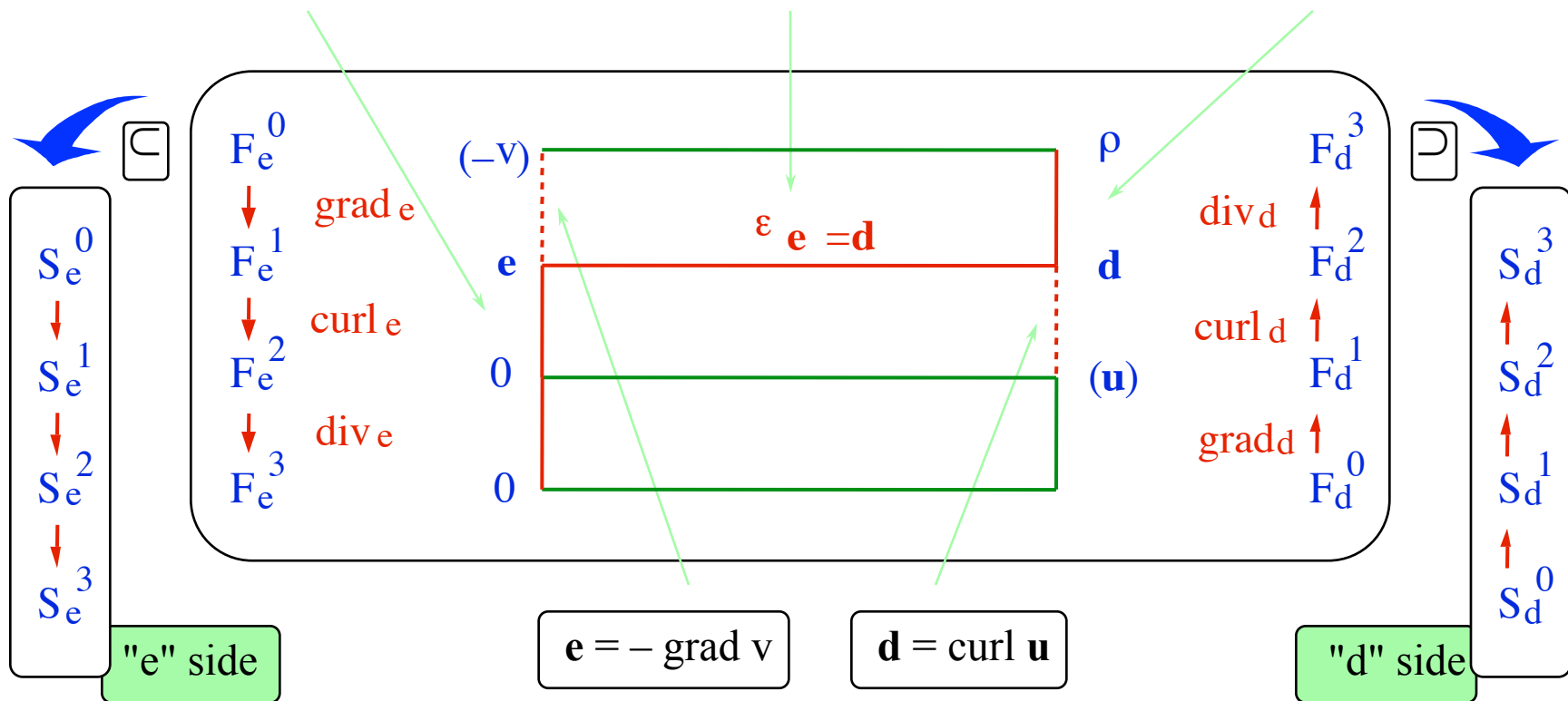


## Basis equations

$$\text{curl } \mathbf{e} = 0$$

$$\mathbf{d} = \epsilon \mathbf{e}$$

$$\text{div } \mathbf{d} = \rho$$



# Electrokinetic problem

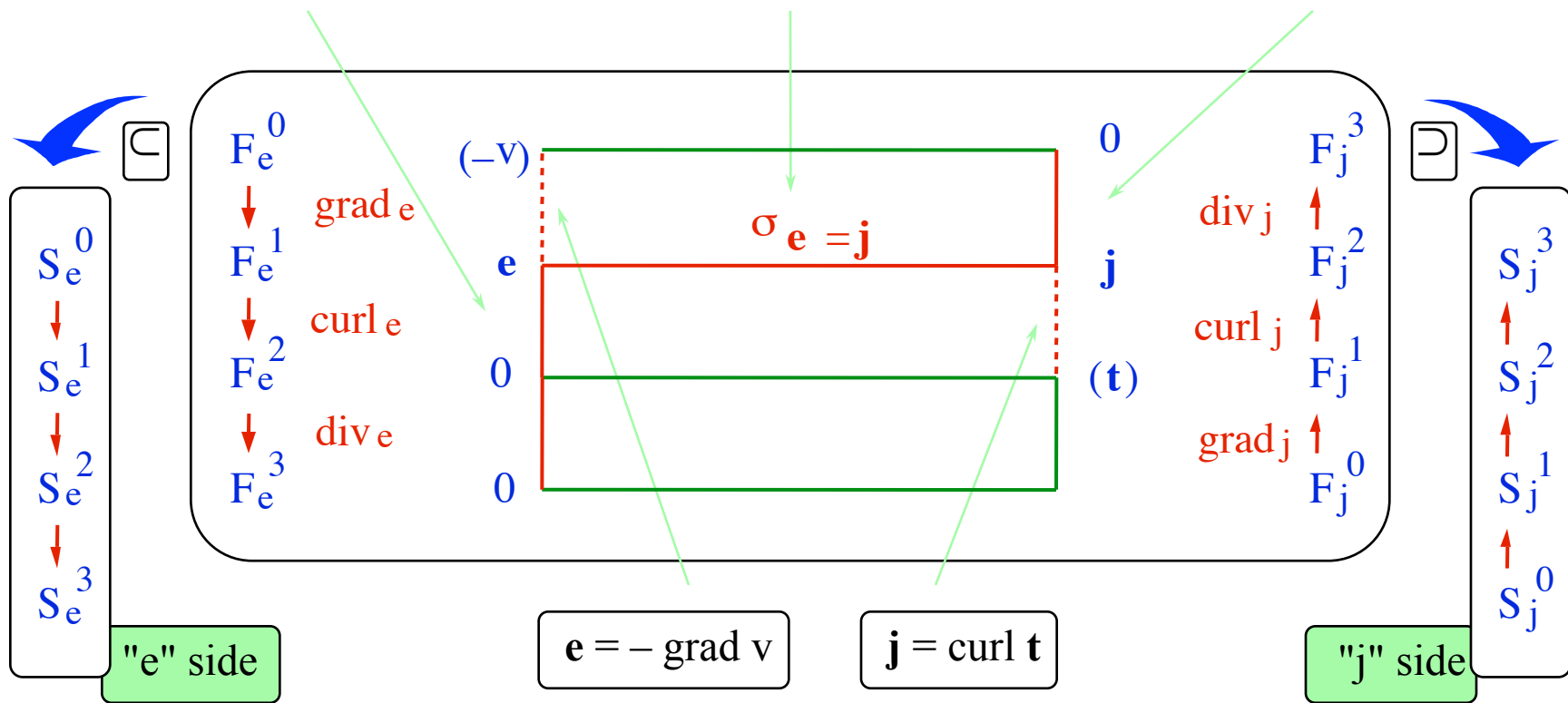


## Basis equations

$$\text{curl } \mathbf{e} = 0$$

$$\mathbf{j} = \sigma \mathbf{e}$$

$$\text{div } \mathbf{j} = 0$$



# Magnetostatic problem

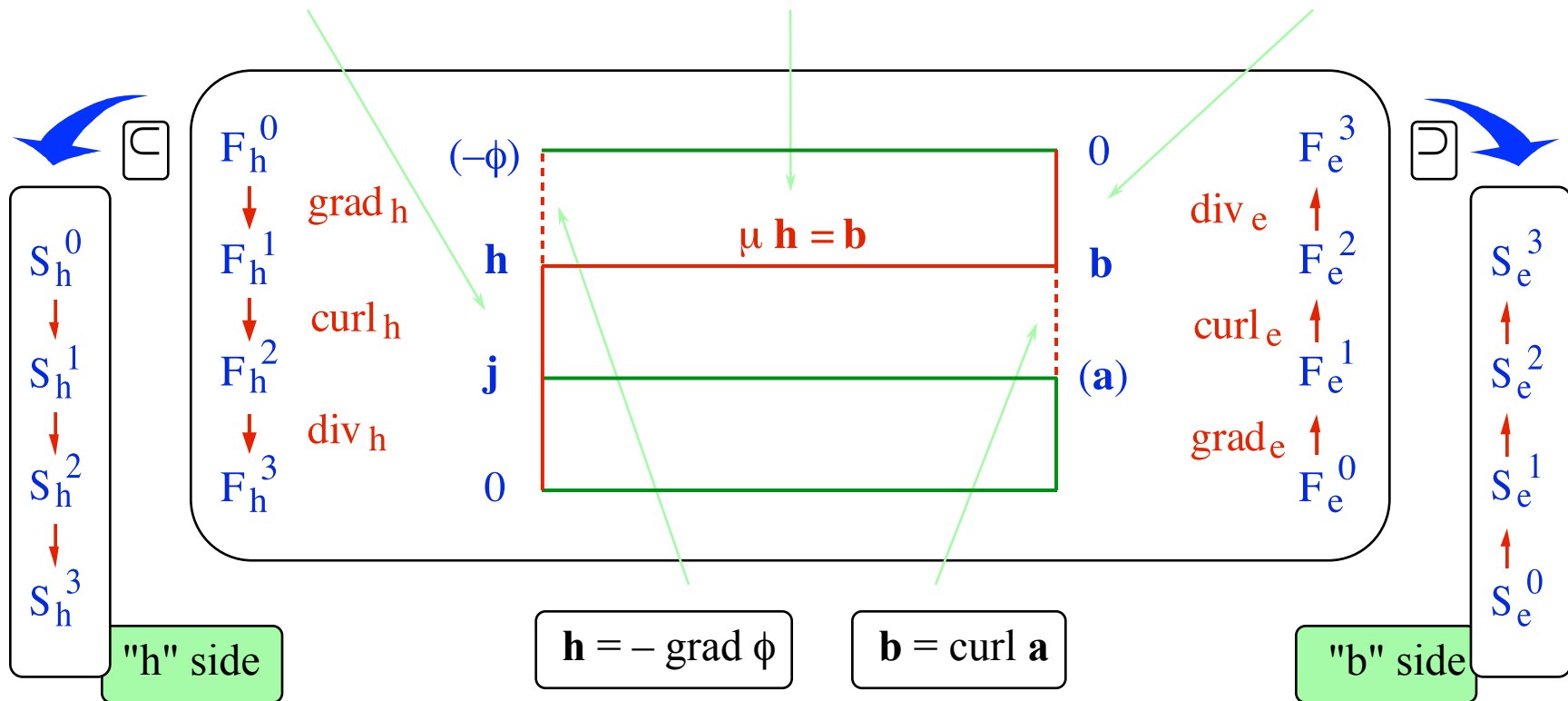


## Basis equations

$$\text{curl } \mathbf{h} = \mathbf{j}$$

$$\mathbf{b} = \mu \mathbf{h}$$

$$\text{div } \mathbf{b} = 0$$



# Magnetodynamic problem



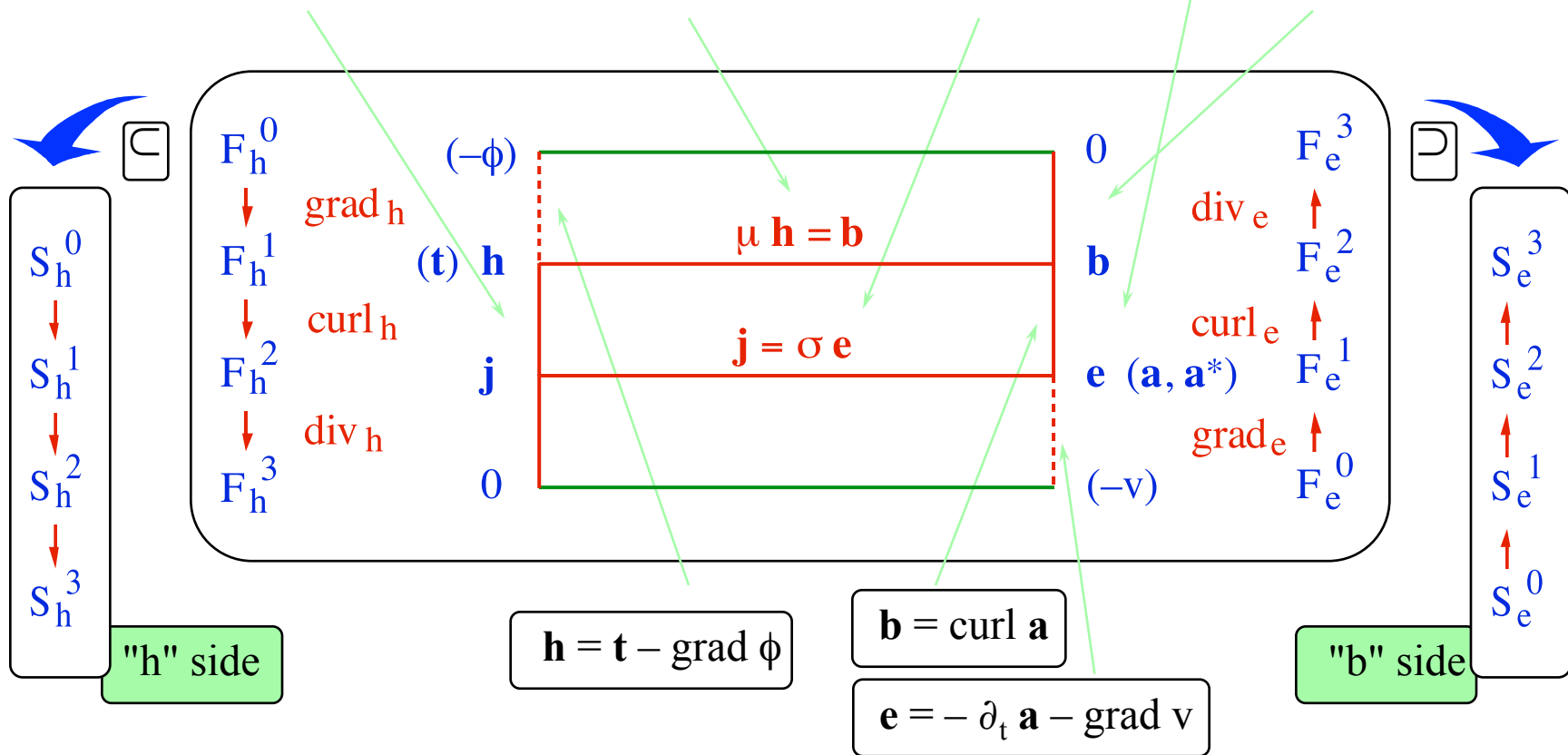
## Basis equations

$$\text{curl } \mathbf{h} = \mathbf{j}$$

$$\mathbf{b} = \mu \mathbf{h}$$

$$\mathbf{j} = \sigma \mathbf{e}$$

$$\begin{aligned} \text{curl } \mathbf{e} &= -\partial_t \mathbf{b} \\ \text{div } \mathbf{b} &= 0 \end{aligned}$$



# Continuous mathematical structure

Domain  $\Omega$ , Boundary  $\partial\Omega = \Gamma_h \cup \Gamma_e$

**Basis structure**

Function spaces  $F_h^0 \subset L^2, F_h^1 \subset L^2, F_h^2 \subset L^2, F_h^3 \subset L^2$

