Part III

Introduction to the Finite Element Technique for Electromagnetic Modelling

Classical and weak formulations

Partial differential problem

Classical formulation

$$L\,u = f \qquad \text{in } \Omega$$
$$B\,u = g \qquad \text{on } \Gamma = \partial\Omega$$

$u \equiv$ classical solution

Notations

$$(u, v) = \int_\Omega u(\mathbf{x})\, v(\mathbf{x})\, d\mathbf{x} \quad , \quad u, v \in L^2(\Omega)$$

$$(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u}(\mathbf{x}) \,.\, \mathbf{v}(\mathbf{x})\, d\mathbf{x} \quad , \quad \mathbf{u}, \mathbf{v} \in \mathbf{L}^2(\Omega)$$

Weak formulation

$$(u, L^* v) - (f, v) + \int_\Gamma Q_g(v)\, ds = 0 \quad , \quad \forall\, v \in V(\Omega)$$

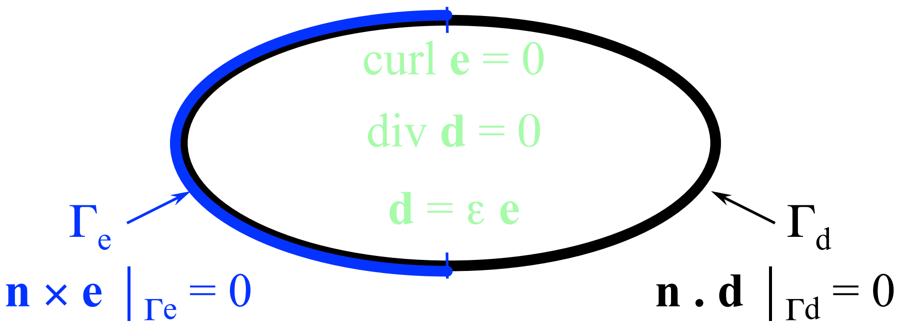
$u \equiv$ weak solution

$v \equiv$ **test function**

Continuous level : $\infty \times \infty$ system
Discrete level : $n \times n$ system
$\Rightarrow$ **numerical solution**

Classical and weak formulations

Application to the electrostatic problem



Electrostatic
classical formulation

$(\mathbf{d} , \mathrm{grad}\, v') = 0 \quad , \quad \forall v' \in V(\Omega)$ with $V(\Omega) = \{ v \in H^0(\Omega) \, ; \, v|_{\Gamma_e} = 0 \}$

$\Rightarrow (\mathrm{div}\, \mathbf{d} , v') + \langle \mathbf{n} \cdot \mathbf{d} , v' \rangle_{\Gamma} = 0 \quad , \quad \forall v' \in V(\Omega)$

$\Downarrow \qquad\qquad \Downarrow$

$\mathrm{div}\, \mathbf{d} = 0 \qquad \mathbf{n} \cdot \mathbf{d} |_{\Gamma d} = 0$

Weak formulation
of div **d** = 0
(+ boundary condition)

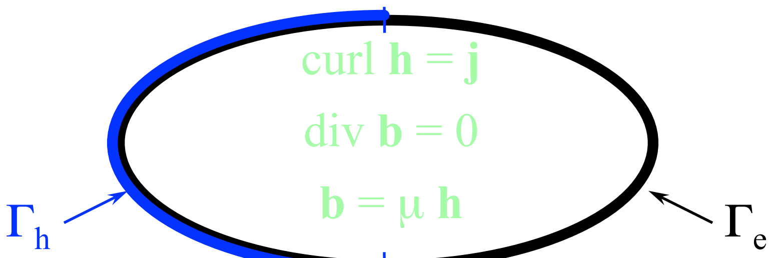
$\mathbf{d} = \varepsilon \, \mathbf{e} \quad \& \quad \mathbf{e} = - \mathrm{grad}\, v \quad \Leftrightarrow \quad \mathrm{curl}\, \mathbf{e} = 0$

$(- \varepsilon \, \mathrm{grad}\, v , \mathrm{grad}\, v') = 0 \quad , \quad \forall v' \in V(\Omega)$

Electrostatic
weak formulation with v

Classical and weak formulations

Application to the magnetostatic problem



$\mathbf{n} \times \mathbf{h} \,\big|_{\Gamma h} = 0$ $\qquad$ $\mathbf{n} \,.\, \mathbf{b} \,\big|_{\Gamma e} = 0$

Magnetostatic
classical formulation

$(\mathbf{b} , \text{grad}\, \phi') = 0 \quad , \quad \forall\, \phi' \in \Phi(\Omega) \qquad \text{with} \;\; \Phi(\Omega) = \{ \phi \in H^0(\Omega) \,;\, \phi \,\big|_{\Gamma h} = 0 \}$

$$\Rightarrow (\text{div}\, \mathbf{b} , \phi') + < \mathbf{n} \,.\, \mathbf{b} , \phi' >_{\Gamma} = 0 \quad , \quad \forall\, \phi' \in \Phi(\Omega)$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\text{div}\, \mathbf{b} = 0 \qquad \mathbf{n} \,.\, \mathbf{b} \,\big|_{\Gamma e} = 0$$

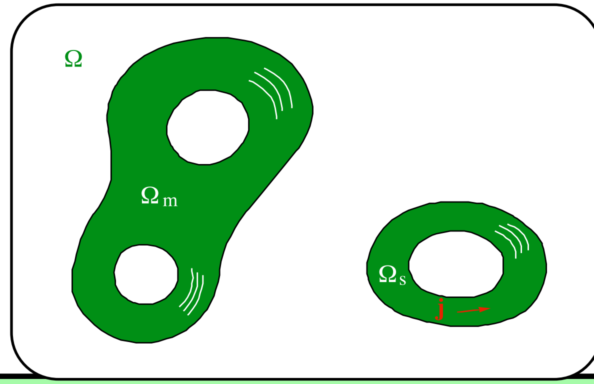
Weak formulation
of div $\mathbf{b} = 0$
(+ boundary condition)

