ELEC0041

Modelling and Design of Electromagnetic Systems

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Electromagnetic Models

Electromagnetic Models

Maxwell's equations →

- Electrostatics
- Electrokinetics
- Electro-quasistatics
- Magnetostatics
- Magneto-quasistatics
- Full wave

Electromagnetic Models

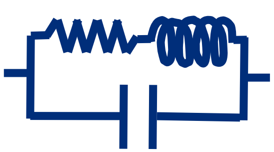
- Electrostatics
  - Distribution of electric field due to static charges and/or levels of electric potential

- Electrokinetics
  - Distribution of static electric current in conductors

- Electro-quasistatics (or "electrodynamics")
  - Distribution of electric field and electric current in materials (insulating and conducting)

- Magnetostatics
  - Distribution of static magnetic field due to magnets and continuous currents

- Magneto-quasistatics (or "magnetodynamics")
  - Distribution of magnetic field and eddy current due to moving magnets and time variable currents



- Full wave
  - Propagation of electromagnetic fields

Maxwell's Equations

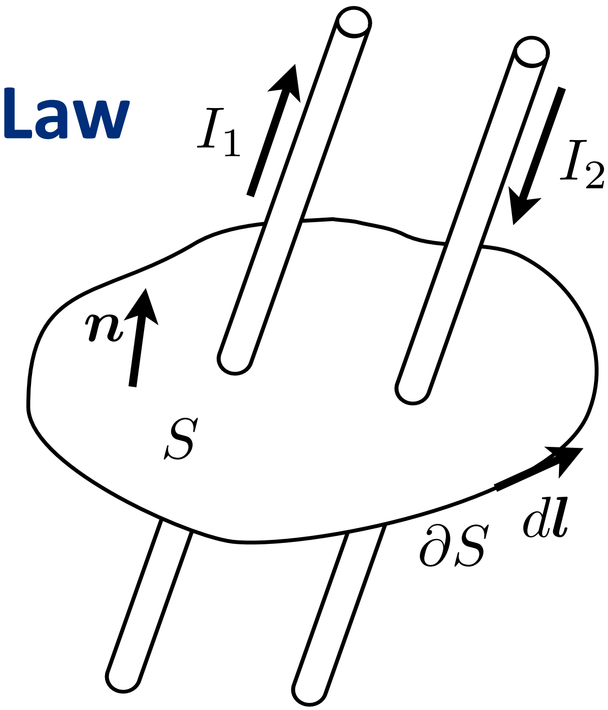
$$\mathbf{curl}\,\boldsymbol{h} = \boldsymbol{j} + \partial_t \boldsymbol{d}$$ Maxwell-Ampère's equation

$$\mathbf{curl}\,\boldsymbol{e} = -\partial_t \boldsymbol{b}$$ Faraday's equation

$$\left.\begin{aligned} \operatorname{div} \boldsymbol{b} &= 0 \\ \operatorname{div} \boldsymbol{d} &= \rho_q \end{aligned}\right\}$$ Conservation equations

$\boldsymbol{h}$	magnetic field (A/m)	$\boldsymbol{e}$	electric field (V/m)
$\boldsymbol{b}$	magnetic flux density (T)	$\boldsymbol{d}$	electric displacement (C/m^2)
$\boldsymbol{j}$	current density (A/m^2)	ρ_q	charge density (C/m^3)

Integral form: Ampère's Law



$$\mathbf{curl}\, \boldsymbol{h} = \boldsymbol{j}$$

$$\Rightarrow \oint_{\partial S} \boldsymbol{h} \cdot d\boldsymbol{l} = I$$

Magnetomotive force (m.m.f.)

Circulation of magnetic field along closed curve equals algebraic sum of currents crossing the underlying surface

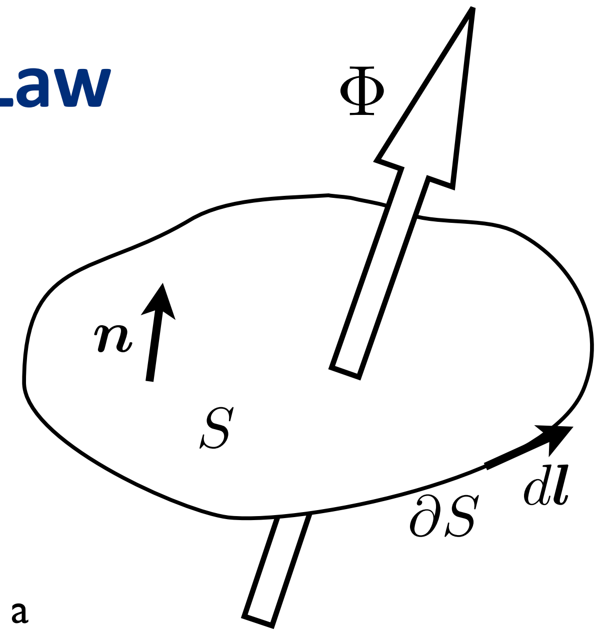
$$\oint_{\partial S} \boldsymbol{h} \cdot d\boldsymbol{l} = I_1 - I_2$$

Conservation of current: $\mathrm{div}\, \boldsymbol{j} = 0$

$$\Rightarrow \oint_{\partial V} \boldsymbol{j} \cdot \boldsymbol{n}\, ds = 0$$

Sum of currents arriving at a node is zero

Integral form: Faraday's Law



$$\mathbf{curl}\, \boldsymbol{e} = -\partial_t \boldsymbol{b}$$

$$\Rightarrow \oint_{\partial S} \boldsymbol{e} \cdot d\boldsymbol{l} = -\partial_t \Phi$$

Electromotive force (e.m.f.)

Any variation of magnetic flux through a circuit gives rise to an electromotive force

For a circuit moving at speed $\boldsymbol{v}$:

f.e.m. = $\oint_{\partial S(t)} \boldsymbol{f}/q \cdot d\boldsymbol{l} = \oint_{\partial S(t)} (\boldsymbol{e} + \boldsymbol{v} \times \boldsymbol{b}) \cdot d\boldsymbol{l} = -d_t \int_{S(t)} \boldsymbol{b} \cdot \boldsymbol{n}\, ds$

Conservation of magnetic flux density: $\mathrm{div}\, \boldsymbol{b} = 0$

$$\Rightarrow \oint_{\partial V} \boldsymbol{b} \cdot \boldsymbol{n}\, ds = 0$$

Magnetic flux lines are closed

Lorentz Force

Interaction of electromagnetic fields with a point charge moving at speed $\boldsymbol{v}$

$$\boldsymbol{f} = q(\boldsymbol{e} + \boldsymbol{v} \times \boldsymbol{b})$$

For a conductor (electrically neutral, only negative charges moving):

$$\boldsymbol{f} = \boldsymbol{j} \times \boldsymbol{b} = \mathbf{curl}\, \boldsymbol{h} \times \boldsymbol{b}$$ Laplace Force

Electromagnetic Power

Poynting vector: $\boldsymbol{s} = \boldsymbol{e} \times \boldsymbol{h}$

Power exchanged with a volume (interior normal):

$$P = \oint_{\partial V} \boldsymbol{s} \cdot \boldsymbol{n} \, ds = -\int_V \operatorname{div} \boldsymbol{s} \, dv = \int_V p \, dv$$

Power density:

$$p = -\operatorname{div} \boldsymbol{e} \times \boldsymbol{h} = -\boldsymbol{h} \cdot \mathbf{rot}\, \boldsymbol{e} + \boldsymbol{e} \cdot \mathbf{rot}\, \boldsymbol{h}$$

$$\Rightarrow p = \boldsymbol{h} \cdot \partial_t \boldsymbol{b} + \boldsymbol{e} \cdot \boldsymbol{j} + \boldsymbol{e} \cdot \partial_t \boldsymbol{d}$$

"magnetic" *"conduction"* *"electric"*

Material Constitutive Relations

$\boldsymbol{b} = \mu \boldsymbol{h}$	Magnetic relation
$\boldsymbol{d} = \varepsilon \boldsymbol{e}$	Dielectric relation
$\boldsymbol{j} = \sigma \boldsymbol{e}$	Ohm's law

Characteristics of materials:

- μ magnetic permeability (H/m)
- ε dielectric permittivity (F/m)
- σ electric conductivity (S/m)

} constants (linear materials), functions of electromagnetic fields (nonlinear materials), tensorial (anisotropic materials), functions of other physical fields (temperature, ...)

Magnetic Relation

$$\boldsymbol{b} = \mu \boldsymbol{h} \qquad \mu = \mu_r \mu_0 \begin{cases} \mu_r & \text{Relative magnetic permeability} \\ \mu_0 & \text{Vacuum permeability } (4\pi 10^{-7} \text{H/m}) \end{cases}$$

- Diamagnetic and paramagnetic materials $\mu_r \approx 1$

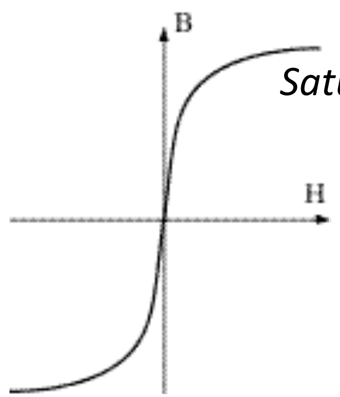
  Linear materials (silver, copper, aluminum)

- Ferromagnetic materials $\mu_r \gg 1, \quad \mu_r = \mu_r(h)$

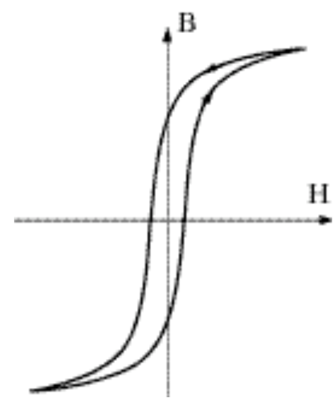
  Nonlinear materials (steel, iron)

  Ferromagnetic-paramagnetic transition for T > T_{Curie} (T_{Curie} of iron : 1043 K)

b(h) curves



Saturation



Hysteresis (non bijective law)

Energy dissipation = cycle area

Steinmetz formula: $p_{\text{hyst}} = \omega k_h b_{\max}^{\nu}$

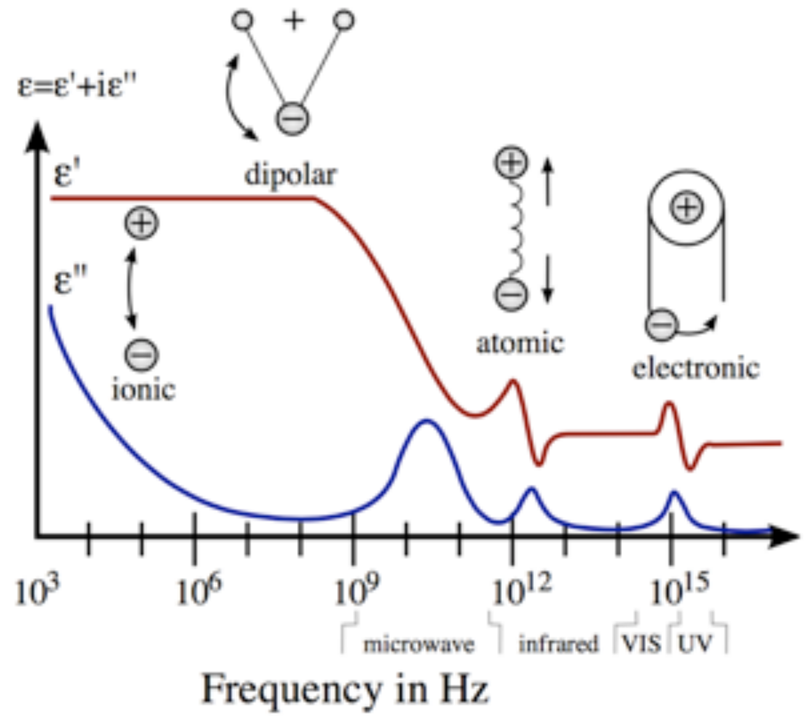
Dielectric Relation

$$\boldsymbol{d} = \varepsilon \boldsymbol{e} \qquad \varepsilon = \varepsilon_r \varepsilon_0 \begin{cases} \varepsilon_r & \text{Relative dielectric permittivity} \\ \varepsilon_0 = \dfrac{1}{\mu_0 c_0^2} & \text{Vacuum permittivity} \end{cases}$$

Vacuum permittivity (8.854187817620... × 10−12 F/m)

ε_r at room temperature for f < 1kHz

Air	1.0006
Teflon	2.1
Polyethylene	2.25
Paper	3.85
Glass	3.7 - 10
Concrete	4.5
Water	80



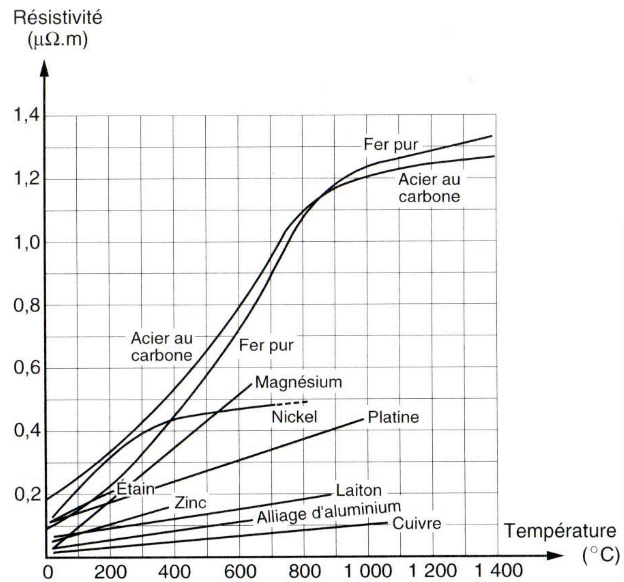
Ohm's Law

$$\boldsymbol{j} = \sigma \boldsymbol{e} \qquad \text{(Resistivity } \rho = \frac{1}{\sigma} \text{)}$$