$$\text{dom}(\text{grad}_h) = F_h^0 = \{ \phi \in L^2(\Omega) ; \text{grad } \phi \in L^2(\Omega), \quad \phi|_{\Gamma_h} = 0 \}$$

$$\text{dom}(\text{curl}_h) = F_h^1 = \{ \mathbf{h} \in L^2(\Omega) ; \text{curl } \mathbf{h} \in L^2(\Omega), \quad \mathbf{n} \wedge \mathbf{h}|_{\Gamma_h} = 0 \}$$

$$\text{dom}(\text{div}_h) = F_h^2 = \{ \mathbf{j} \in L^2(\Omega) ; \text{div } \mathbf{j} \in L^2(\Omega), \quad \mathbf{n} \cdot \mathbf{j}|_{\Gamma_h} = 0 \}$$

$$\text{grad}_h F_h^0 \subset F_h^1, \quad \text{curl}_h F_h^1 \subset F_h^2, \quad \text{div}_h F_h^2 \subset F_h^3$$

Boundary conditions on  $\Gamma_h$

Sequence

$$F_h^0 \xrightarrow{\text{grad}_h} F_h^1 \xrightarrow{\text{curl}_h} F_h^2 \xrightarrow{\text{div}_h} F_h^3$$

**Basis structure**

Function spaces  $F_e^0 \subset L^2, F_e^1 \subset L^2, F_e^2 \subset L^2, F_e^3 \subset L^2$

$$\text{dom}(\text{grad}_e) = F_e^0 = \{ \mathbf{v} \in L^2(\Omega) ; \text{grad } \mathbf{v} \in L^2(\Omega), \quad \mathbf{v}|_{\Gamma_e} = 0 \}$$