$\mathbf{b} = \mu\, \mathbf{h} \qquad \& \qquad \mathbf{h} = \mathbf{h}_s - \text{grad}\, \phi \quad (\text{with curl}\; \mathbf{h}_s = \mathbf{j}) \quad \Leftrightarrow \quad \text{curl}\; \mathbf{h} = \mathbf{j}$

$$\boxed{(\mu\, (\mathbf{h}_s - \text{grad}\, \phi) , \text{grad}\, \phi') = 0 \quad , \quad \forall\, \phi' \in \Phi(\Omega)}$$

Magnetostatic
weak formulation with ϕ

Magnetostatic formulations



Maxwell equations
(magnetic - static)

curl $\mathbf{h} = \mathbf{j}$
div $\mathbf{b} = 0$

$\mathbf{b} = \mu\, \mathbf{h}$

ϕ Formulation

a Formulation

*"**h**" side* ⟵ | ⟶ *"**b**" side*

Magnetostatic formulations

Basis equations

$\mathrm{curl}\ \mathbf{h} = \mathbf{j}$ (h)

$\mathbf{b} = \mu\ \mathbf{h}$ (m)

$\mathrm{div}\ \mathbf{b} = 0$ (b)

ϕ Formulation

Magnetic scalar potential ϕ

$\mathbf{h} = \mathbf{h}_s - \mathrm{grad}\ \phi$ $\Rightarrow$ (h) OK

$\mathbf{h}_s$ given such as $\mathrm{curl}\ \mathbf{h}_s = \mathbf{j}$ (non-unique)

$\mathrm{div}\ (\ \mu\ (\ \mathbf{h}_s - \mathrm{grad}\ \phi\ )\ ) = 0$ $\Leftarrow$ (b) & (m)

Multivalued potential

Cuts

a Formulation

Magnetic vector potential a

(b) OK $\Leftarrow$ $\mathbf{b} = \mathrm{curl}\ \mathbf{a}$

(h) & (m) $\rightarrow$ $\mathrm{curl}\ (\ \mu^{-1}\ \mathrm{curl}\ \mathbf{a}\ ) = \mathbf{j}$

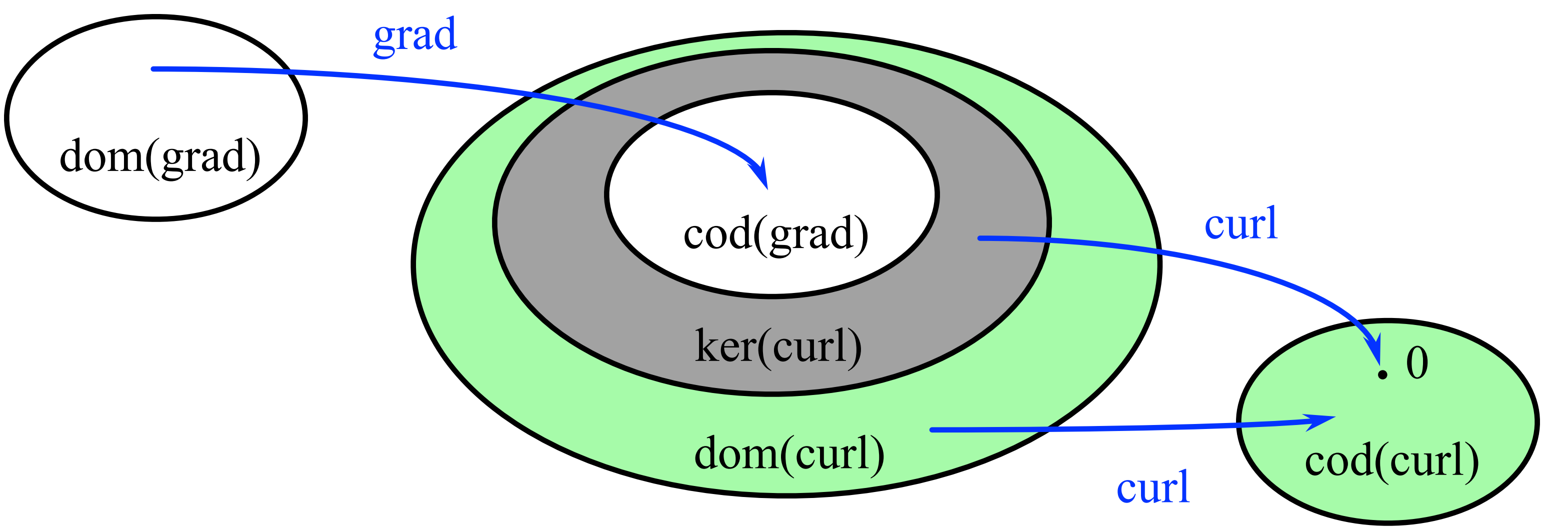
Non-unique potential

Gauge condition

Multivalued scalar potential

Kernel of the curl (in a domain Ω)

ker (curl) = { $\mathbf{v}$: curl $\mathbf{v}$ = 0 }



$$\boxed{\text{cod (grad)} \subset \text{ker (curl)}}$$

Multivalued scalar potential - Cut

curl $\mathbf{h} = 0$ in Ω

OK $\Leftarrow$

$\Rightarrow$?

Scalar potential ϕ

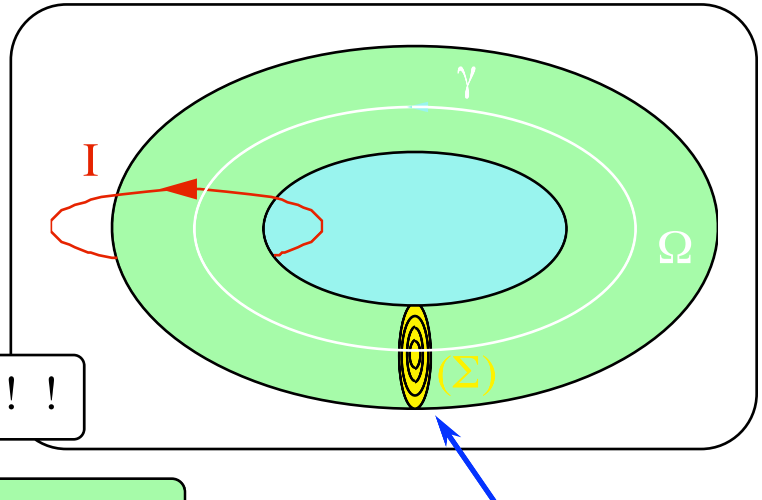
$\mathbf{h} = -\operatorname{grad} \phi$ in Ω