Simple models for temperature dependency

- Metals : $\rho = \rho_0(1 + \alpha(T - T_0))$

	ρ_0 $(T_0 = 20°C)$ (Ωm)	α $(°C^{-1})$
Aluminum	2.7 10^{-8}	4 10^{-3}
Copper	1.7 10^{-8}	3.9 10^{-3}
Iron	9.6 10^{-8}	6.5 10^{-3}



- Glass : $\ln \rho = A + \frac{B}{T}$

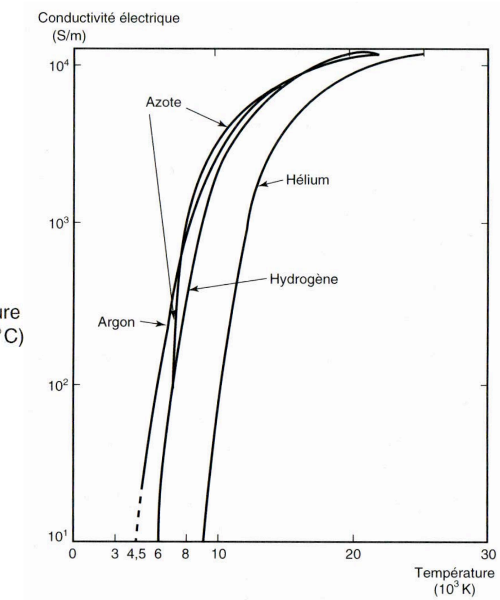
Common glass: $\ln \rho = -4.6 + \frac{7678}{T}$

- Ionic solutions : $\sigma = \sigma_0 + \alpha(T - T_0)$

Tap water: $\sigma_0 = 0.055\,\Omega^{-1} m^{-1}$

$\alpha = 1.65\,10^{-3}\,°C^{-1}\Omega^{-1}m^{-1}$

$T_0 = 20°C$



Model Choice

Maxwell's equations & constitutive relations in frequency domain, without sources:

$$\Delta \boldsymbol{e} - i\omega\sigma\mu \boldsymbol{e} + \omega^2 \varepsilon\mu \boldsymbol{e} = 0$$

Using characteristic lengths

- domain size L
- skin depth $\delta = \sqrt{\dfrac{2}{\omega\sigma\mu}}$
- wavelength $\lambda = \dfrac{2\pi}{k}$, with $\begin{cases} \text{wave number } k = \dfrac{\omega}{c} \\ \text{speed of light } c = \dfrac{1}{\sqrt{\varepsilon\mu}} \end{cases}$

allows to write in non-dimensional form:

$$\left(\frac{3}{L^2} - \frac{2i}{\delta^2} + \frac{4\pi}{\lambda^2} \right) \boldsymbol{e} = 0$$

Model Choice

$$\left(\frac{3}{L^2} - \frac{2i}{\delta^2} + \frac{4\pi}{\lambda^2}\right)\boldsymbol{e} = 0$$

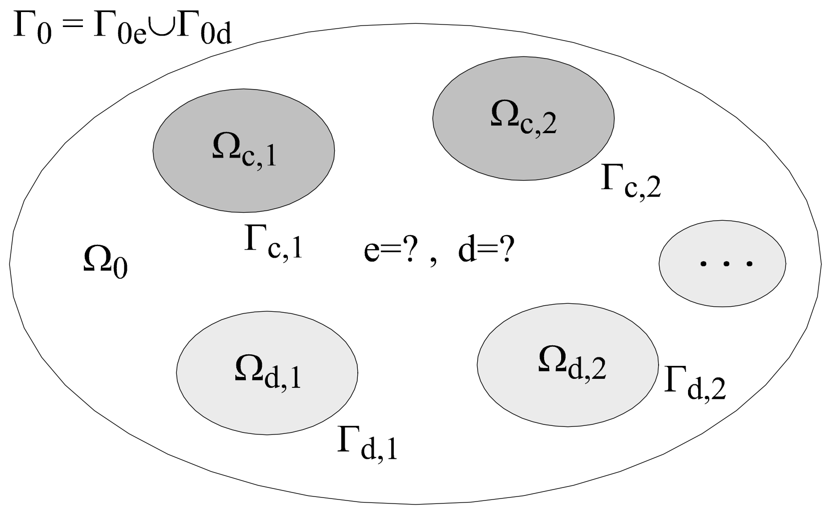
Non-dimensional numbers

$$\begin{cases} g_1 = \left(\dfrac{\lambda}{L}\right)^2 \\ g_2 = \left(\dfrac{\delta}{L}\right)^2 \\ g_3 = \left(\dfrac{\lambda}{\delta}\right)^2 \end{cases}$$

- $g_1 \gg 1$ uncoupled electric or magnetic problems
  - $g_2 \gg 1$ magnetostatics
  - $g_2 \lesssim 1$ magnetodynamics
  - $g_3 \gg 1$ electrokinetics
  - $g_3 \approx 1$ electrodynamics
  - $g_3 \ll 1$ electrostatics
- $g_1 \lesssim 1$ full wave ($g_1 \to 0$ high-frequency asymptotics)

Electrostatics

$$\mathbf{curl}\, \boldsymbol{e} = 0$$
$$\operatorname{div} \boldsymbol{d} = \rho_q$$
$$\boldsymbol{d} = \varepsilon \boldsymbol{e}$$



Ω_0 Exterior region
$\Omega_{c,i}$ Conductors
$\Omega_{d,i}$ Dielectrics

Boundary conditions

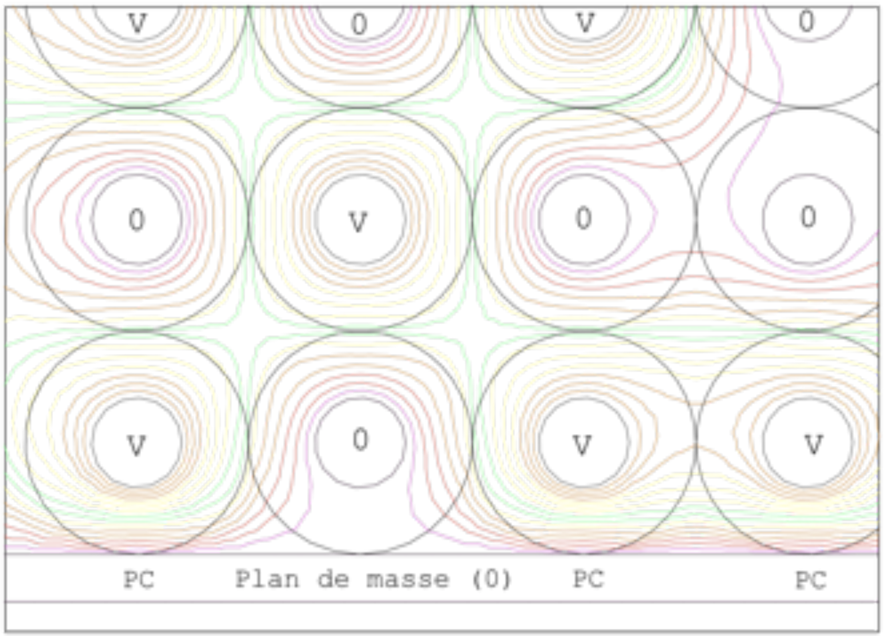
$$\boldsymbol{n} \times \boldsymbol{e}|_{\Gamma_{0e}} = 0$$
$$\boldsymbol{n} \cdot \boldsymbol{d}|_{\Gamma_{0d}} = 0$$

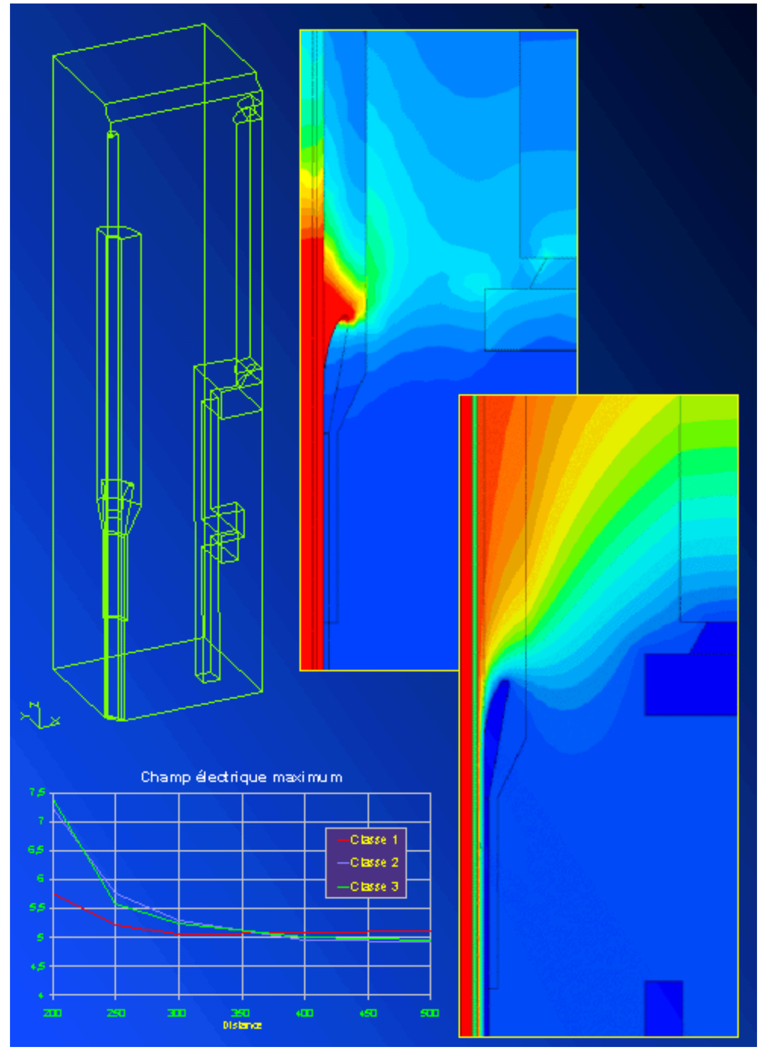
Example: electric scalar potential formulation

$$\operatorname{div} \varepsilon\, \mathbf{grad}\, v = -\rho_q \quad \text{with} \quad \boldsymbol{e} = -\mathbf{grad}\, v$$

- Formulation for
  - the exterior region Ω_0
  - the dielectric regions $\Omega_{d,i}$
- In each conducting region $\Omega_{c,i}$, $v = v^i \Rightarrow v|_{\Gamma_{c,i}} = v^i$