$$\text{dom}(\text{curl}_e) = F_e^1 = \{ \mathbf{a} \in L^2(\Omega) ; \text{curl } \mathbf{a} \in L^2(\Omega), \quad \mathbf{n} \wedge \mathbf{a}|_{\Gamma_e} = 0 \}$$

$$\text{dom}(\text{div}_e) = F_e^2 = \{ \mathbf{b} \in L^2(\Omega) ; \text{div } \mathbf{b} \in L^2(\Omega), \quad \mathbf{n} \cdot \mathbf{b}|_{\Gamma_e} = 0 \}$$

$$\text{grad}_h F_e^0 \subset F_e^1, \quad \text{curl}_e F_e^1 \subset F_e^2, \quad \text{div}_e F_e^2 \subset F_e^3$$

Boundary conditions on  $\Gamma_e$

Sequence

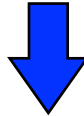
$$F_e^3 \xleftarrow{\text{div}_e} F_e^2 \xleftarrow{\text{curl}_e} F_e^1 \xleftarrow{\text{grad}_e} F_e^0$$

# Discrete mathematical structure

## Continuous problem

Continuous function spaces & domain  
Classical and weak formulations

Discretization



Approximation

## Discrete problem

Discrete function spaces piecewise defined  
in a discrete domain (mesh)

## Finite element method

### Questions

Classical & weak formulations → ?  
Properties of the fields → ?

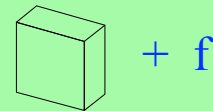
### Objective

To build a discrete structure  
as similar as possible  
as the continuous structure