Circulation of h along path γ_{AB} in Ω

$$\int_{\gamma_{AB}} \mathbf{h} \cdot d\mathbf{l} = \int_{\gamma_{AB}} -\operatorname{grad} \phi \cdot d\mathbf{l} = \phi_A - \phi_B$$

Closed path γ_{AB} ($A \equiv B$)
surrounding a conductor (with current I)

$\Rightarrow \quad \phi_A - \phi_B = 0 \neq I \quad ! \; ! \; !$



ϕ must be discontinuous ... through a **cut**

$\Delta \phi = I$

Vector potential - gauge condition

div $\mathbf{b} = 0$ in Ω

OK $\Leftarrow$

$\Rightarrow$?

Vector potential a

$\mathbf{b} = \text{curl } \mathbf{a}$ in Ω

Non-uniqueness of vector potential **a**

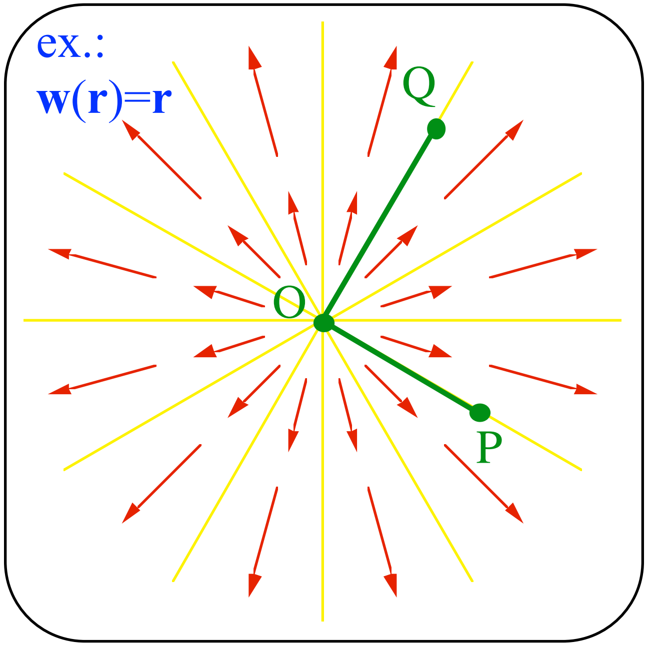
$\mathbf{b} = \text{curl } \mathbf{a} = \text{curl } (\mathbf{a} + \text{grad } \eta)$

Gauge condition

Coulomb gauge div $\mathbf{a} = 0$

Gauge $\mathbf{a} \cdot \omega = 0$

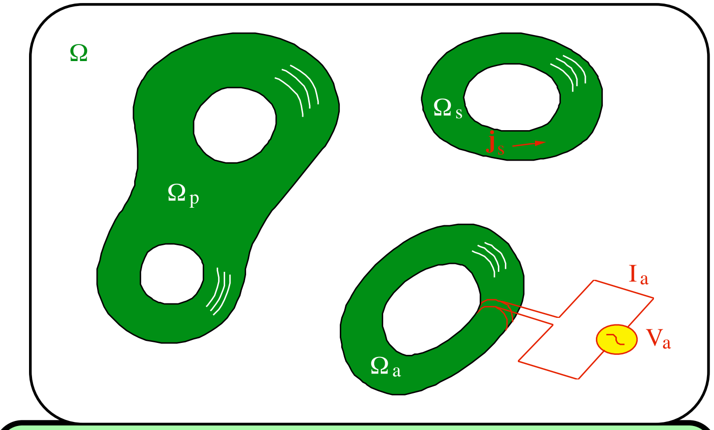
ω vector field with non-closed lines linking any 2 points in Ω



Magnetodynamic formulations

h-ϕ Formulation

a^* Formulation



Maxwell equations
(quasi-stationary)

$$\mathrm{curl}\,\mathbf{h} = \mathbf{j}$$
$$\mathrm{curl}\,\mathbf{e} = -\,\partial_t\,\mathbf{b}$$
$$\mathrm{div}\,\mathbf{b} = 0$$

$$\mathbf{b} = \mu\,\mathbf{h}$$
$$\mathbf{j} = \sigma\,\mathbf{e}$$

t-ω Formulation

a-v Formulation

*"**h**" side* ⟵ | ⟶ *"**b**" side*

Magnetodynamic formulations

Basis equations

$\mathrm{curl}\ \mathbf{h} = \mathbf{j}$ (h)

$\mathbf{b} = \mu\ \mathbf{h}$

$\mathbf{j} = \sigma\ \mathbf{e}$

$\mathrm{curl}\ \mathbf{e} = -\ \partial_t\ \mathbf{b}$

$\mathrm{div}\ \mathbf{b} = 0$ (b)

h-ϕ Formulation

Magnetic field h

Magnetic scalar potential ϕ

$\mathbf{h}$ ds Ω_c

$\mathbf{h} = \mathbf{h}_s - \mathrm{grad}\ \phi$ ds Ω_c^C

$\Rightarrow$ (h) OK

$\mathrm{curl}\ \mathbf{h}_s = \mathbf{j}_s$

$\mathrm{curl}\ (\sigma^{-1}\ \mathrm{curl}\ \mathbf{h}) + \partial_t\ (\mu\ \mathbf{h}) = 0$

$\mathrm{div}\ (\mu\ (\mathbf{h}_s - \mathrm{grad}\ \phi)) = 0$

$\leftarrow$ (b) $\rightarrow$

$\leftarrow$ in Ω_c $\rightarrow$

$\leftarrow$ in Ω_c^C $\rightarrow$

t-ω Formulation

Electric vector potential t

Magnetic scalar potential ω

$\mathbf{j} = \mathrm{curl}\ \mathbf{t}$

(h) OK $\Leftarrow$

$\mathbf{h} = \mathbf{t} - \mathrm{grad}\ \omega$

$\mathrm{curl}\ (\sigma^{-1}\ \mathrm{curl}\ \mathbf{t}) + \partial_t\ (\mu\ (\mathbf{t} - \mathrm{grad}\ \omega)) = 0$