Electrostatics

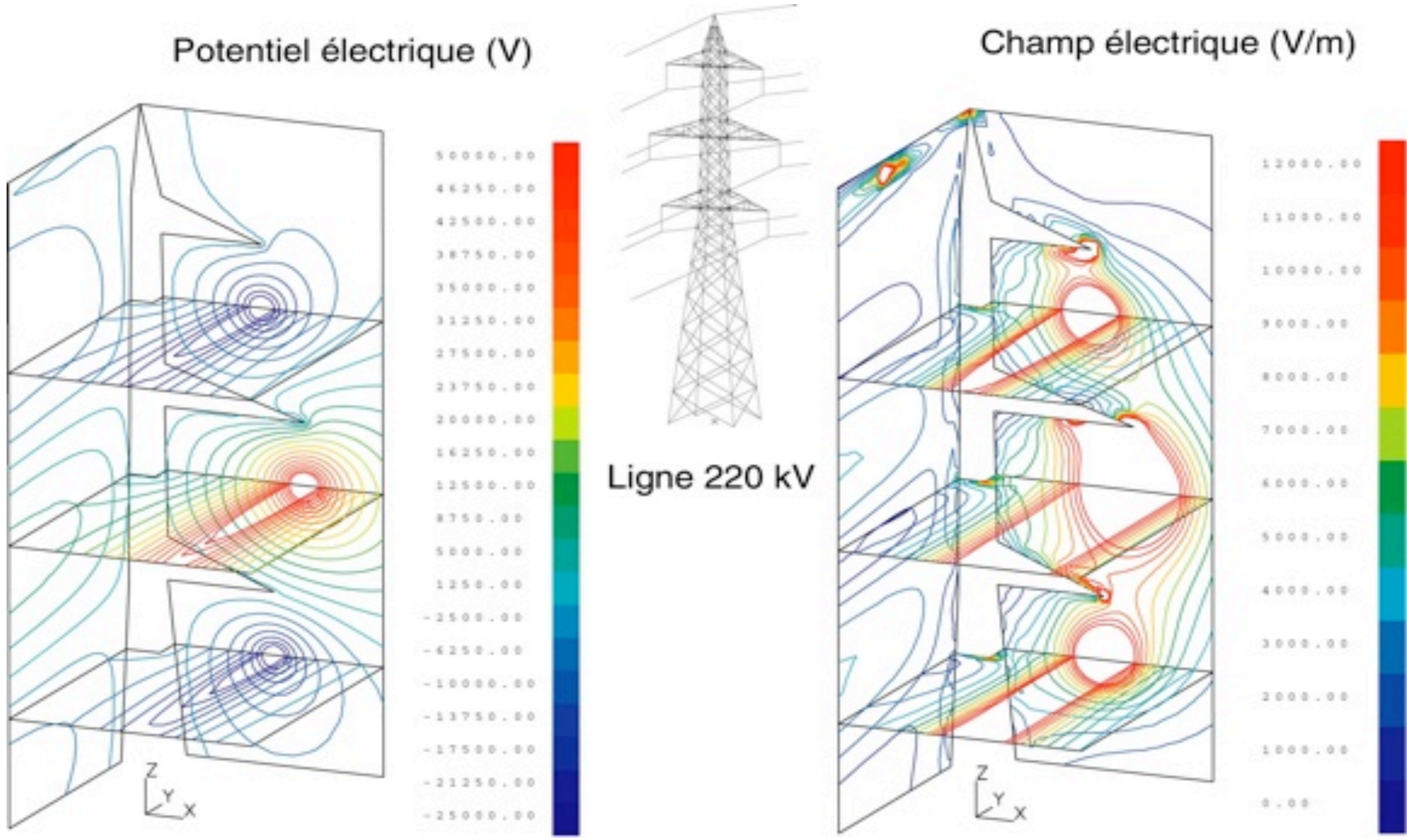




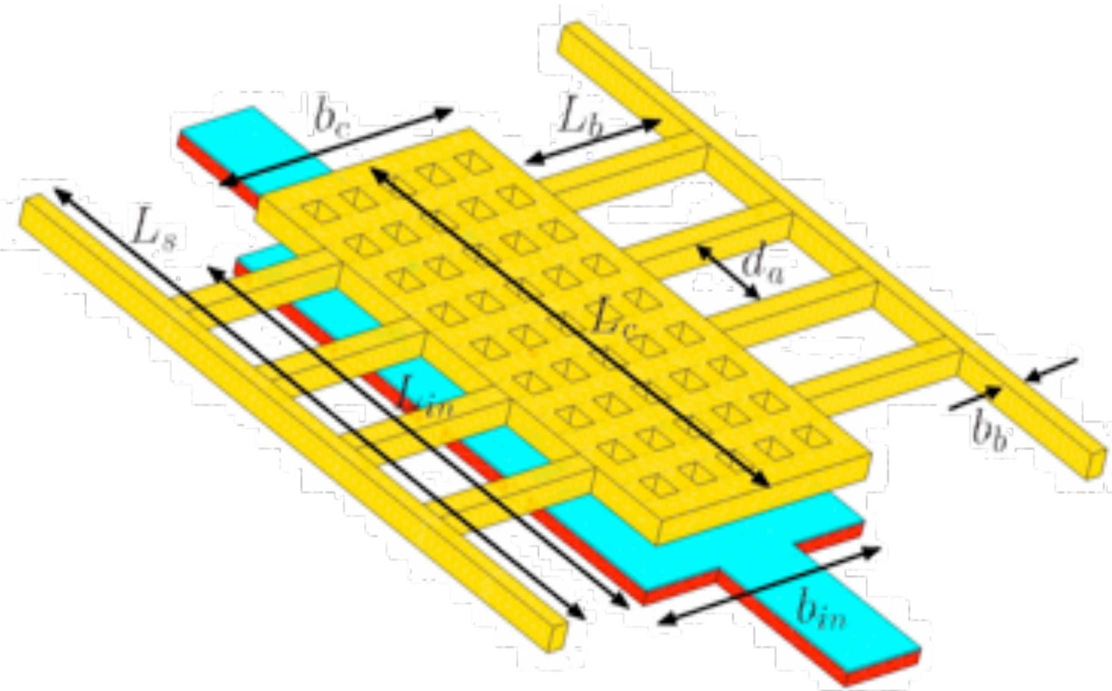
Cable bundles and high-voltage isolators

Electrostatics

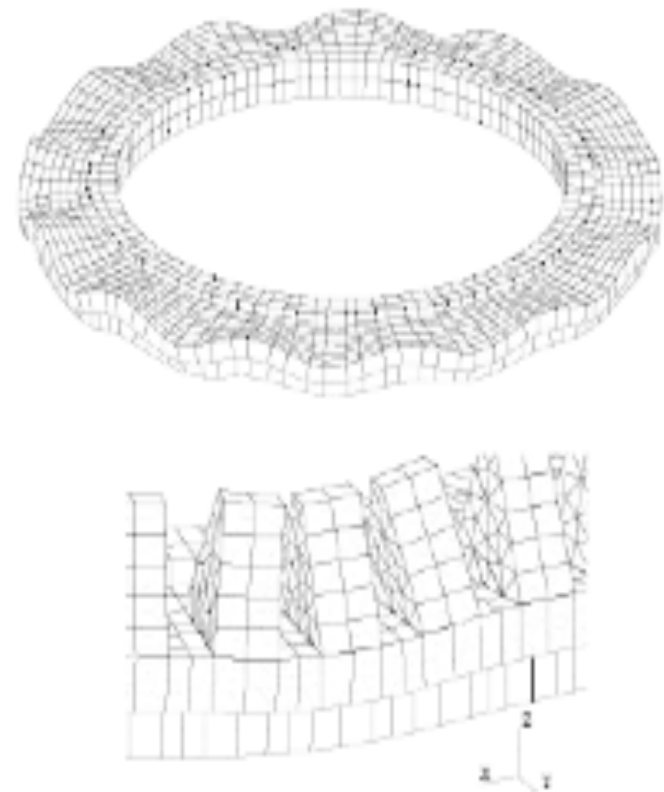
Potential and electric field next to a 220 kV high voltage tower



Electrostatics

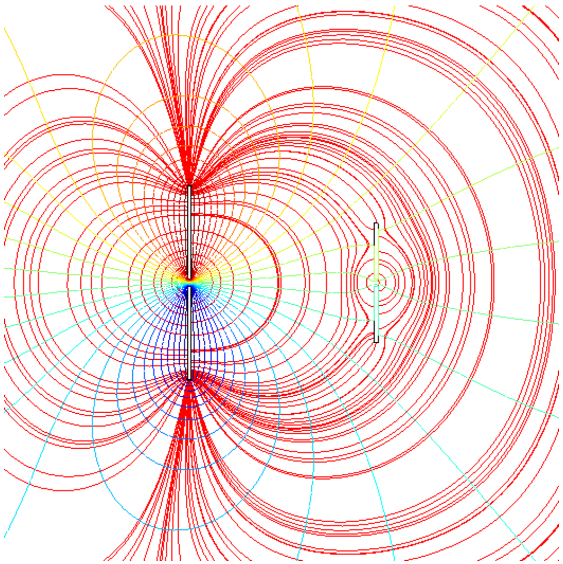


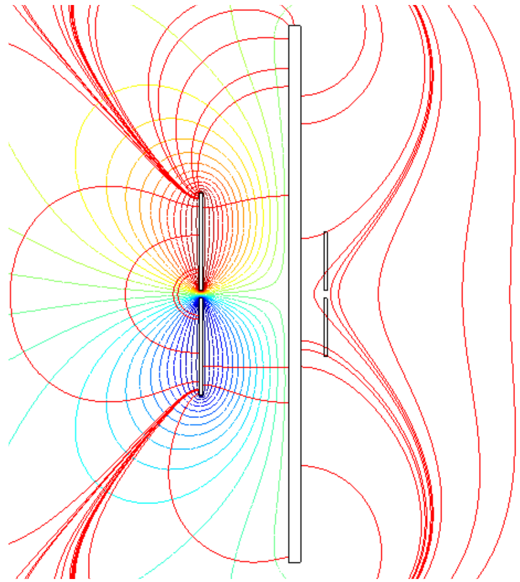
Shunt Capacitive
MEM switch

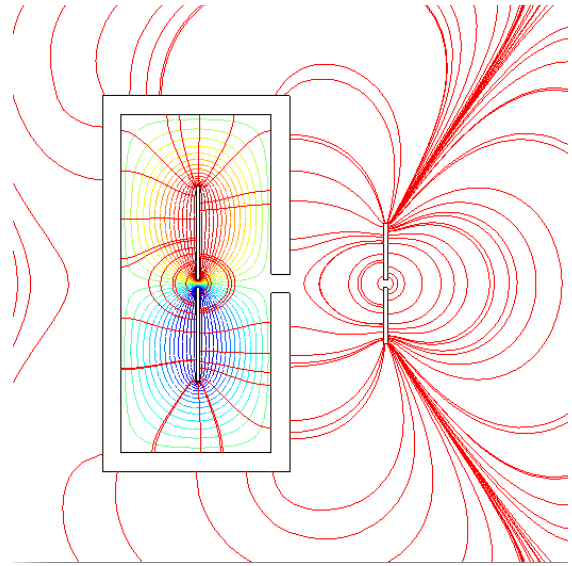


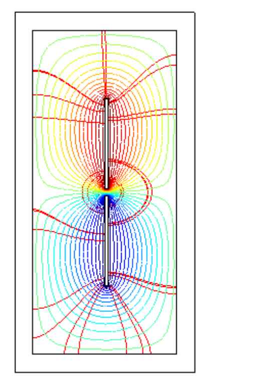
Piezoelectric Motor

Electrostatics







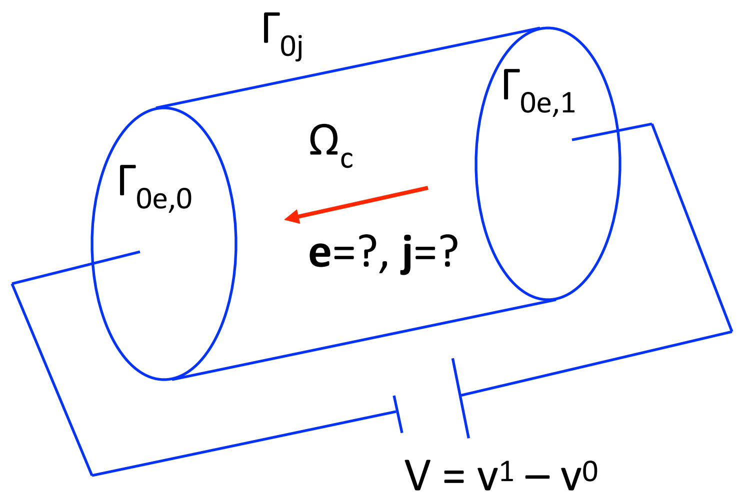


Effectiveness of electric field shields

(T. Hubing, Clemson University)

Electrokinetics

$$\mathbf{curl}\,\boldsymbol{e} = 0$$
$$\mathbf{curl}\,\boldsymbol{h} = \boldsymbol{j} \Rightarrow \mathrm{div}\,\boldsymbol{j} = 0$$
$$\boldsymbol{j} = \sigma \boldsymbol{e}$$



Ω_c Conductor

Boundary conditions

$$\boldsymbol{n} \times \boldsymbol{e}|_{\Gamma_{0e}} = 0$$
$$\boldsymbol{n} \cdot \boldsymbol{j}|_{\Gamma_{0j}} = 0$$

Example: electric scalar potential formulation

$$\mathrm{div}\,\sigma\,\mathbf{grad}\,v = 0 \quad \text{with} \quad \boldsymbol{e} = -\mathbf{grad}\,v$$

- Formulation for the conducting region Ω_c
- On each electrode $\Gamma_{0e,i}$, $v = v^i \Rightarrow v|_{\Gamma_{0e,i}} = v^i$

Electro-quasistatics

$$\mathbf{curl}\,\boldsymbol{e} = 0$$

$$\mathbf{curl}\,\boldsymbol{h} = \boldsymbol{j} + \partial_t \boldsymbol{d} \Rightarrow \mathrm{div}\,(\boldsymbol{j} + \partial_t \boldsymbol{d}) = 0$$

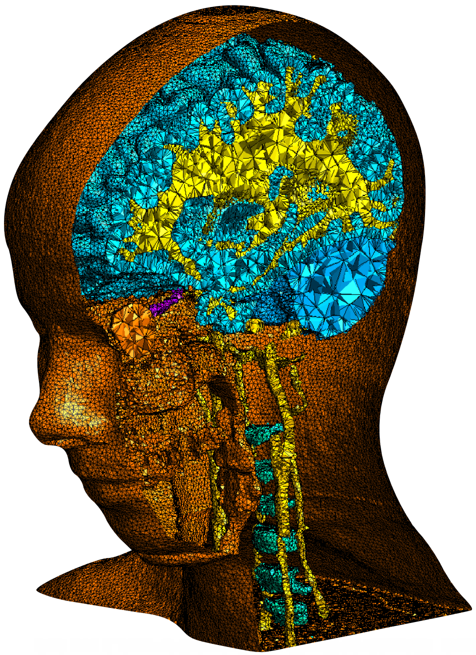
$$\boldsymbol{j} = \sigma \boldsymbol{e}$$

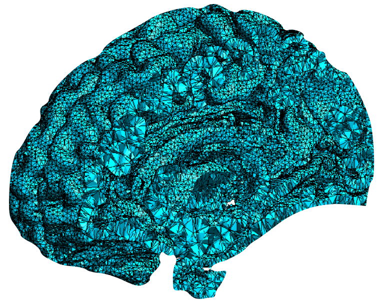
$$\boldsymbol{d} = \varepsilon \boldsymbol{e}$$

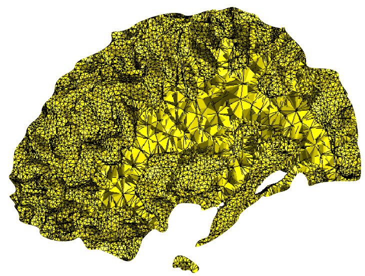
Example: electric scalar potential formulation

$$\mathrm{div}\,(\sigma\,\mathbf{grad}\,v + \varepsilon\mathbf{grad}\,\partial_t v) = 0 \quad \text{with} \quad \boldsymbol{e} = -\mathbf{grad}\,v$$