# Discrete mathematical structure

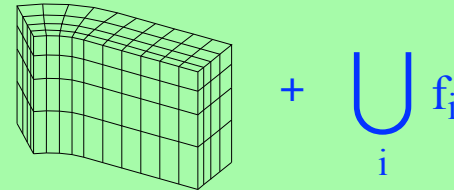
## Finite element

Interpolation in a geometric element of simple shape



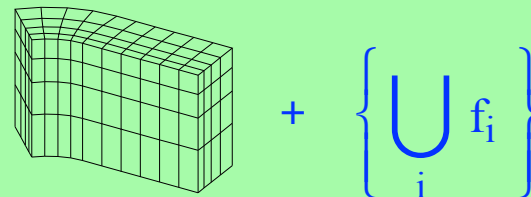
## Finite element space

Function space  
& Mesh



## Sequence of finite element spaces

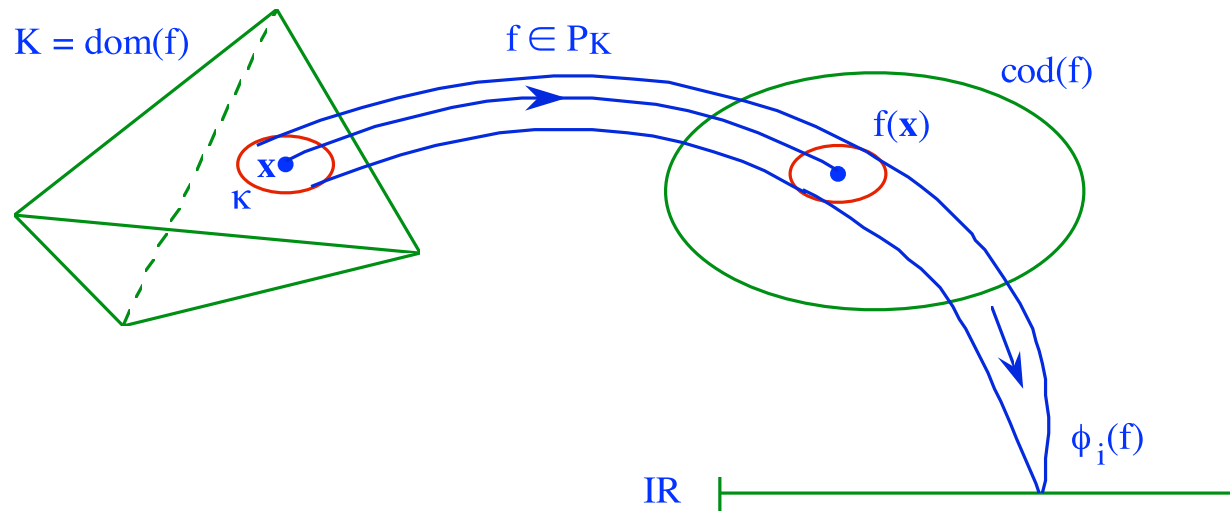
Sequence of function spaces  
& Mesh



# Finite elements

## ❖ Finite element $(K, P_K, \Sigma_K)$

- ◆  $K$  = domain of space (tetrahedron, hexahedron, prism)
- ◆  $P_K$  = function space of finite dimension  $n_K$ , defined in  $K$
- ◆  $\Sigma_K$  = set of  $n_K$  degrees of freedom  
represented by  $n_K$  linear functionals  $\phi_i$ ,  $1 \leq i \leq n_K$ ,  
defined in  $P_K$  and whose values belong to  $\mathbb{R}$