$\mathrm{div}\ (\mu\ (\mathbf{t} - \mathrm{grad}\ \omega)) = 0$

+ Gauge

Magnetodynamic formulations

Basis equations

$\mathrm{curl}\,\mathbf{h} = \mathbf{j}$ (h)

$\mathbf{b} = \mu\,\mathbf{h}$

$\mathbf{j} = \sigma\,\mathbf{e}$

$\mathrm{curl}\,\mathbf{e} = -\partial_t\,\mathbf{b}$

$\mathrm{div}\,\mathbf{b} = 0$ (b)

a^* Formulation

Magnetic vector potential a^*

$\mathbf{b} = \mathrm{curl}\,\mathbf{a}^*$

$\mathbf{e} = -\partial_t\,\mathbf{a}^*$

$\Rightarrow$ (b) OK

$\mathrm{curl}\,(\mu^{-1}\,\mathrm{curl}\,\mathbf{a}^*) + \sigma\,\partial_t\,\mathbf{a}^* = \mathbf{j}_s$

+ Gauge in Ω_c^C

$\leftarrow$ (h) $\rightarrow$

a-v Formulation

Magnetic vector potential a
Electric scalar potential v

$\mathbf{b} = \mathrm{curl}\,\mathbf{a}$

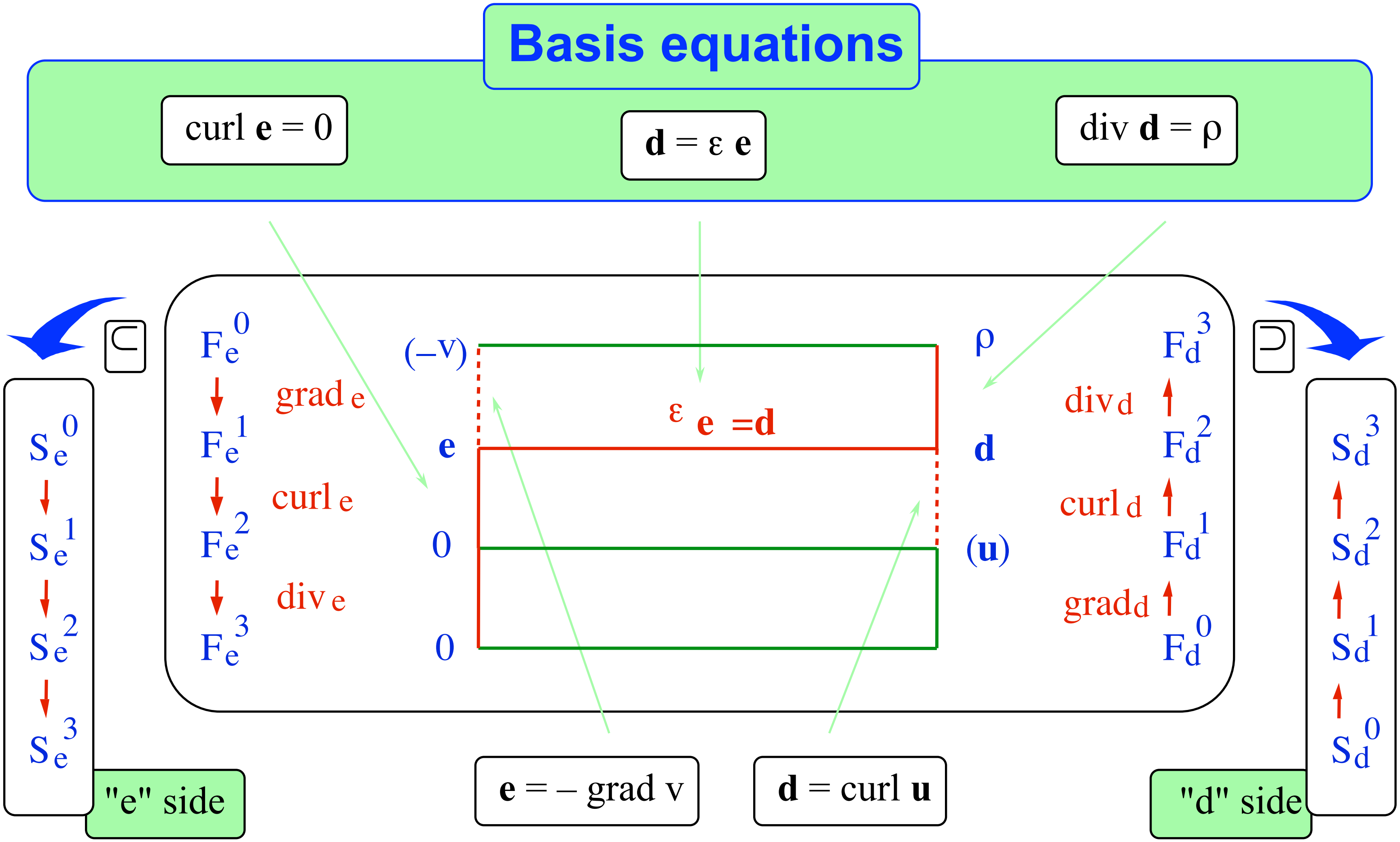
$\mathbf{e} = -\partial_t\,\mathbf{a} - \mathrm{grad}\,v$

(b) OK $\Leftarrow$

$\mathrm{curl}\,(\mu^{-1}\,\mathrm{curl}\,\mathbf{a}) + \sigma\,(\partial_t\,\mathbf{a} + \mathrm{grad}\,v)) = \mathbf{j}_s$