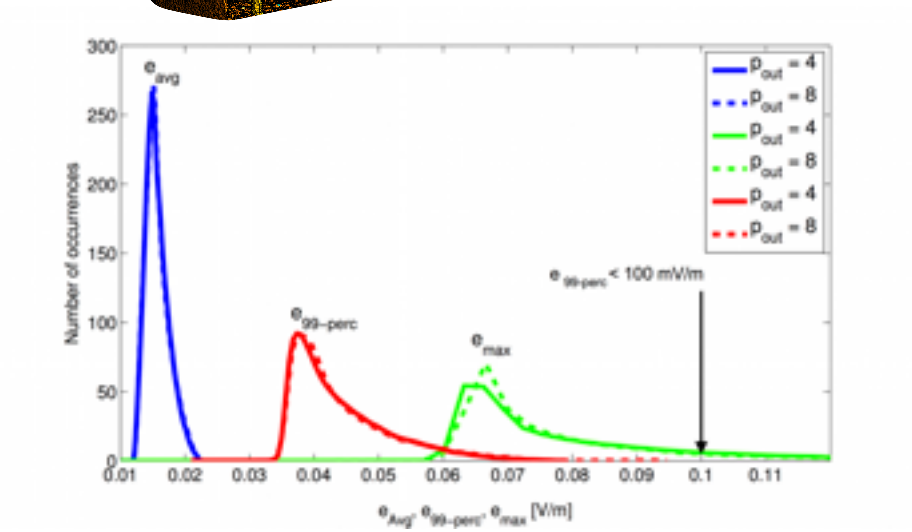
Electrokinetics & Electro-quasistatics

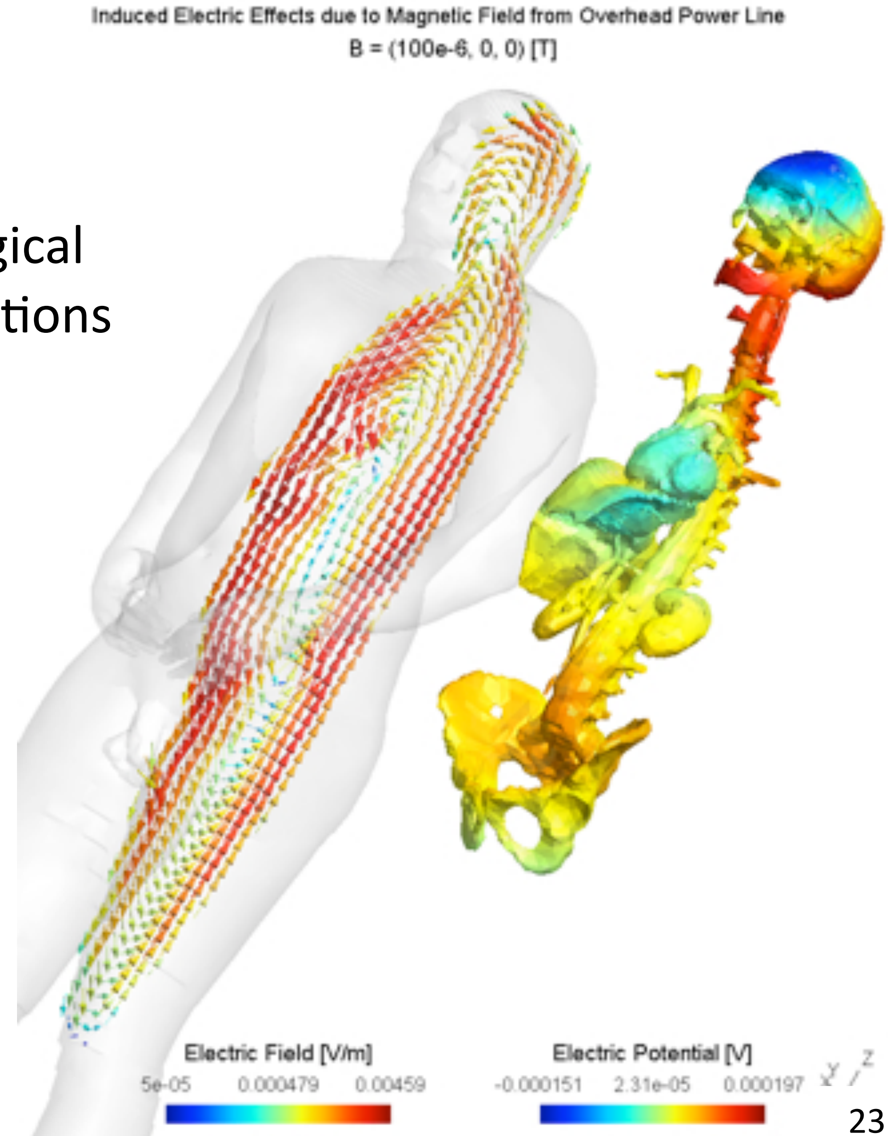






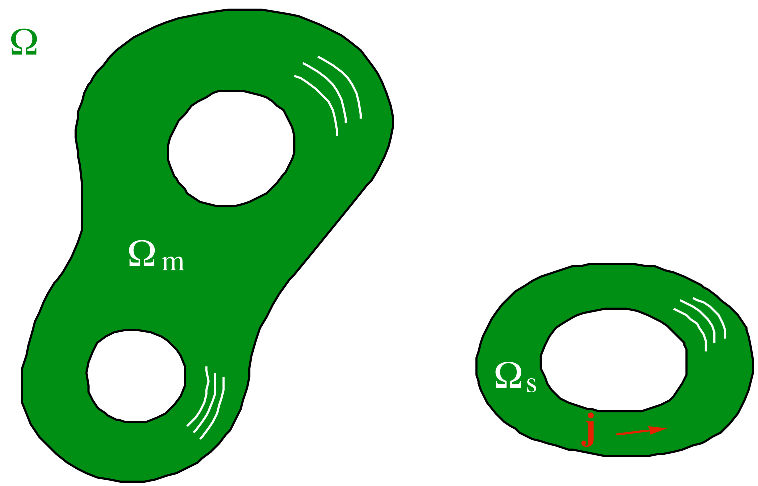
Biological applications





Magnetostatics

$$\mathbf{curl}\, \boldsymbol{h} = \boldsymbol{j}$$
$$\mathrm{div}\, \boldsymbol{b} = 0$$
$$\boldsymbol{b} = \mu \boldsymbol{h}$$



Ω Studied domain
Ω_m Magnetic domain
Ω_s Inductor

$\boldsymbol{j} = \boldsymbol{j}_s$ imposed source current density in inductor

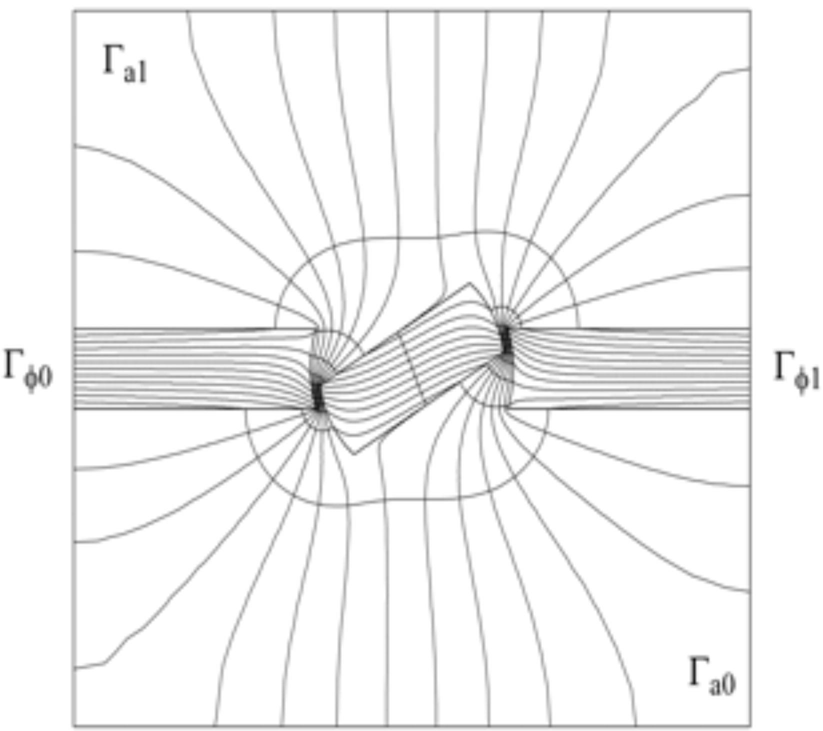
With magnets:

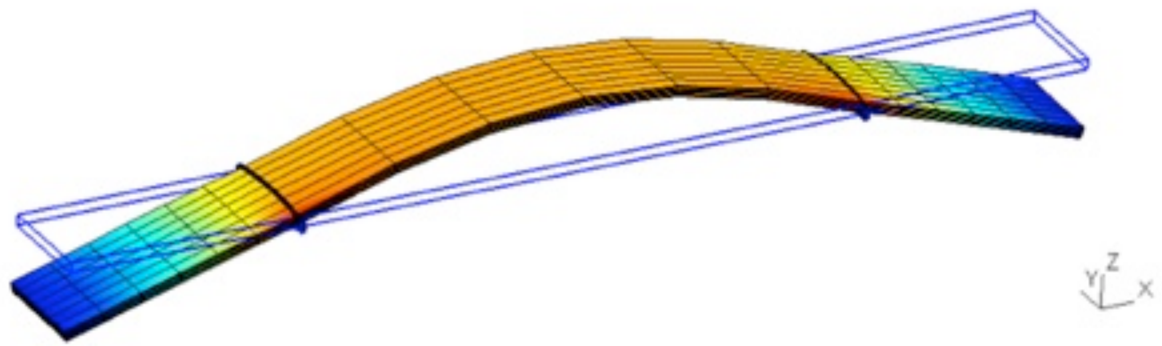
$$\boldsymbol{b} = \mu \boldsymbol{h} + \boldsymbol{b}_s$$
$$\boldsymbol{h} = \frac{1}{\mu}\boldsymbol{b} + \boldsymbol{h}_s$$

Example: magnetic vector potential formulation

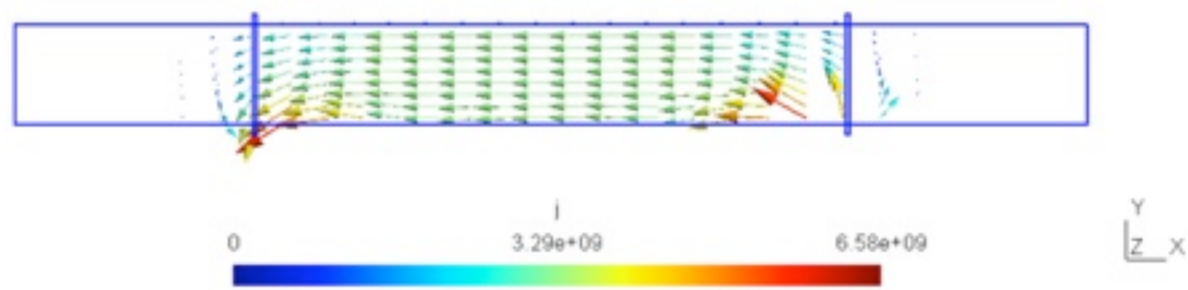
$$\mathbf{curl}\, \frac{1}{\mu} \mathbf{curl}\, \boldsymbol{a} = \boldsymbol{j}_s$$ with $$\boldsymbol{b} = \mathbf{curl}\, \boldsymbol{a}$$

Magnetostatics



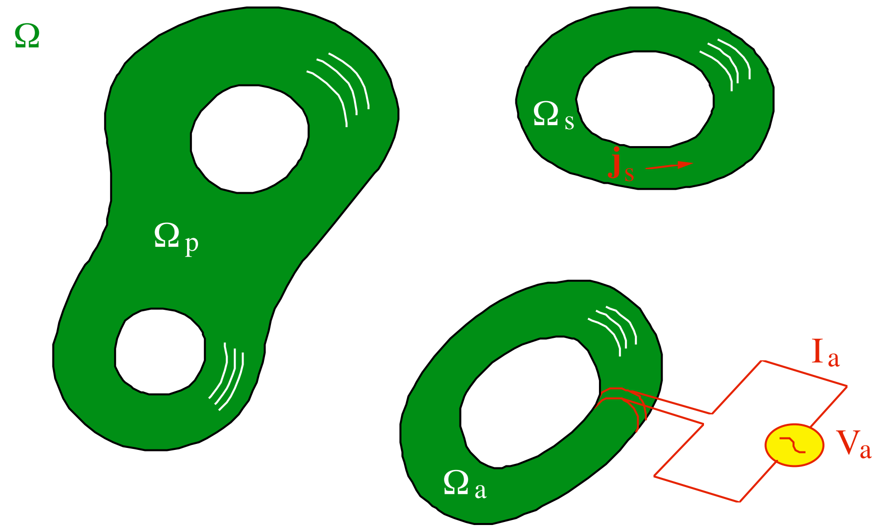


MEMS magnetometer



Magneto-quasistatics

$$\mathbf{curl}\,\boldsymbol{h} = \boldsymbol{j}$$
$$\mathbf{curl}\,\boldsymbol{e} = -\partial_t \boldsymbol{b}$$
$$\mathrm{div}\,\boldsymbol{b} = 0$$
$$\boldsymbol{b} = \mu \boldsymbol{h} + \boldsymbol{b}_s$$
$$\boldsymbol{j} = \sigma \boldsymbol{e} + \boldsymbol{j}_s$$



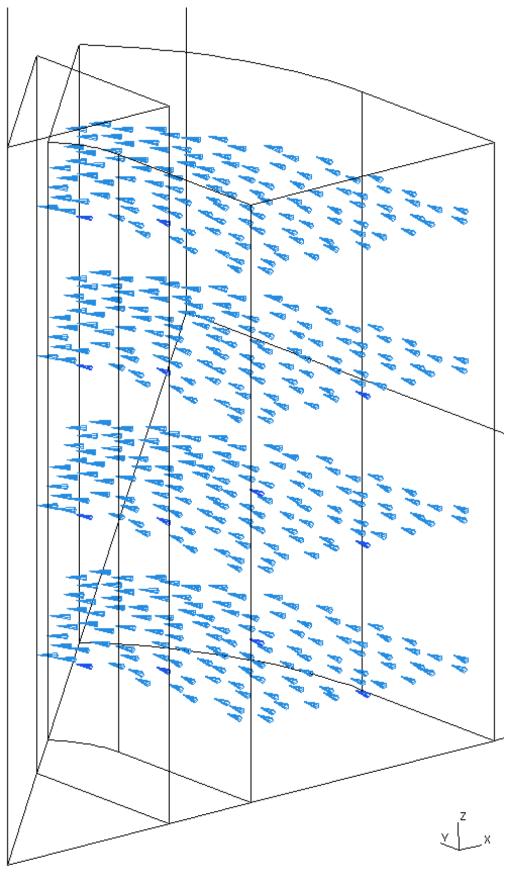
Ω Studied domain
Ω_p Passive conductor and/or magnetic domain
Ω_a Active conductor
Ω_s Inductor