# Finite elements

## ❖ Unisolvance

$\forall u \in P_K$ ,  $u$  is uniquely defined by the degrees of freedom

## ❖ Interpolation

$$u_K = \sum_{i=1}^{n_K} \phi_i(u) p_i$$

Degrees of freedom

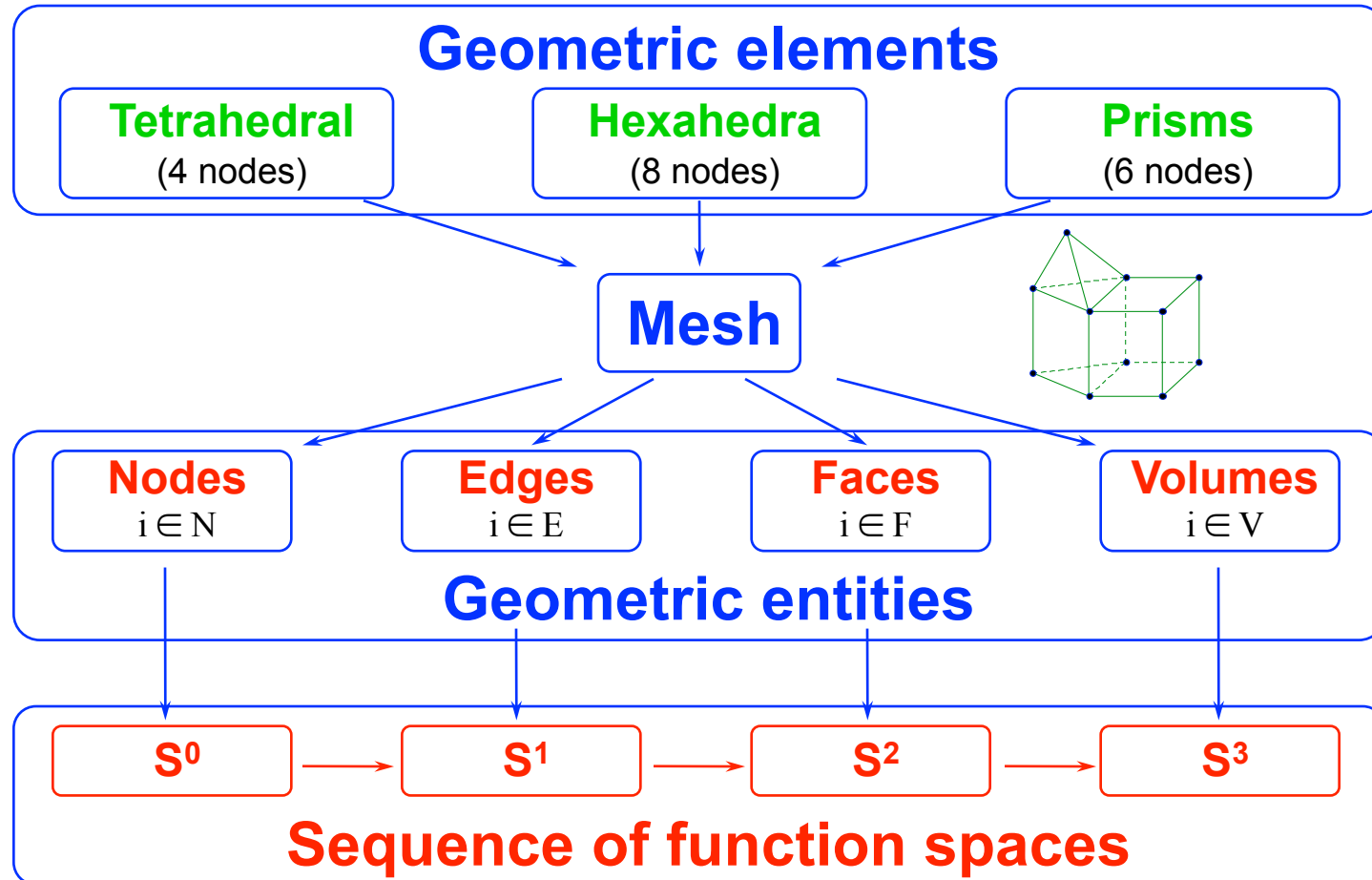
Basis functions

## ❖ Finite element space

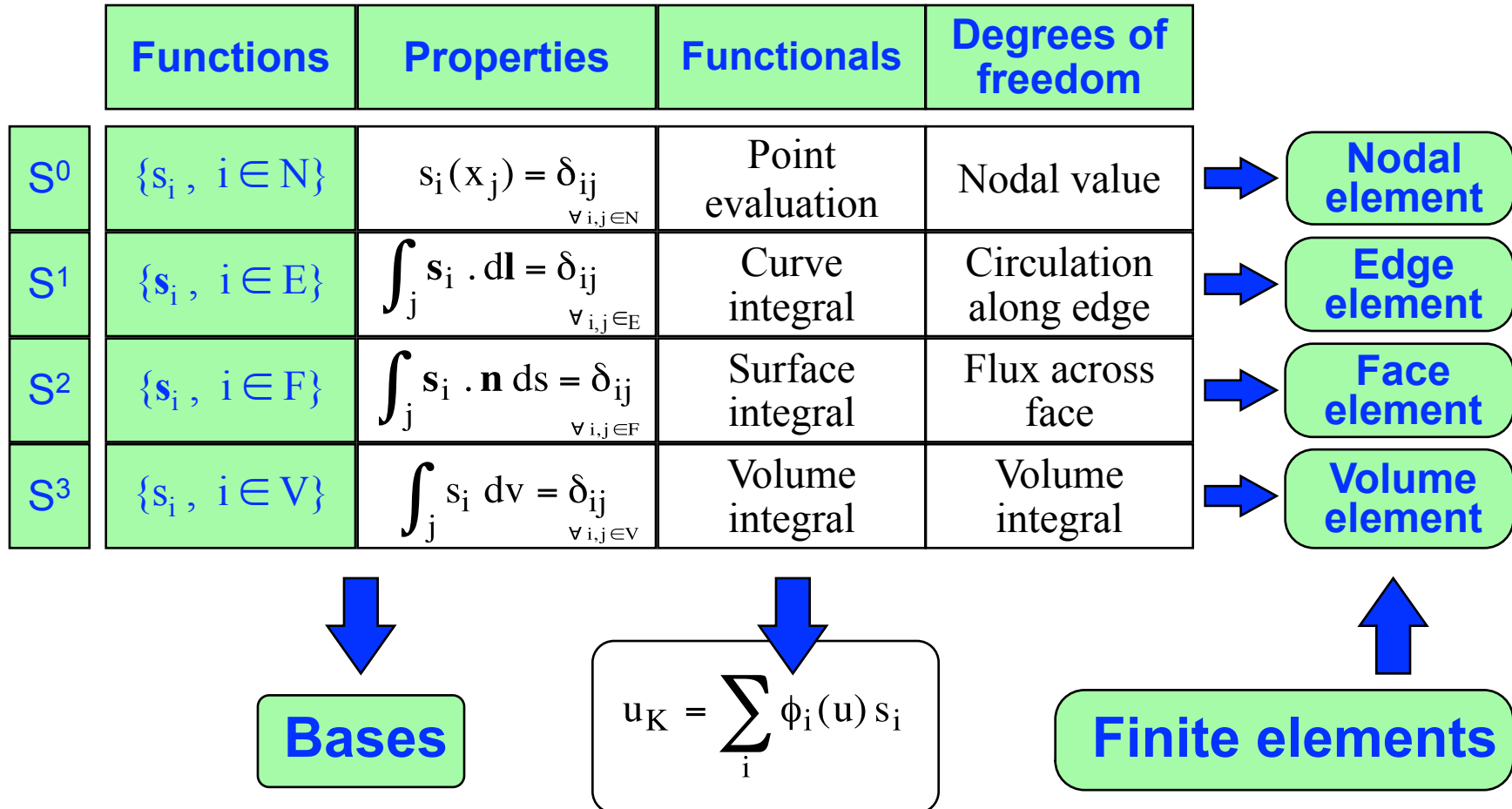
Union of finite elements  $(K_j, P_{K_j}, \Sigma_{K_j})$  such as :

- the union of the  $K_j$  fill the studied domain ( $\equiv$  mesh)
- some continuity conditions are satisfied across the element interfaces