+ Gauge in Ω

Electrostatic problem



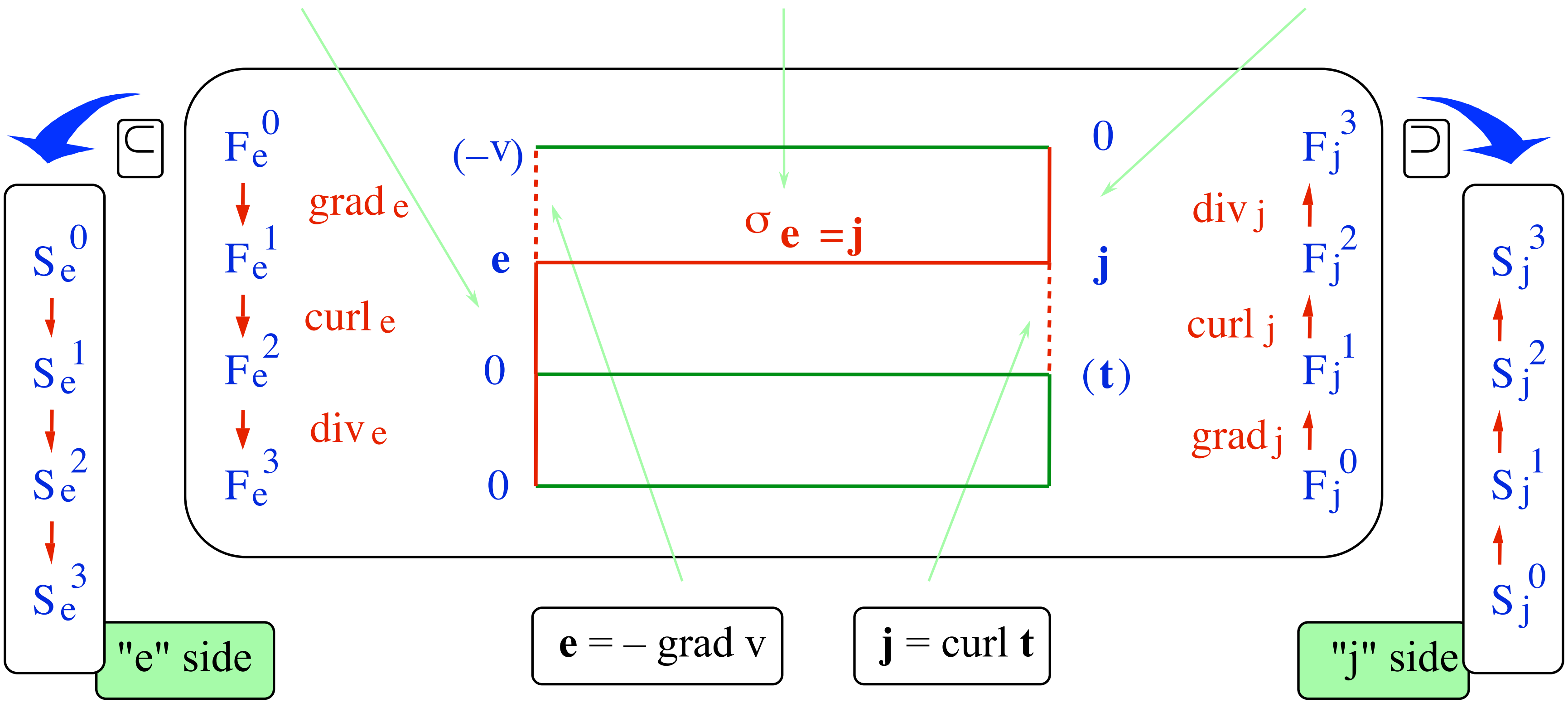
Electrokinetic problem

Basis equations

$\text{curl}\ \mathbf{e} = 0$

$\mathbf{j} = \sigma\ \mathbf{e}$

$\text{div}\ \mathbf{j} = 0$



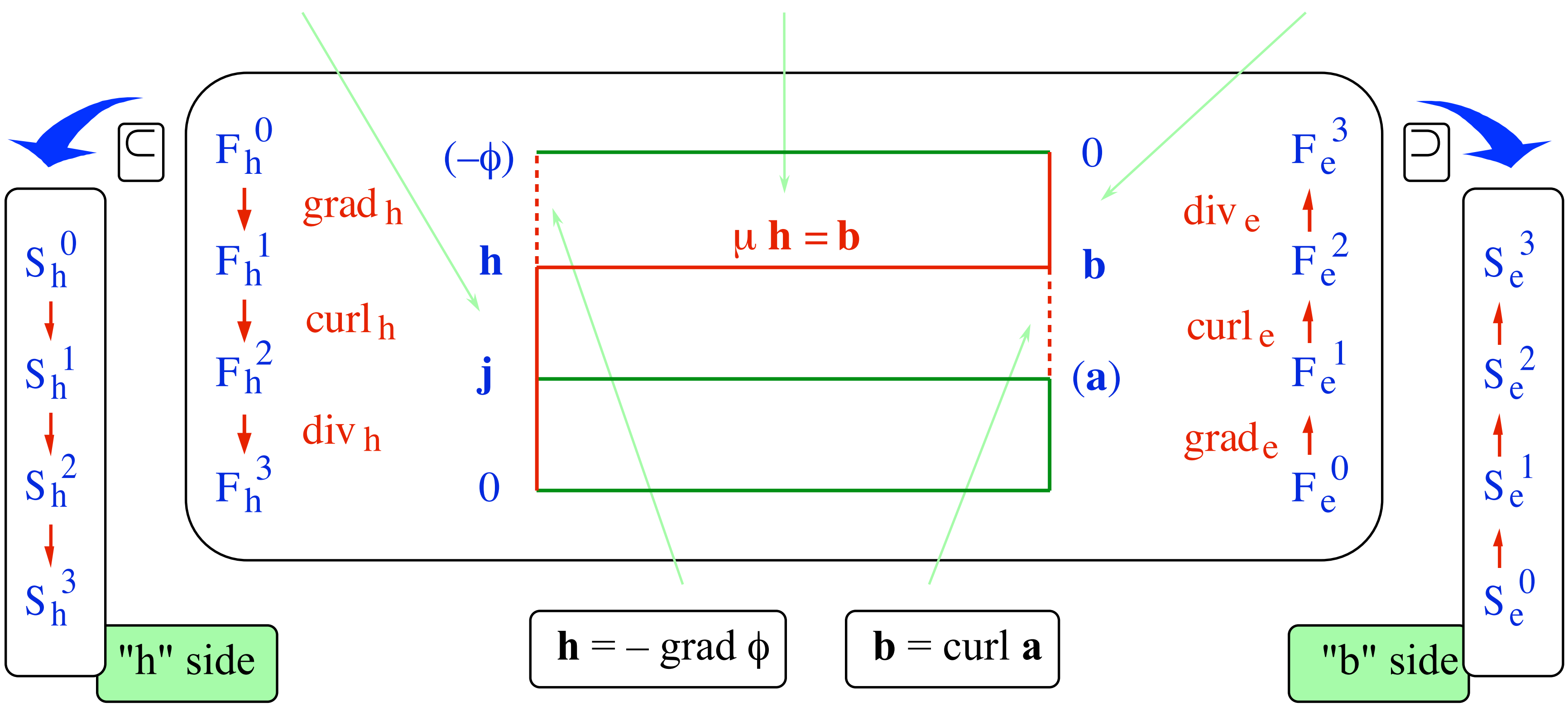
Magnetostatic problem

Basis equations

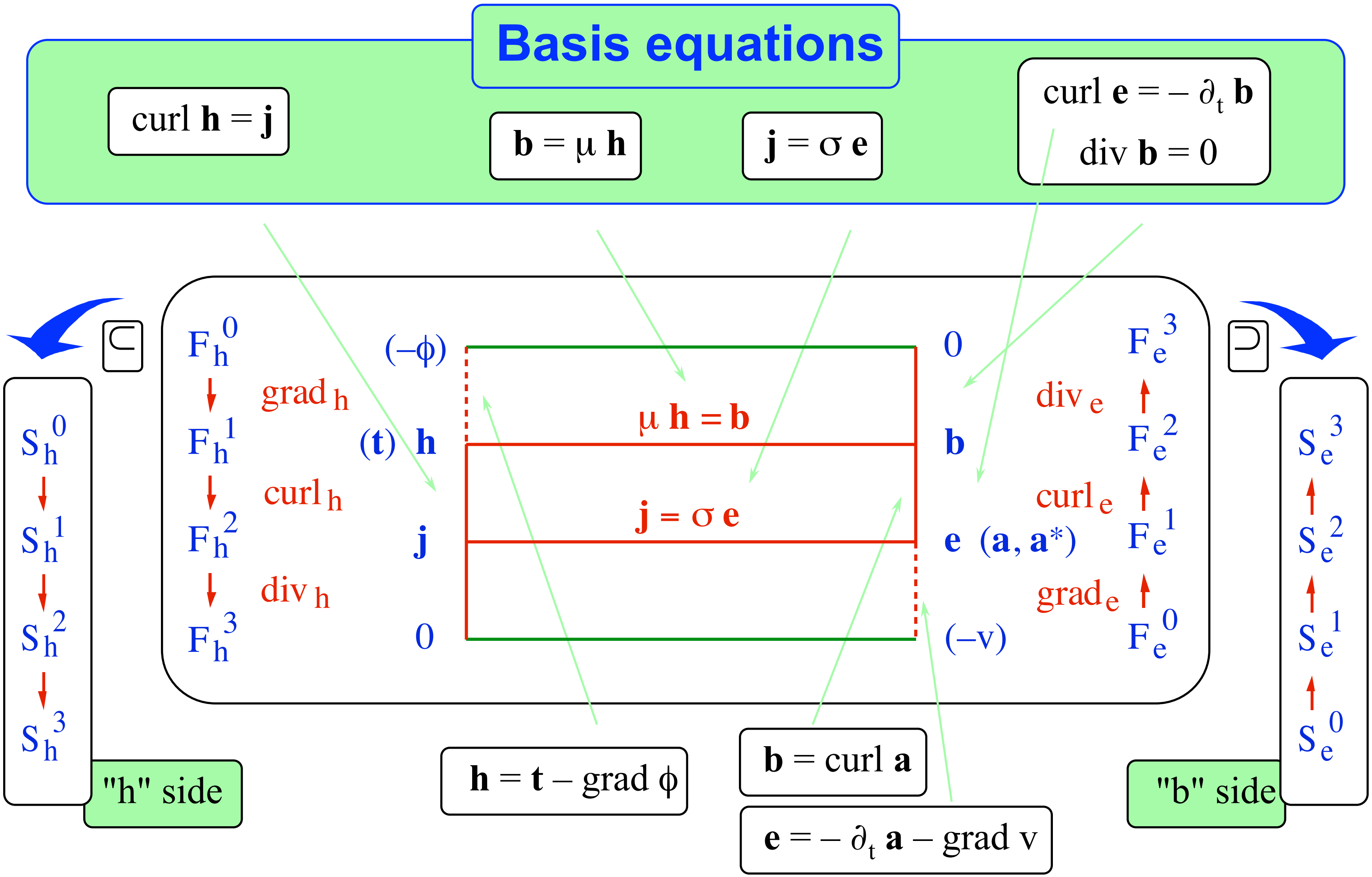
$\mathrm{curl}\,\mathbf{h} = \mathbf{j}$

$\mathbf{b} = \mu\,\mathbf{h}$

$\mathrm{div}\,\mathbf{b} = 0$



Magnetodynamic problem



Continuous mathematical structure

Domain Ω, Boundary $\partial\Omega = \Gamma_h \cup \Gamma_e$

Basis structure

Function spaces $F_h^0 \subset L^2$, $F_h^1 \subset \mathbf{L}^2$, $F_h^2 \subset \mathbf{L}^2$, $F_h^3 \subset L^2$

$$\mathrm{dom}\,(\mathrm{grad}_h) = F_h^0 = \{ \phi \in L^2(\Omega) \,;\, \mathrm{grad}\,\phi \in \mathbf{L}^2(\Omega) \,,\; \phi\big|_{\Gamma h} = 0 \}$$
$$\mathrm{dom}\,(\mathrm{curl}_h) = F_h^1 = \{ \mathbf{h} \in \mathbf{L}^2(\Omega) \,;\, \mathrm{curl}\,\mathbf{h} \in \mathbf{L}^2(\Omega) \,,\; \mathbf{n} \wedge \mathbf{h}\big|_{\Gamma h} = 0 \}$$
$$\mathrm{dom}\,(\mathrm{div}_h) = F_h^2 = \{ \mathbf{j} \in \mathbf{L}^2(\Omega) \,;\, \mathrm{div}\,\mathbf{j} \in L^2(\Omega) \,,\; \mathbf{n} \cdot \mathbf{j}\big|_{\Gamma h} = 0 \}$$