Example: magnetic vector potential formulation

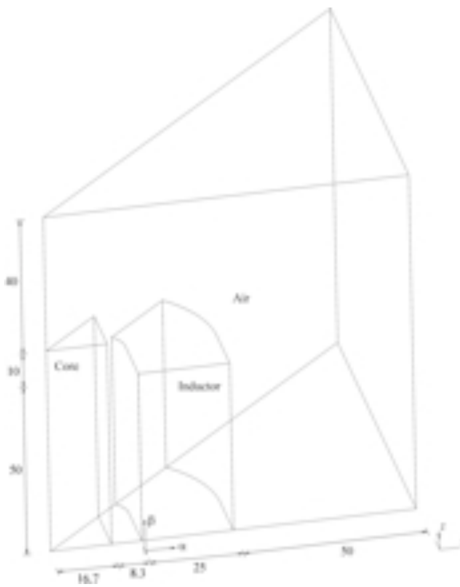
$$\mathbf{curl}\,\frac{1}{\mu}\mathbf{curl}\,\boldsymbol{a} + \sigma(\partial_t \boldsymbol{a} + \mathbf{grad}\,v) = \boldsymbol{j}_s \quad \text{with} \quad \boldsymbol{b} = \mathbf{curl}\,\boldsymbol{a}$$
$$\boldsymbol{e} = -\mathbf{grad}\,v - \partial_t \boldsymbol{a}$$

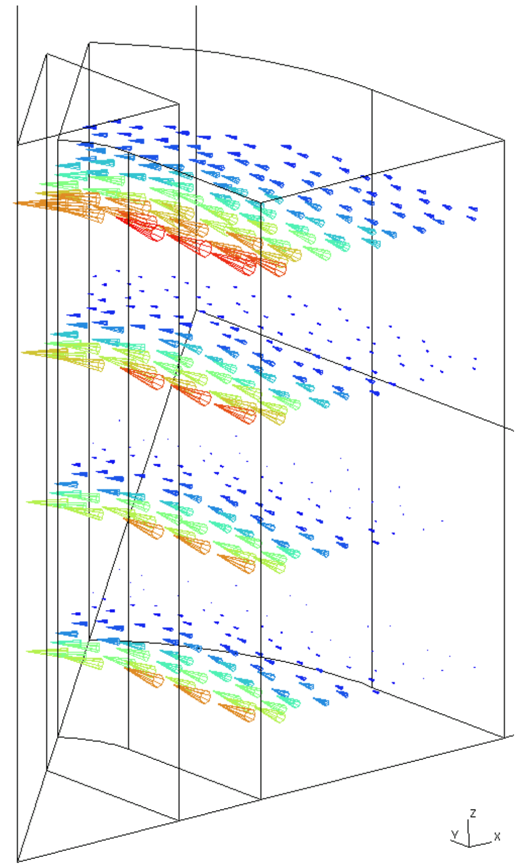
Magneto-quasistatics

Inductor (portion : 1/8th)



Stranded inductor - uniform current density ($\mathbf{j}_s$)

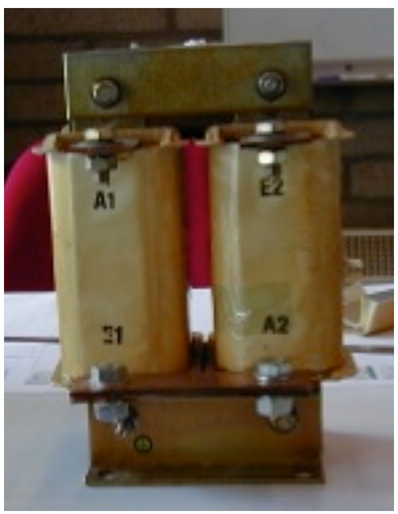


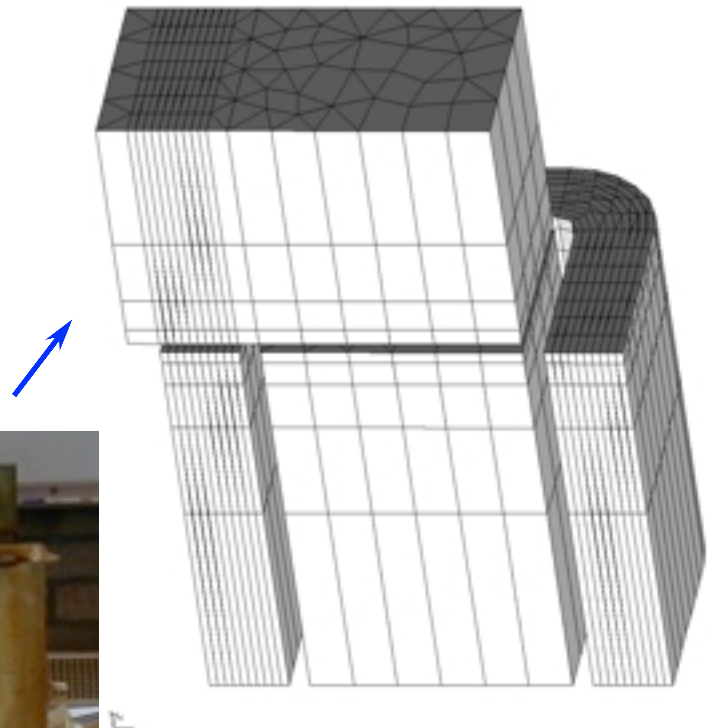


Massive inductor - non-uniform current density ($\mathbf{j}$)

Magneto-quasistatics

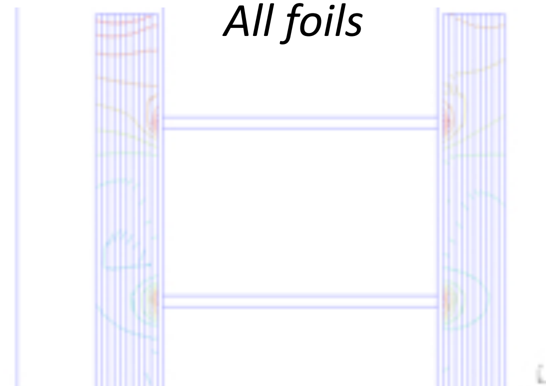
Foil winding inductance - current density (in a cross-section)