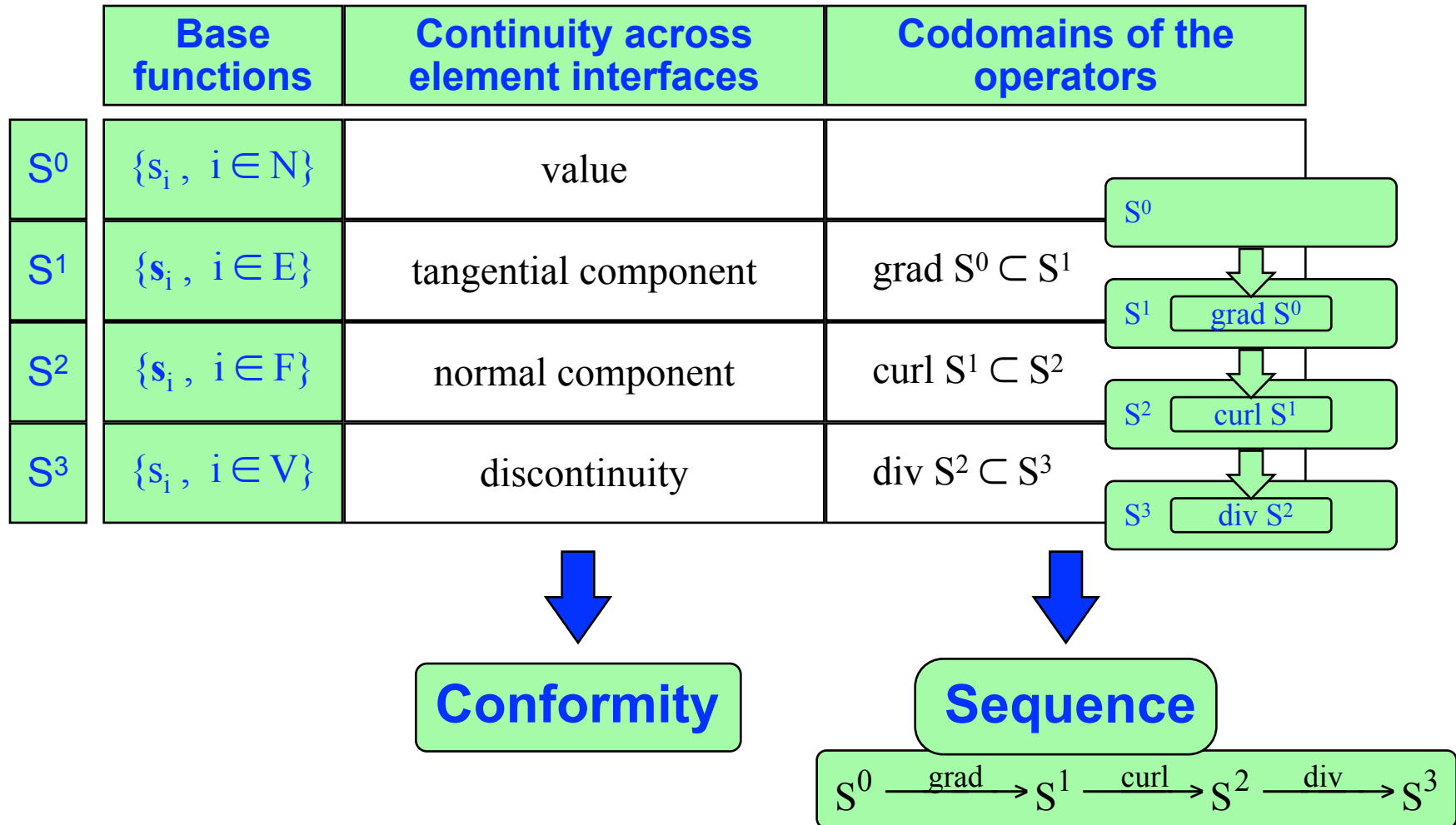
# Sequence of finite element spaces



# Sequence of finite element spaces



# Sequence of finite element spaces



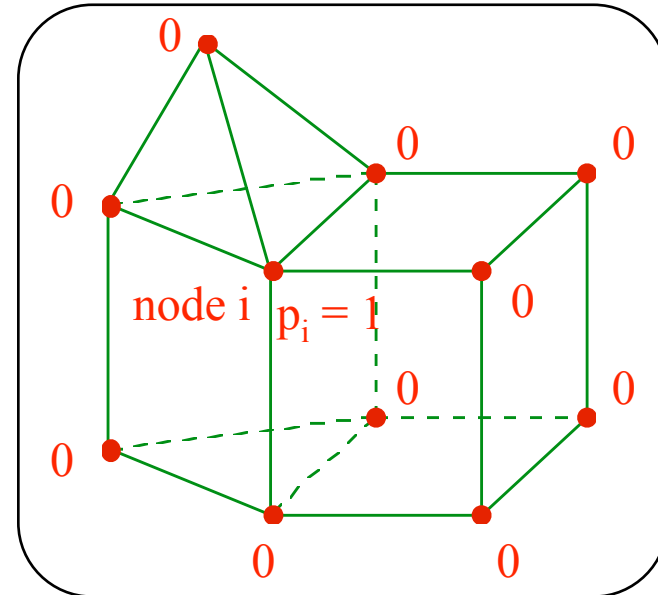
# Function spaces $S^0$ et $S^3$

For each **node**  $i \in \mathbf{N} \rightarrow$  **scalar field**

$$s_i(\mathbf{x}) = p_i(\mathbf{x}) \in S^0$$

$$p_i = \begin{cases} 1 & \text{at node } i \\ 0 & \text{at all other nodes} \end{cases}$$

$p_i$  continuous in  $\Omega$



For each **Volume**  $v \in \mathbf{V} \rightarrow$  **scalar field**

$$s_v = 1 / \text{vol}(v) \in S^3$$

# Edge function space $S^1$

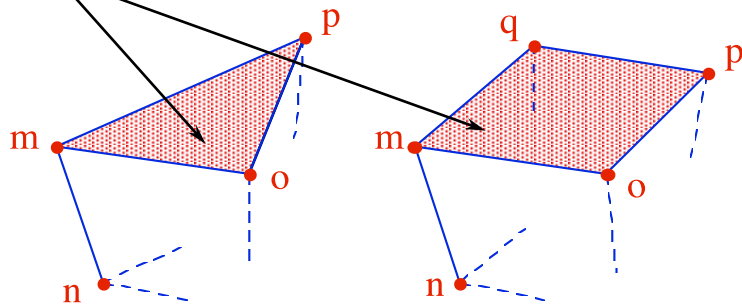
For each **edge**  $e_{ij} = \{i, j\} \in E \rightarrow$  **vector field**

$$s_{e_{ij}} = p_j \operatorname{grad} \sum_{r \in N_{F, \bar{j}i}} p_r - p_i \operatorname{grad} \sum_{r \in N_{F, \bar{i}j}} p_r$$

$$s_e \in S^1$$

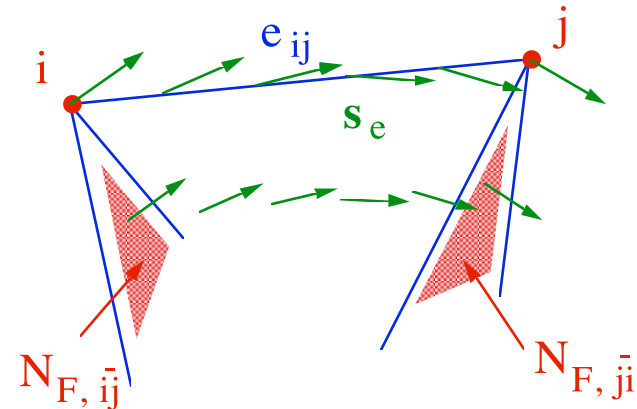
Definition of the set of nodes  $N_{F, m\bar{n}}$

$$N_{F, m\bar{n}} = \{i \in N; i \in f_{\operatorname{mop}(q)}, o, p, q \neq n\}$$

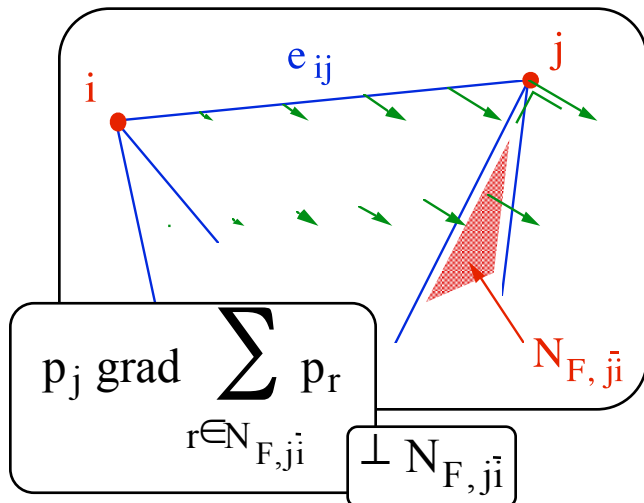


N.B.: In an element : 3 edges/node

Illustration of the vector field  $s_e$

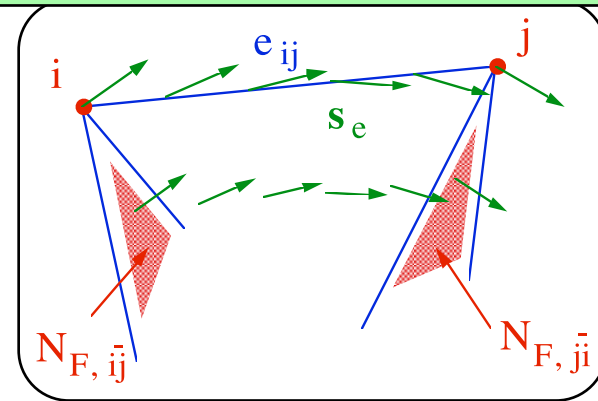
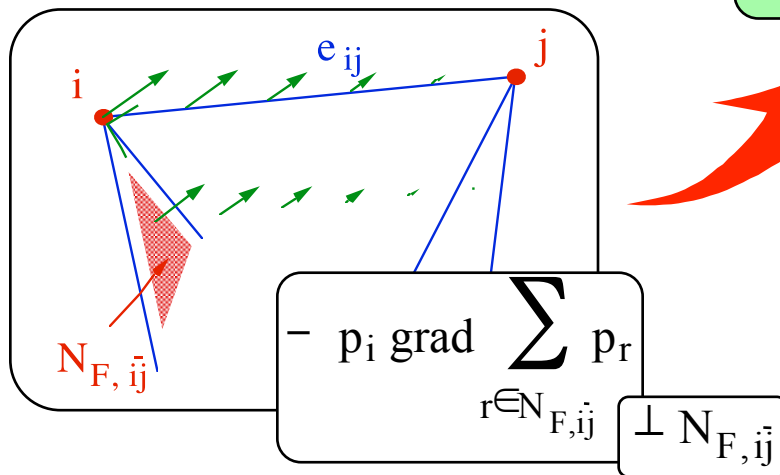