$\mathrm{grad}_h\, F_h^0 \subset F_h^1$, $\mathrm{curl}_h\, F_h^1 \subset F_h^2$, $\mathrm{div}_h\, F_h^2 \subset F_h^3$

Boundary conditions on Γ_h

Sequence

$$F_h^0 \xrightarrow{\mathrm{grad}\ h} F_h^1 \xrightarrow{\mathrm{curl}\ h} F_h^2 \xrightarrow{\mathrm{div}\ h} F_h^3$$

Basis structure

Function spaces $F_e^0 \subset L^2$, $F_e^1 \subset \mathbf{L}^2$, $F_e^2 \subset \mathbf{L}^2$, $F_e^3 \subset L^2$

$$\mathrm{dom}\,(\mathrm{grad}_e) = F_e^0 = \{ v \in L^2(\Omega) \,;\, \mathrm{grad}\,v \in \mathbf{L}^2(\Omega) \,,\; v\big|_{\Gamma e} = 0 \}$$
$$\mathrm{dom}\,(\mathrm{curl}_e) = F_e^1 = \{ \mathbf{a} \in \mathbf{L}^2(\Omega) \,;\, \mathrm{curl}\,\mathbf{a} \in \mathbf{L}^2(\Omega) \,,\; \mathbf{n} \wedge \mathbf{a}\big|_{\Gamma e} = 0 \}$$
$$\mathrm{dom}\,(\mathrm{div}_e) = F_e^2 = \{ \mathbf{b} \in \mathbf{L}^2(\Omega) \,;\, \mathrm{div}\,\mathbf{b} \in L^2(\Omega) \,,\; \mathbf{n} \cdot \mathbf{b}\big|_{\Gamma e} = 0 \}$$

$\mathrm{grad}_h\, F_e^0 \subset F_e^1$, $\mathrm{curl}_e\, F_e^1 \subset F_e^2$, $\mathrm{div}_e\, F_e^2 \subset F_e^3$

Boundary conditions on Γ_e

Sequence

$$F_e^3 \xleftarrow{\mathrm{div}\ e} F_e^2 \xleftarrow{\mathrm{curl}\ e} F_e^1 \xleftarrow{\mathrm{grad}\ e} F_e^0$$

Discrete mathematical structure

Continuous problem

Continuous function spaces & domain
Classical and weak formulations

Discretization ⬇ **Approximation**

Discrete problem

Discrete function spaces piecewise defined
in a discrete domain (mesh)

Finite element method

Questions

Classical & weak formulations → ?
Properties of the fields → ?

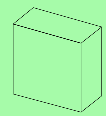
Objective

To build a discrete structure
as similar as possible
as the continuous structure

Discrete mathematical structure

Finite element

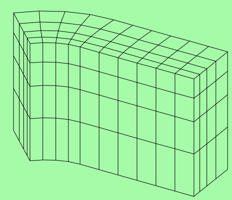
Interpolation in a geometric element of simple shape



$+ \, f$

Finite element space

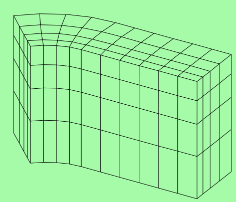
Function space
& Mesh



$+ \bigcup_i f_i$

Sequence of finite element spaces

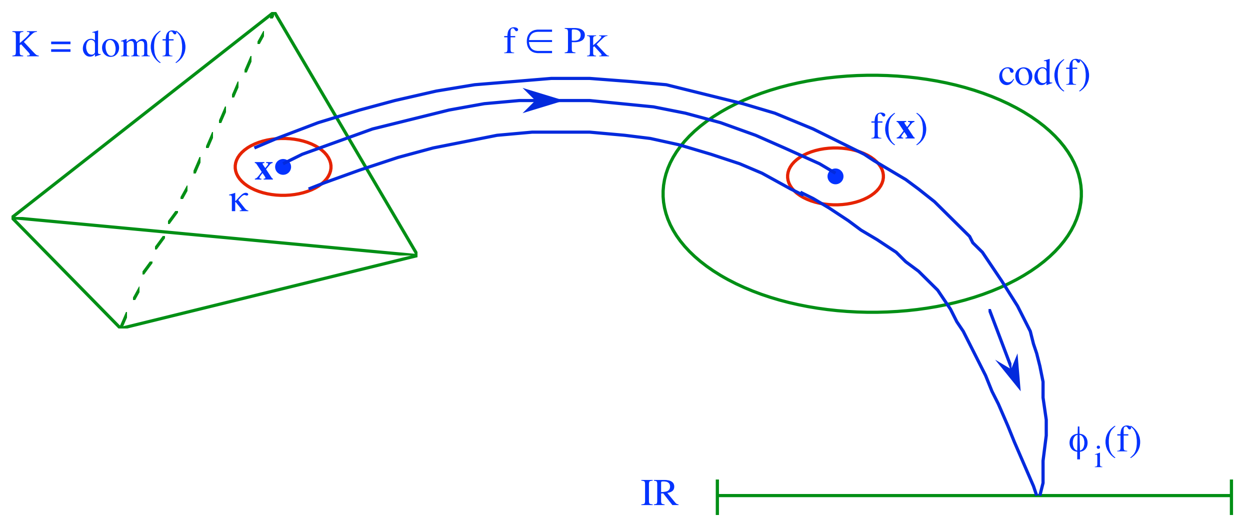
Sequence of function spaces
& Mesh



$+ \left\{ \bigcup_i f_i \right\}$

Finite elements

- **Finite element (K, P_K, Σ_K)**
  - K = domain of space (tetrahedron, hexahedron, prism)
  - P_K = function space of finite dimension n_K, defined in K
  - Σ_K = set of n_K degrees of freedom
    represented by n_K linear functionals ϕ_i, $1 \le i \le n_K$,
    defined in P_K and whose values belong to IR