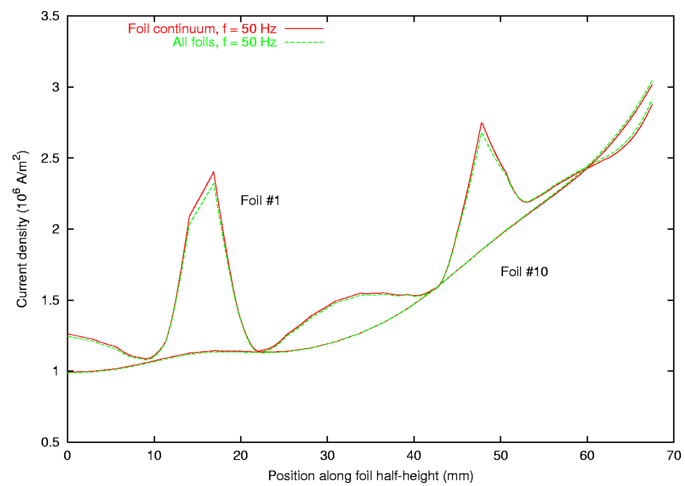




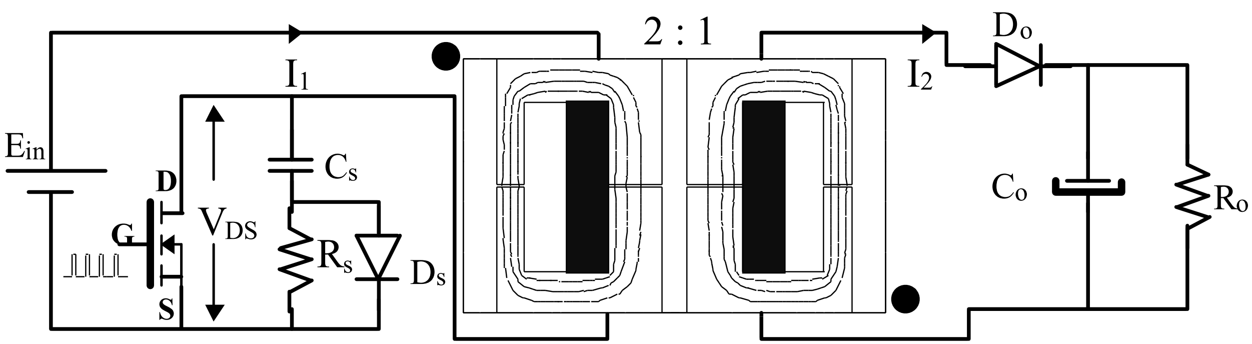
With air gaps, Frequency f = 50 Hz

All foils

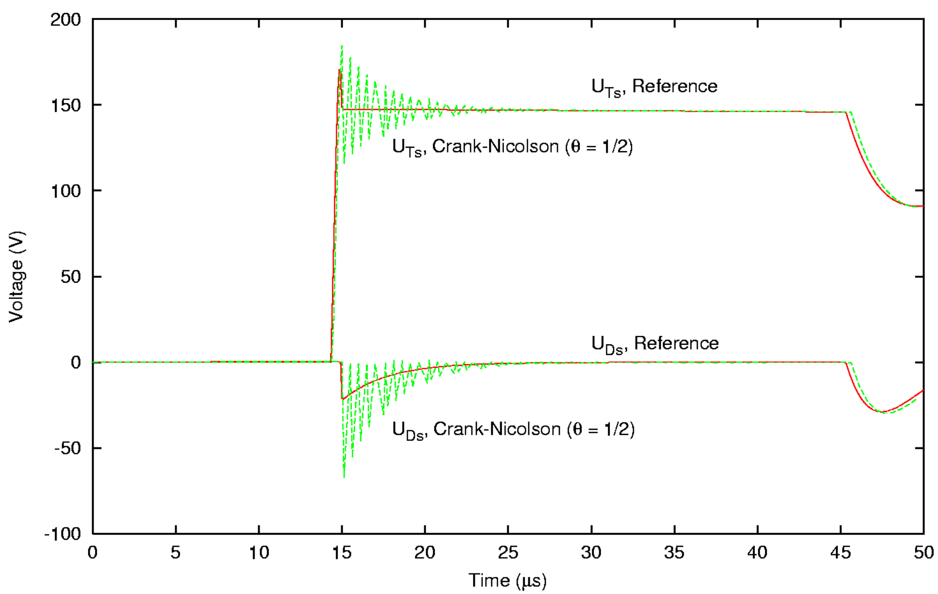


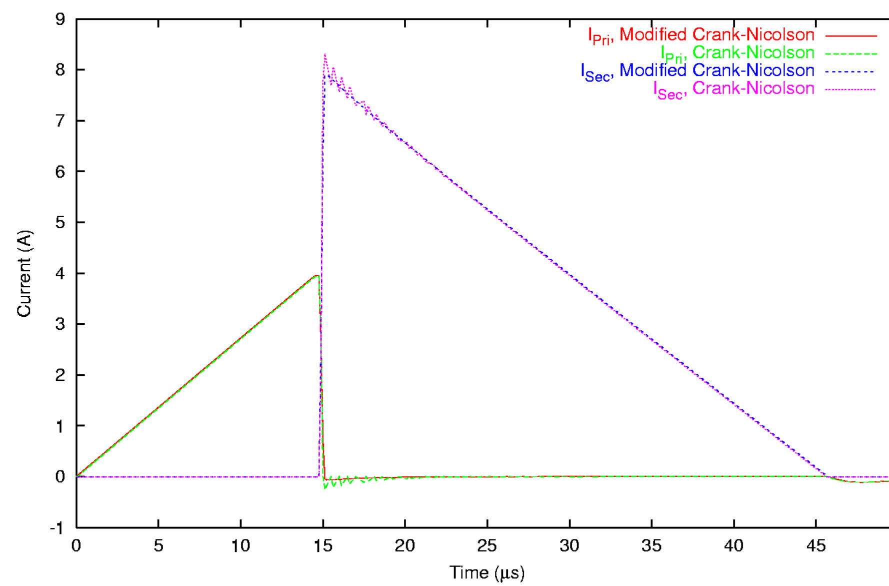


Magneto-quasistatics

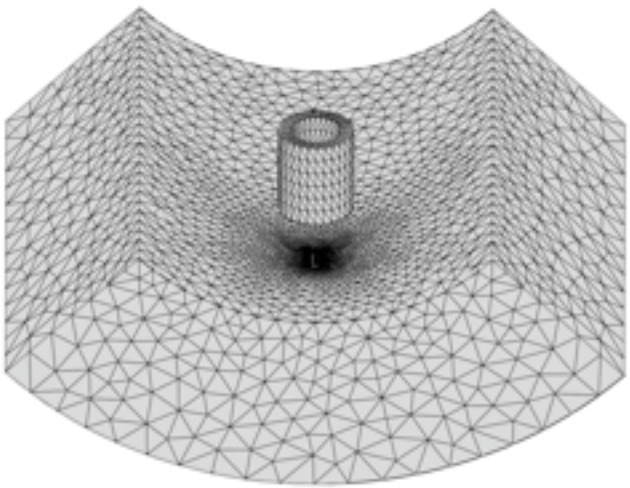


Sudden primary and secondary current changes in the transformer

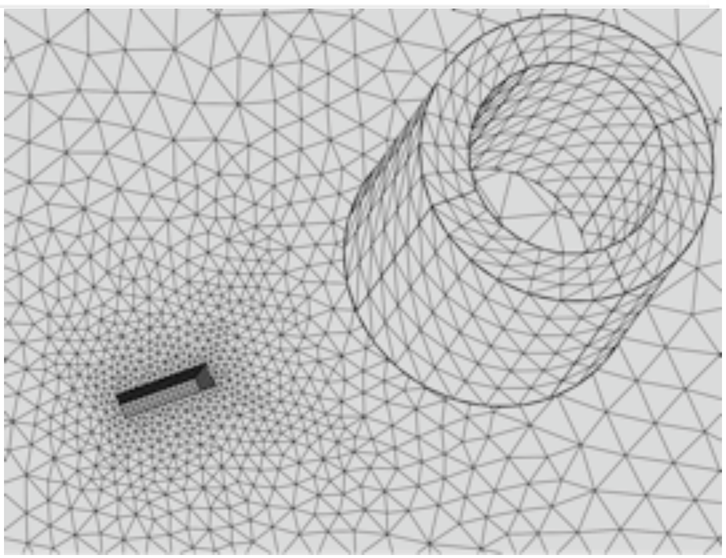


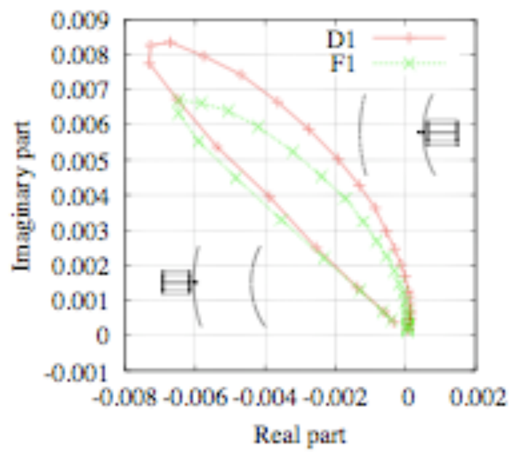


Magneto-quasistatics

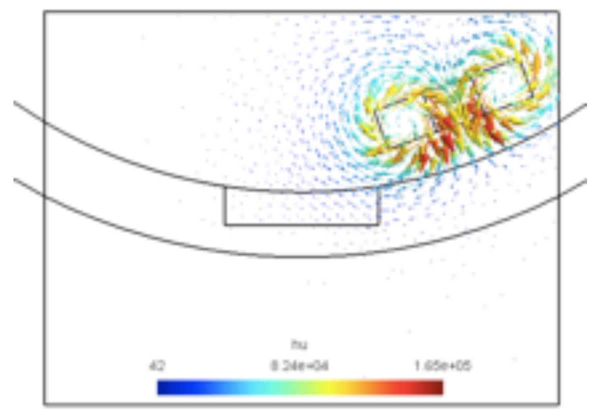


Eddy-current non-destructive testing



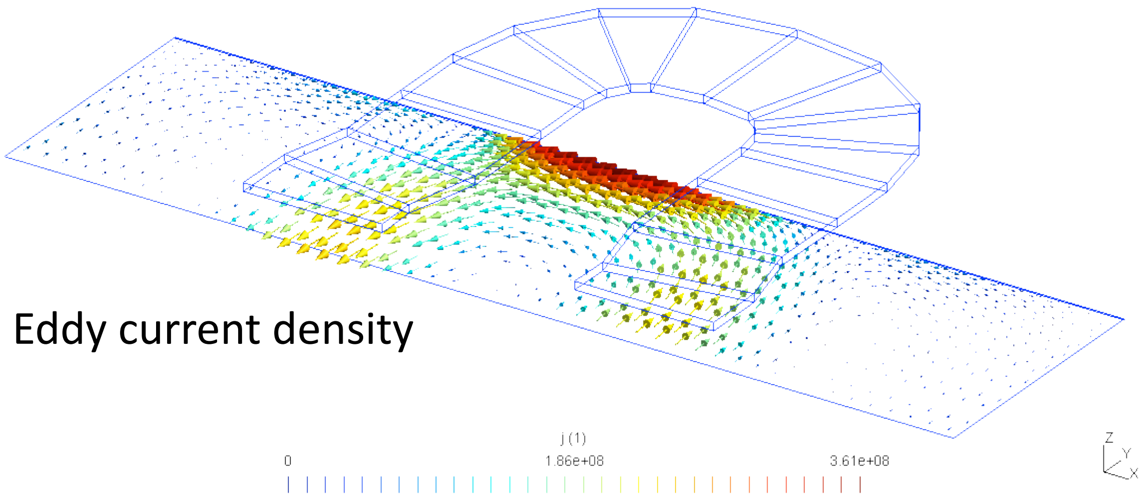


Impedance change



Magnetic field without defect

Magneto-quasistatics

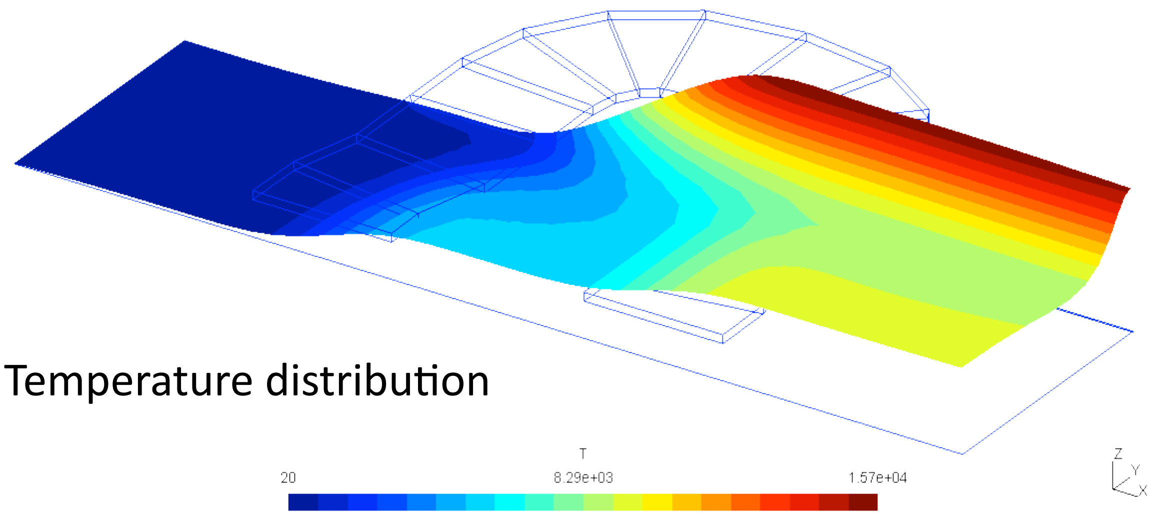


Eddy current density

Transverse induction heating

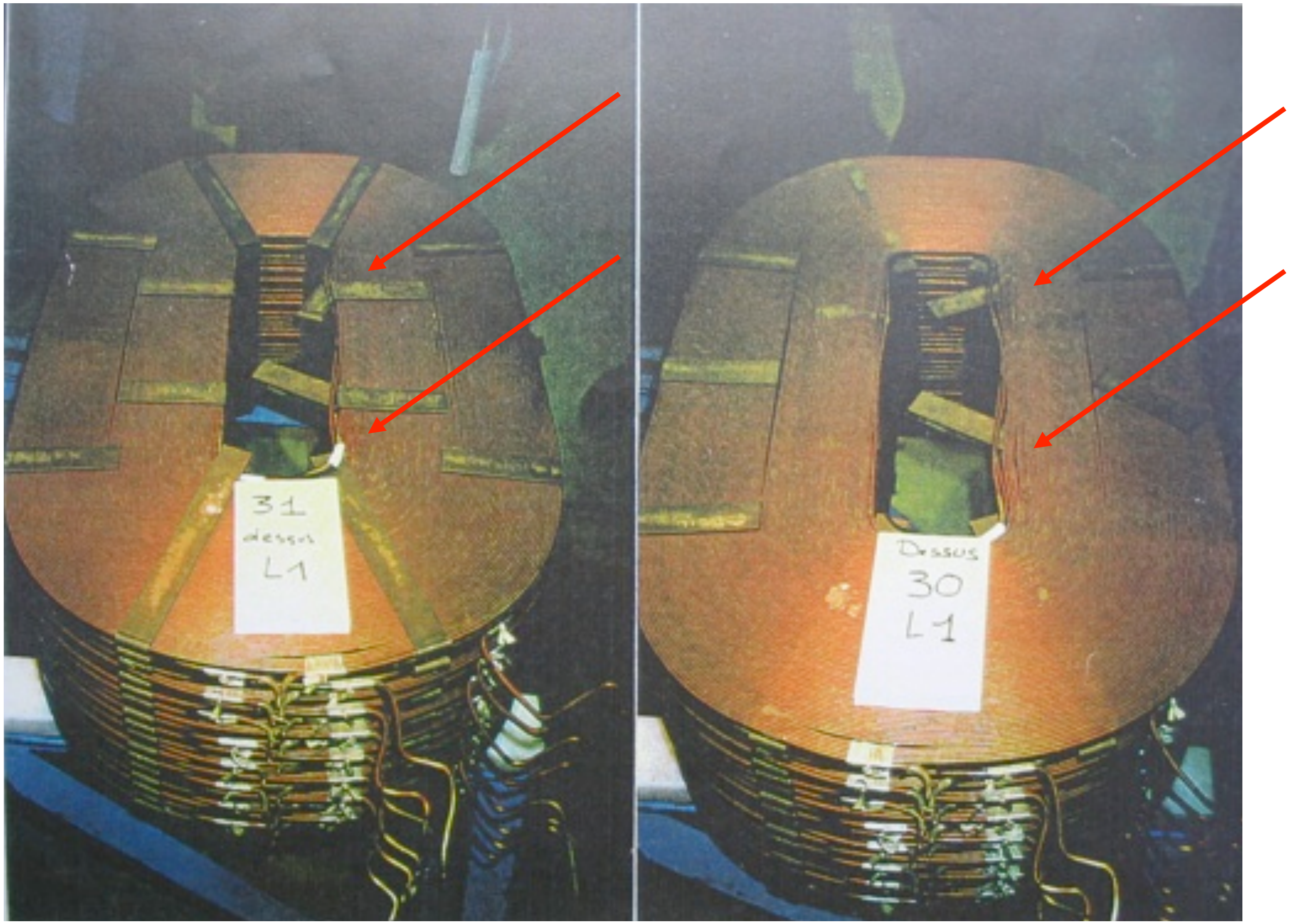
(nonlinear physical characteristics, moving plate, global quantities)

Search for optimization of temperature profile



Temperature distribution

Magneto-quasistatics

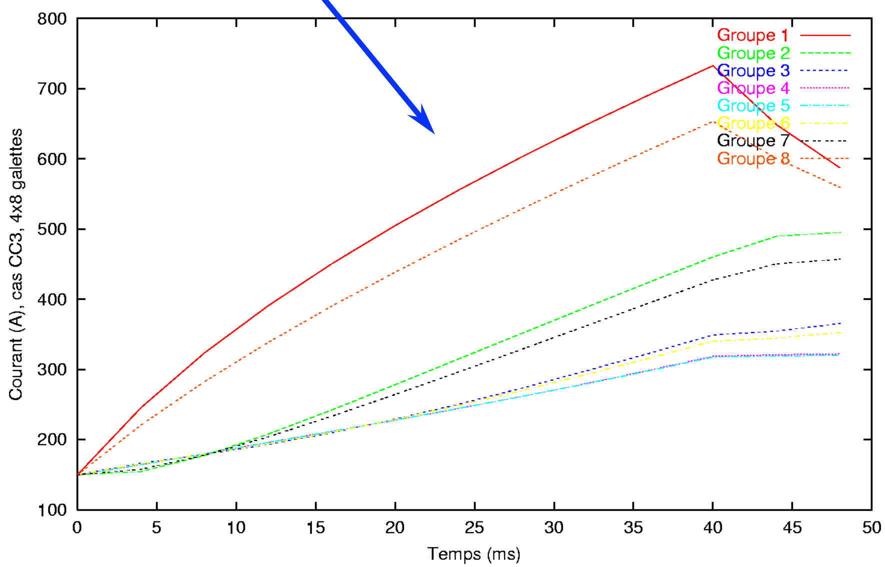


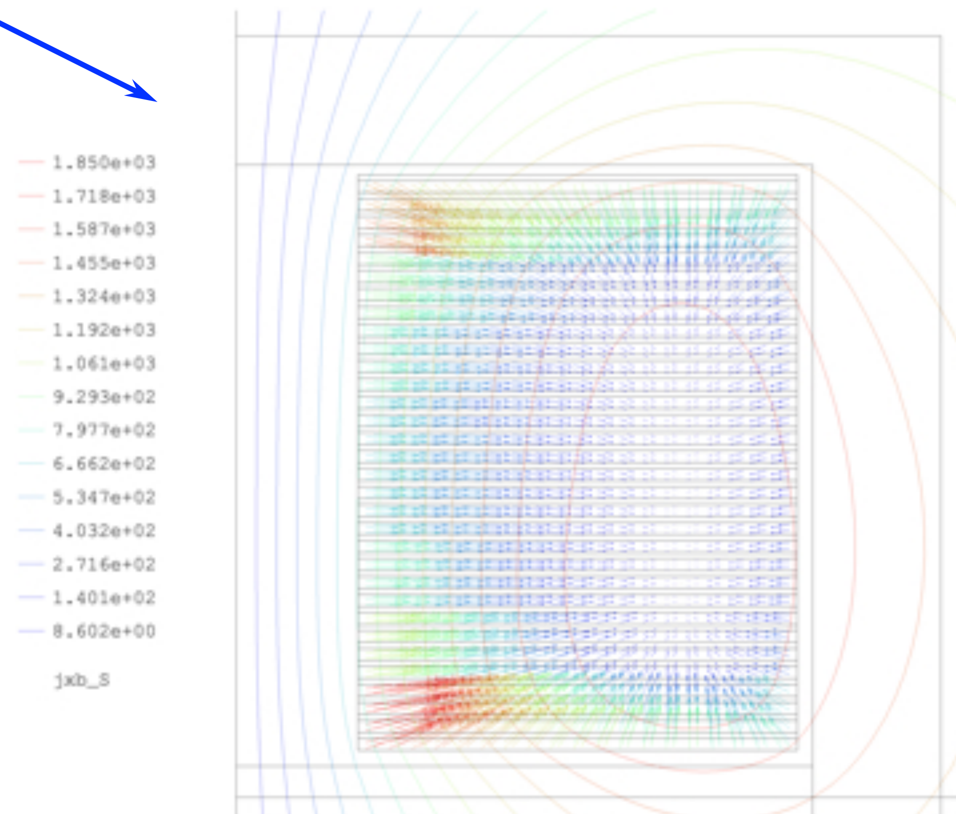
Forces

Magneto-quasistatics

Magnetic field lines and electromagnetic force (N/m)
(8 groups, total current 3200 A)

Currents in each of the
8 groups in parallel
non-uniformly distributed!