# Edge function space $S^1$



Geometric interpretation  
of the vector field  $s_e$

$$s_{e_{ij}} = p_j \text{grad} \sum_{r \in N_{F, ji}} p_r - p_i \text{grad} \sum_{r \in N_{F, ij}} p_r$$



# Function space $S^2$

For each facet  $f \in F \rightarrow$  vector field

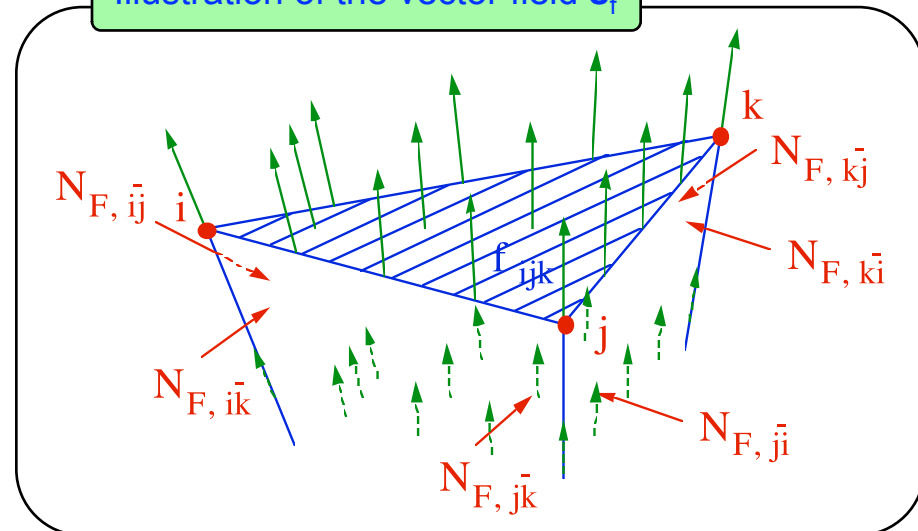
$$f = f_{ijk(l)} = \{i, j, k, (l)\} = \{q_1, q_2, q_3, (q_4)\}$$

$$s_f = a_f \sum_{c=1}^{\#N_f} p_{q_c} \text{grad} \left( \sum_{r \in N_{F, q_c}} p_r \right) \wedge \text{grad} \left( \sum_{r \in N_{F, q_c}} p_r \right)$$

$$s_f \in S^2$$

$$\#N_f = \begin{cases} \blacktriangle & 3 \rightarrow a_f = 2 \\ \blacksquare & 4 \rightarrow a_f = 1 \end{cases}$$

Illustration of the vector field  $s_f$



# Particular subspaces of $S^1$

## Kernel of the curl operator

$$\mathbf{h} \in S^1(\Omega) ; \text{curl } \mathbf{h} = 0 \text{ in } \Omega_c^c \subset \Omega \rightarrow \mathbf{h} \equiv ?$$

## Applications

Ampere equation  
in a domain  $\Omega_c^c$   
without current  
( $\leftrightarrow \Omega_c$ )

## Gauged subspace

$$\mathbf{a} \in S^1(\Omega) ; \mathbf{b} = \text{curl } \mathbf{a} \in S^2(\Omega) \rightarrow \mathbf{a} \equiv ?$$

Gauge  $\mathbf{a} \cdot \boldsymbol{\omega} = 0$   
to fix  $\mathbf{a}$

Gauge condition  
on a vector potential

Definition of a  
generalized source field  $\mathbf{h}_s$   
such that  $\text{curl } \mathbf{h}_s = \mathbf{j}_s$

# Kernel of the curl operator

## Case of simply connected domains

$$H = \{ \mathbf{h} \in S^1(\Omega) ; \text{curl } \mathbf{h} = 0 \text{ in } \Omega_c^C \}$$

$$\mathbf{h} = -\text{grad } \phi \text{ in } \Omega_c^C$$

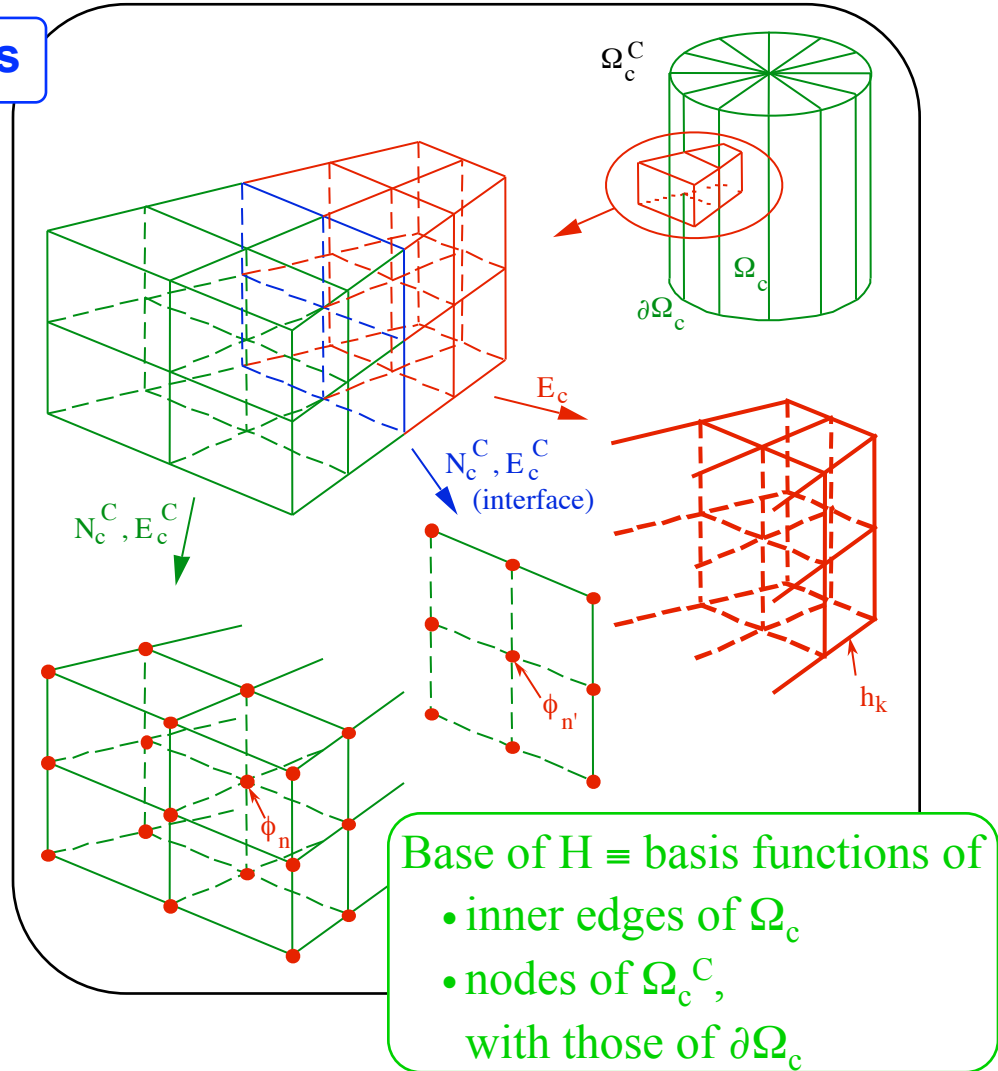
$$\mathbf{h} = \sum_{e \in E} h_a \mathbf{s}_a = \sum_{k \in E_c} h_k \mathbf{s}_k + \sum_{l \in E_c^C} h_l \mathbf{s}_l$$

$$h_l = \int_{l_{ab}} \mathbf{h} \cdot d\mathbf{l} = \int_{l_{ab}} -\text{grad } \phi \cdot d\mathbf{l} = \phi_{a_1} - \phi_{b_1}$$

$$\mathbf{h} = \sum_{k \in E_c} h_k \mathbf{s}_k + \sum_{l \in E_c^C} (\phi_{a_1} - \phi_{b_1}) \mathbf{s}_l$$

$$\mathbf{h} = \sum_{k \in E_c} h_k \mathbf{s}_k + \sum_{n \in N_c^C} \phi_n \mathbf{v}_n$$

with  $\mathbf{v}_n = \sum_{nj \in E_c^C} \mathbf{s}_{nj}$



# Kernel of the curl operator

## Case of multiply connected domains

$$H = \{ \mathbf{h} \in S^1(\Omega); \text{curl } \mathbf{h} = 0 \text{ in } \Omega_c^C \}$$

$$\phi = \phi^{\text{cont}} + \phi^{\text{disc}}$$

$$\mathbf{h} = -\text{grad } \phi \text{ in } \Omega_c^C \text{ (cuts)}$$

$$\phi^+ - \phi^- = \phi|_{\Gamma^+_{eci}} - \phi|_{\Gamma^-_{eci}} = I_i$$

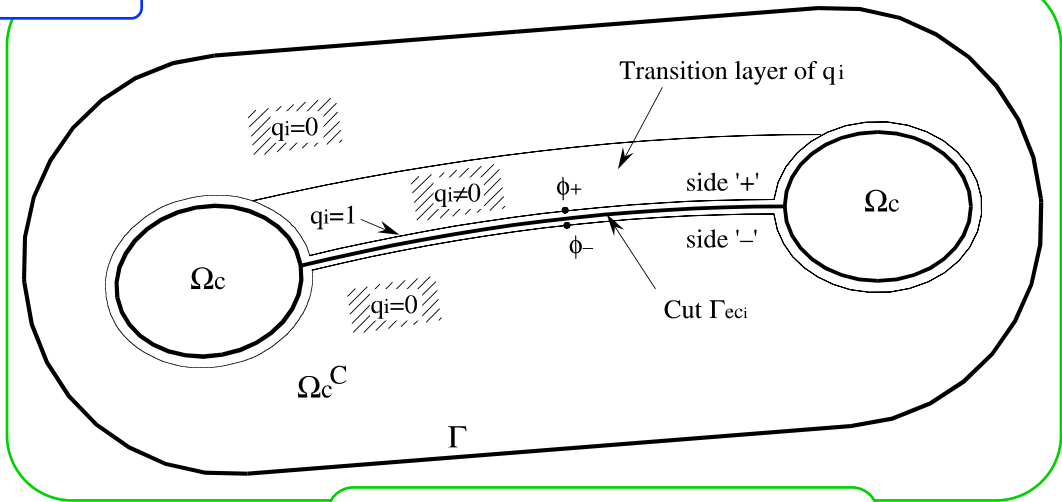
$$\phi^{\text{disc}} = \sum_{i \in C} I_i q_i$$

discontinuity of  $\phi^{\text{disc}}$

$$\mathbf{h} = \sum_{k \in A_c} h_k \mathbf{s}_k + \sum_{n \in N_c^C} \phi_n^{\text{cont}} \mathbf{v}_n + \sum_{i \in C} I_i \mathbf{c}_i$$

$$\text{with } \mathbf{c}_i = \sum_{n \in N_{eci}} s_{nj}$$

edges of  $\Omega_c^C$  ←  $n_j \in A_c^C$   
 starting from a node of the cut ←  $n \in N_{eci}$   
 and located on side '+' ←  $j \in N_c^C$   
 but not on the cut ←  $j \notin N_{eci}$



- $q_i$  • defined in  $\Omega_c^C$
- unit discontinuity across  $\Gamma_{eci}$
- continuous in a transition layer
- zero out of this layer

Basis of  $H \equiv$  basis functions of

- inner edges of  $\Omega_c$
- nodes of  $\Omega_c^C$
- cuts of  $C$

# Gauged subspace of $S^1$

## Gauged space in $\Omega$

$$\mathbf{b} = \text{curl } \mathbf{a} \quad \text{with} \quad \mathbf{a} = \sum_{e \in E} a_e \mathbf{s}_e \in S^1(\Omega) \quad , \quad \mathbf{b} = \sum_{f \in F} b_f \mathbf{s}_f \in S^2(\Omega)$$

$$\searrow \quad b_f = \sum_{e \in E} i(e, f) a_e \quad , \quad f \in F \quad \rightarrow \quad \text{matrix form:} \quad \boxed{b_f} = \boxed{C_{FE}} \quad \boxed{a_e}$$

Face-edge  
incidence matrix

**Tree**  $\equiv$  set of edges connecting  
(in  $\Omega$ ) all the nodes of  $\Omega$  without  
forming any loop ( $\hat{E}$ )

**Co-tree**  $\equiv$  complementary set of the tree ( $\check{E}$ )

## Gauged space of $S^1(\Omega)$

$$\check{S}^1(\Omega) = \{ \mathbf{a} \in S^1(\Omega) ; a_j = 0, \forall j \in \hat{E} \}$$

$$\mathbf{a} = \sum_{i \in \check{E}} a_i \mathbf{s}_i \in \check{S}^1(\Omega)$$

Basis of  $\check{S}^1(\Omega) \equiv$  co-tree edge basis functions  
(explicit gauge definition)