Full Wave

$$\mathbf{curl}\,\boldsymbol{h} = \boldsymbol{j} + \partial_t \boldsymbol{d}$$
$$\mathbf{curl}\,\boldsymbol{e} = -\partial_t \boldsymbol{b}$$
$$\boldsymbol{b} = \mu \boldsymbol{h}$$
$$\boldsymbol{d} = \varepsilon \boldsymbol{e}$$
$$\boldsymbol{j} = \sigma \boldsymbol{e}$$

+ Silver-Müller radiation condition at infinity (outgoing waves)

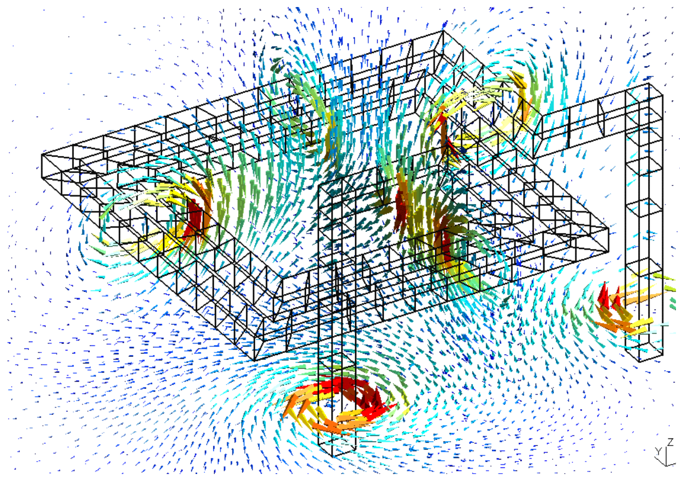
Example: electric or magnetic field formulations

$$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{e} + \sigma\mu\partial_t \boldsymbol{e} + \varepsilon\mu\partial_t^2 \boldsymbol{e} = 0$$
$$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{h} + \sigma\mu\partial_t \boldsymbol{h} + \varepsilon\mu\partial_t^2 \boldsymbol{h} = 0$$

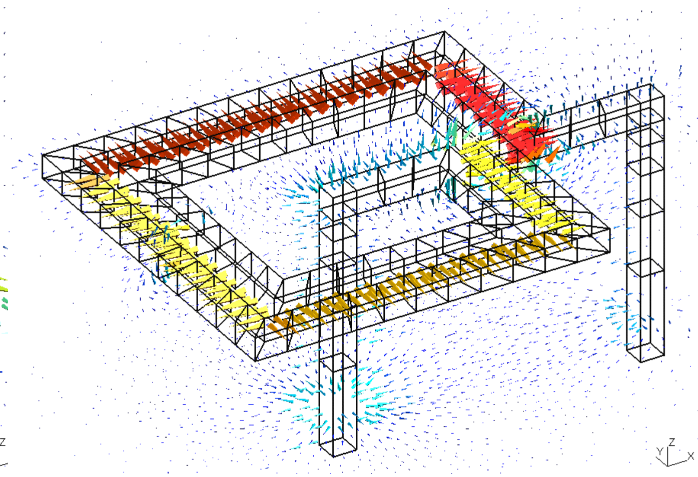
Full Wave

- Frequency and time domain analyses
- Uncoupled resolution

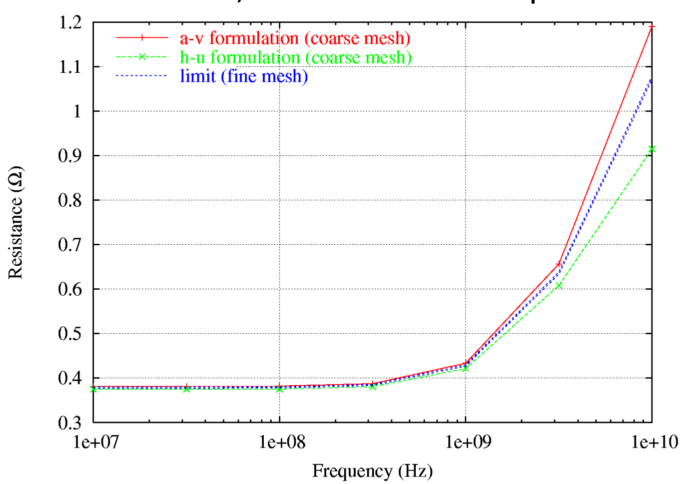
Magnetic flux density

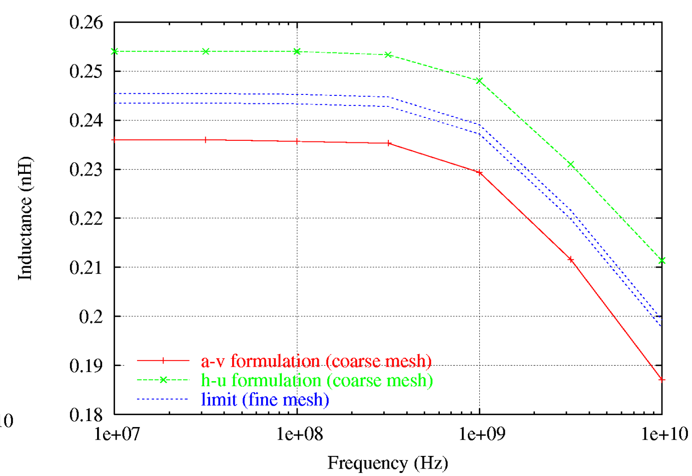


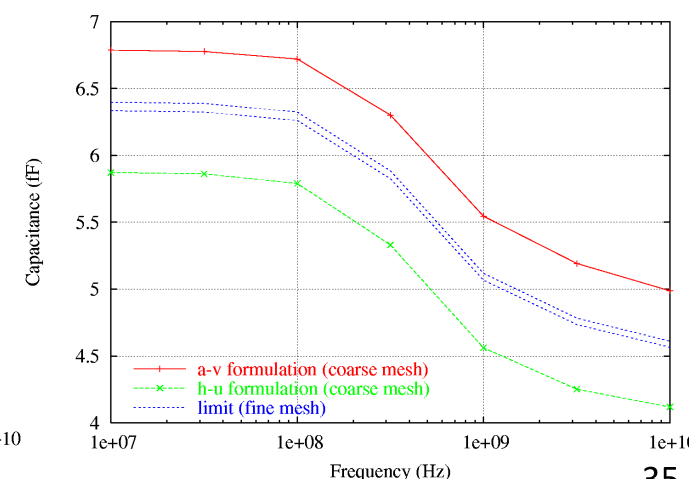
Electric field



Resistance, inductance and capacitance versus frequency

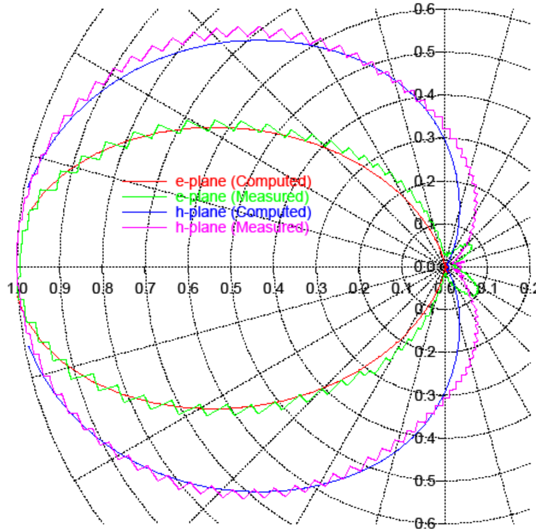


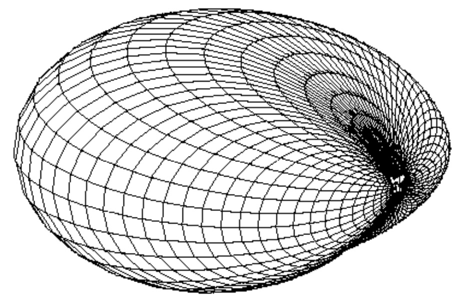


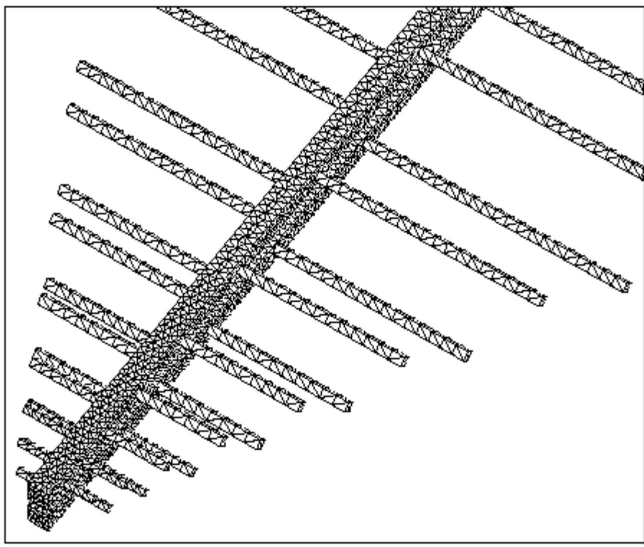


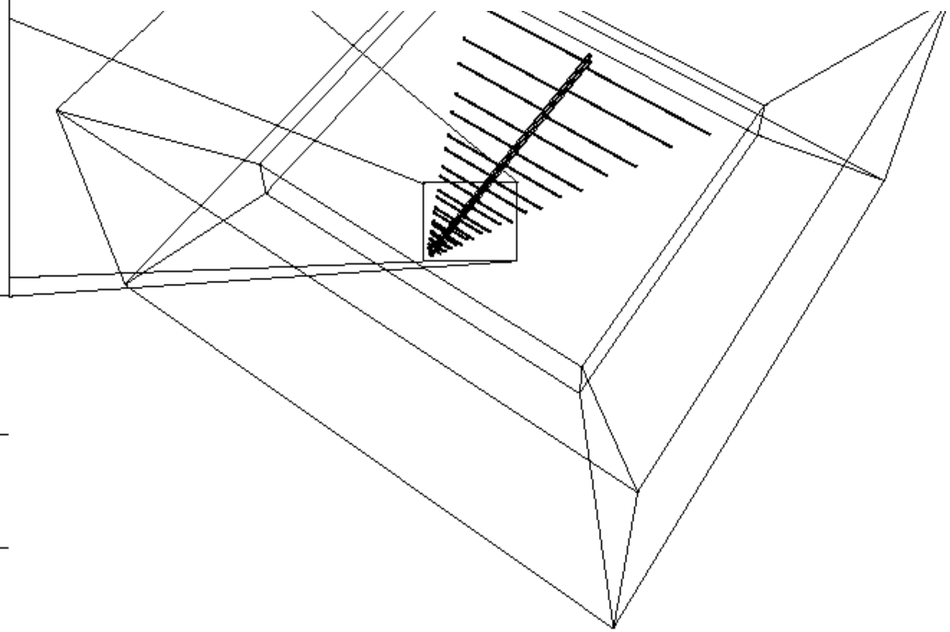
Full Wave

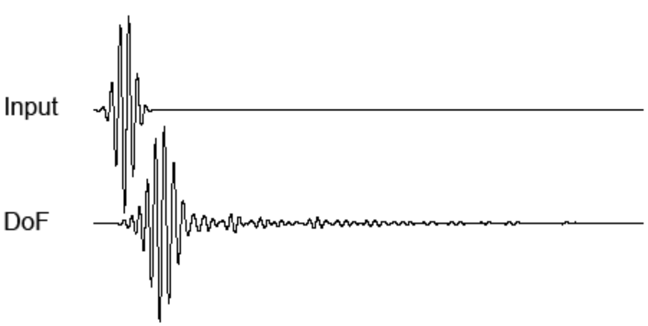
Log-periodic antenna





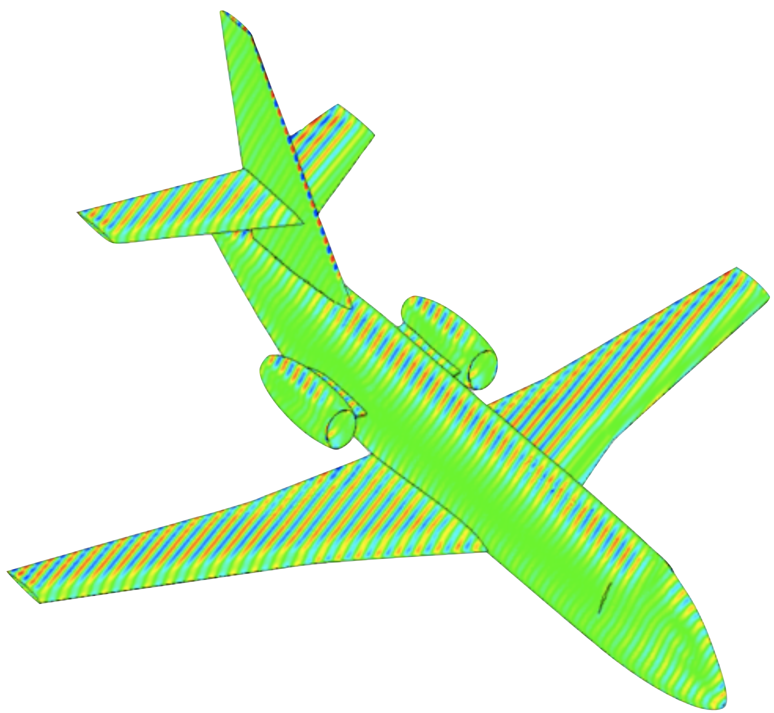


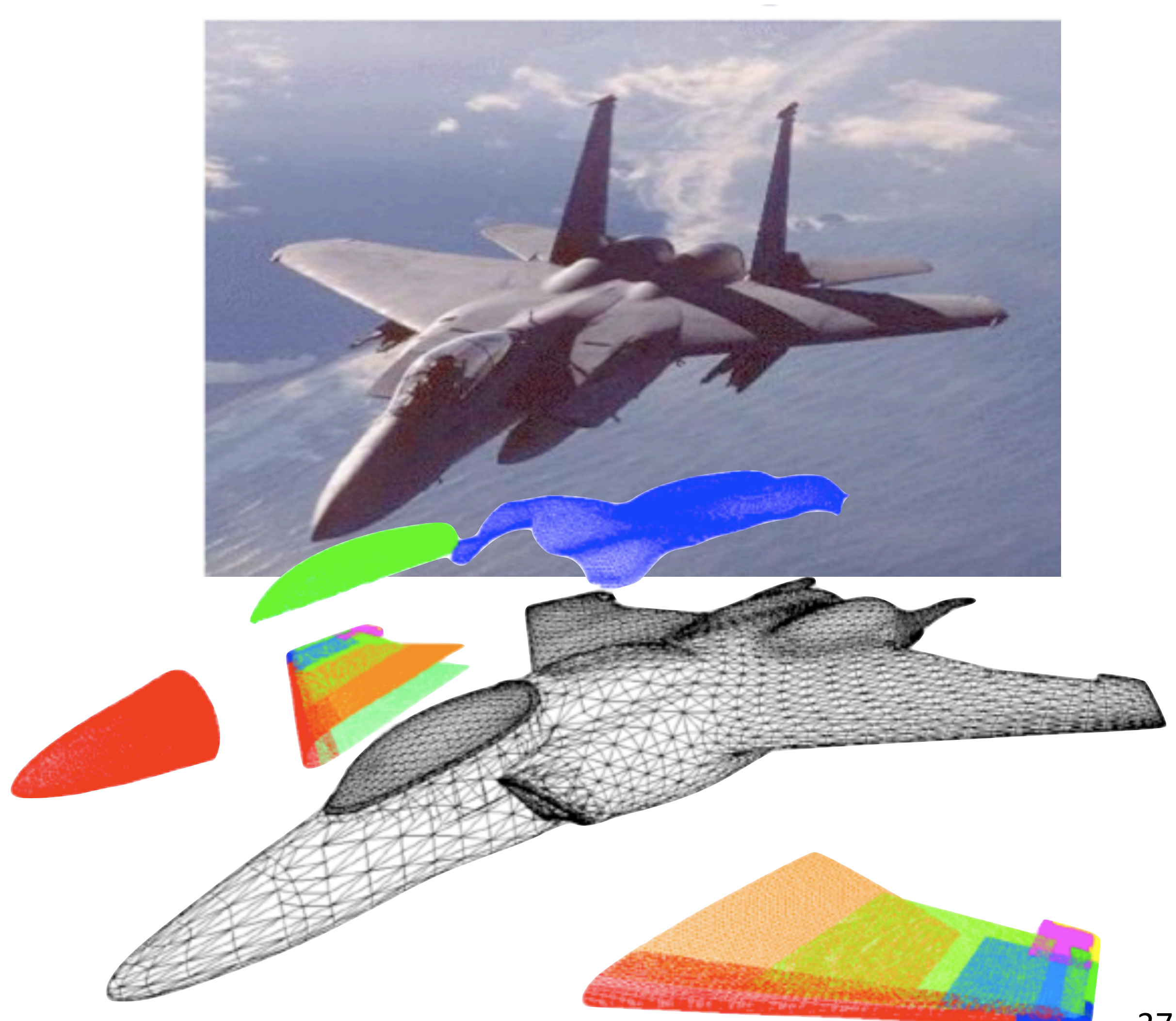




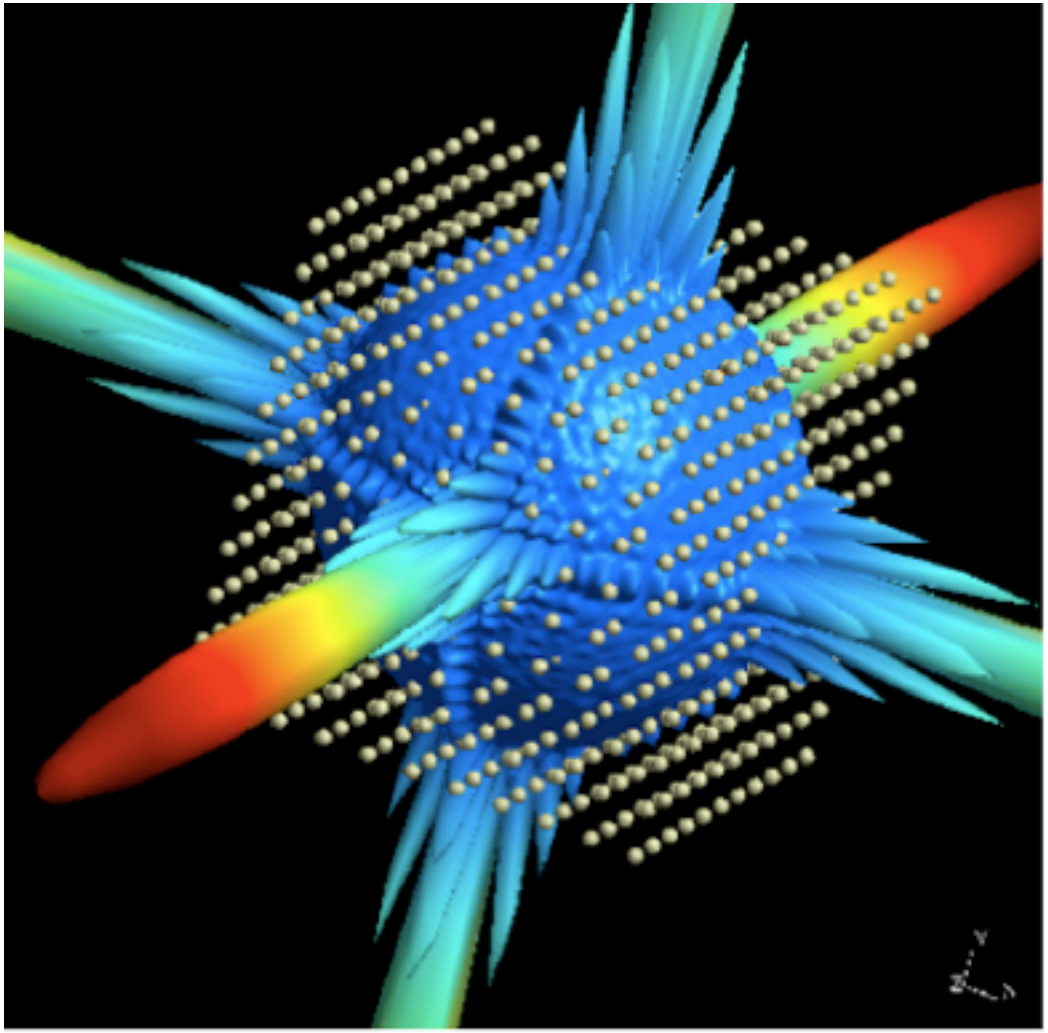
Full Wave

Radar

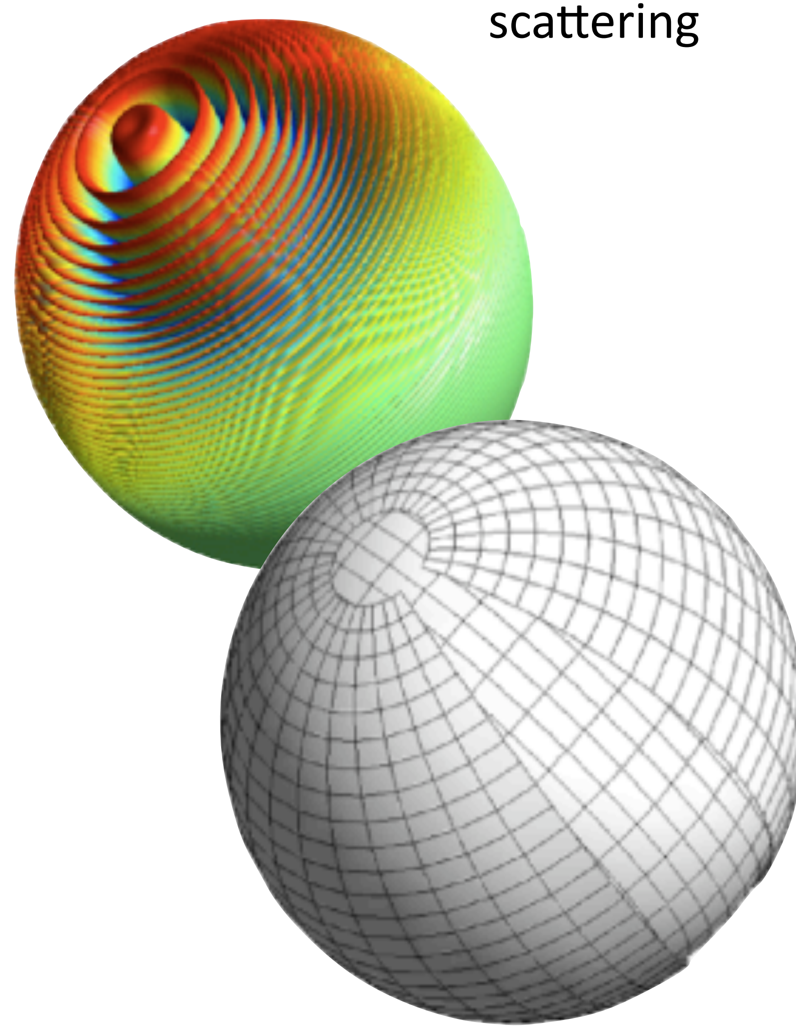




Full Wave

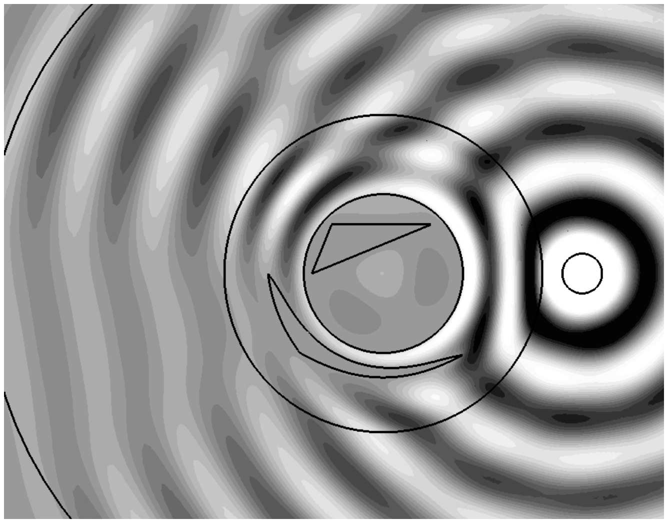


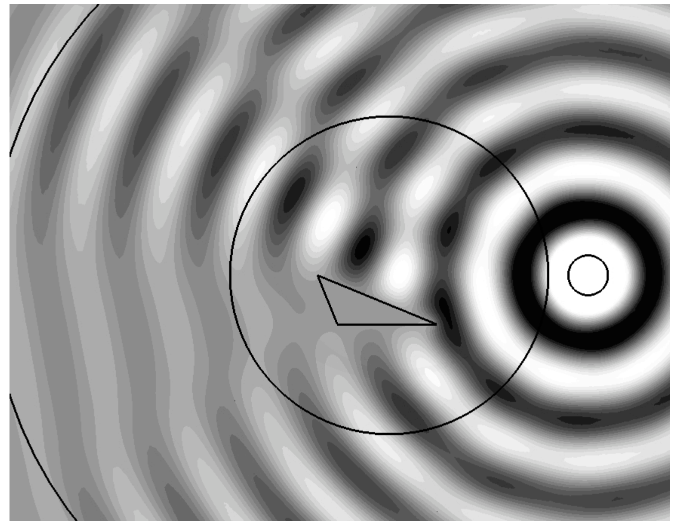
High-frequency scattering



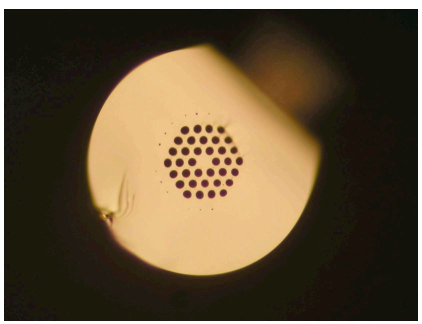
Full Wave

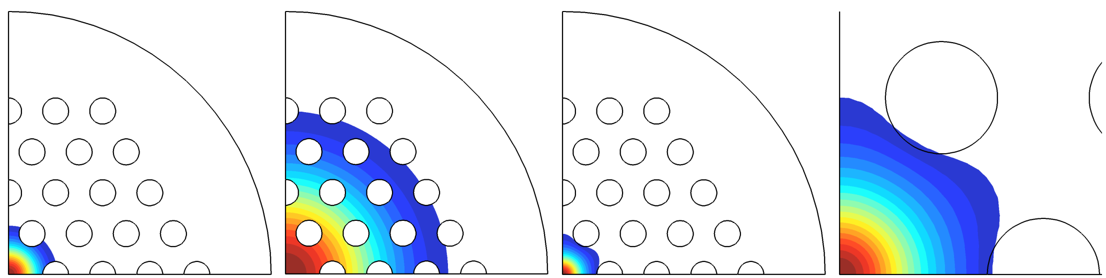
Generalized optical cloaking ("polyjuice")





Microstructured optical fibers: photonic crystal & non-linear (Kerr) effects





Full Wave

Optical Coherence Tomography (OCT) of human retina

Incident Beam