Finite elements

- **Unisolvance**

  $\forall$ **u** $\in$ **P_K , u is uniquely defined by the degrees of freedom**

- **Interpolation**

$$u_K = \sum_{i=1}^{n_K} \phi_i(u)\, p_i$$

Degrees of freedom

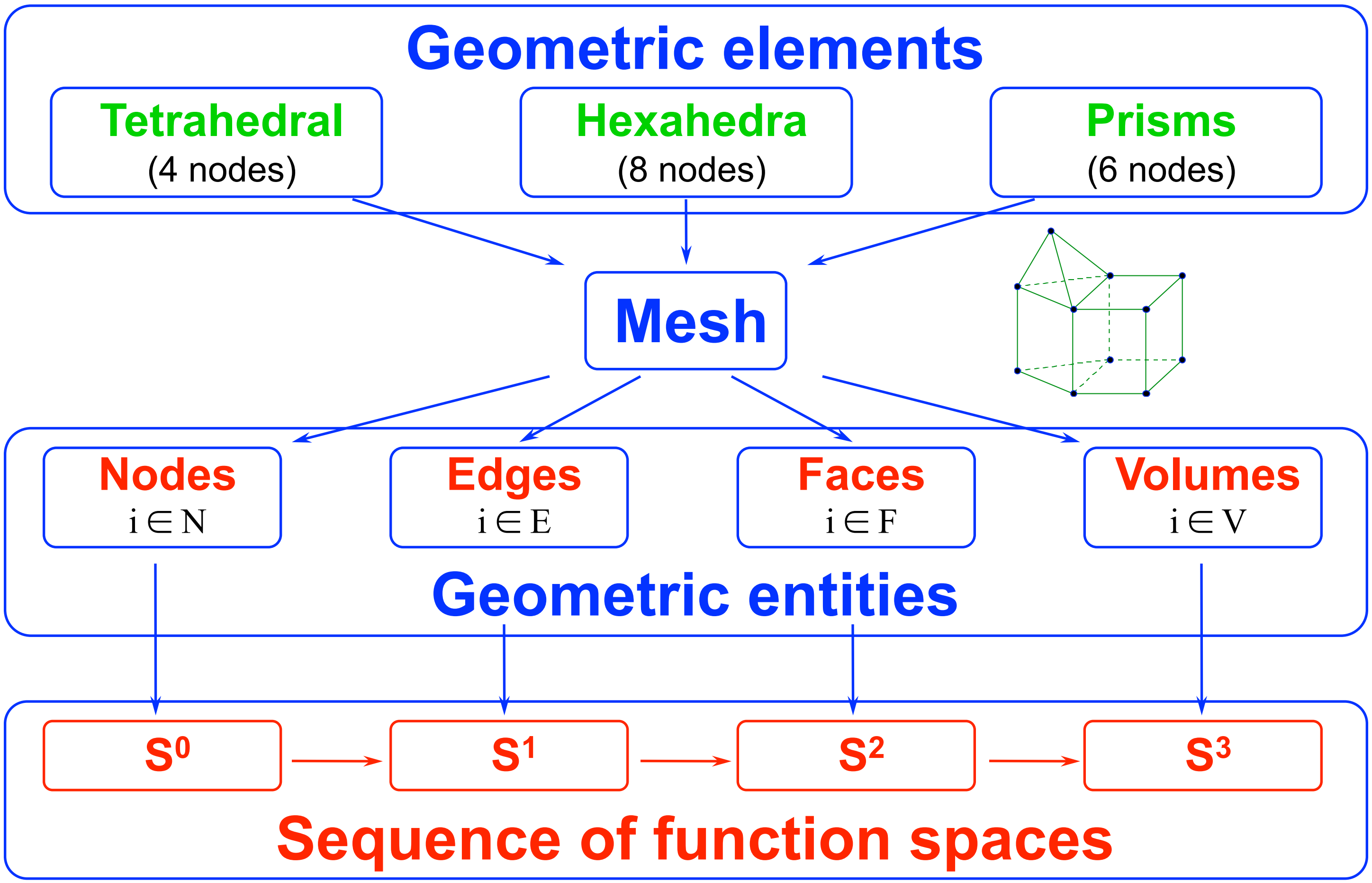
Basis functions

- **Finite element space**

  Union of finite elements (K_j, P_{Kj}, Σ_{Kj}) such as :

  - **the union of the K_j fill the studied domain (≡ mesh)**
  - **some continuity conditions are satisfied across the element interfaces**

Sequence of finite element spaces



Sequence of finite element spaces

	Functions	Properties	Functionals	Degrees of freedom	
S^0	$\{s_i\ ,\ i \in N\}$	$s_i(x_j) = \delta_{ij}\quad \forall\, i,j \in N$	Point evaluation	Nodal value	Nodal element
S^1	$\{\mathbf{s}_i\ ,\ i \in E\}$	$\int_j \mathbf{s}_i \cdot d\mathbf{l} = \delta_{ij}\quad \forall\, i,j \in E$	Curve integral	Circulation along edge	Edge element
S^2	$\{\mathbf{s}_i\ ,\ i \in F\}$	$\int_j \mathbf{s}_i \cdot \mathbf{n}\, ds = \delta_{ij}\quad \forall\, i,j \in F$	Surface integral	Flux across face	Face element
S^3	$\{s_i\ ,\ i \in V\}$	$\int_j s_i\, dv = \delta_{ij}\quad \forall\, i,j \in V$	Volume integral	Volume integral	Volume element

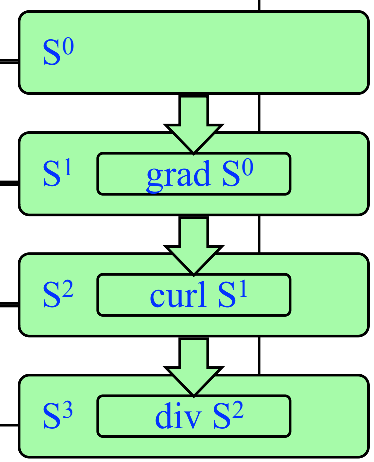
Bases

$$u_K = \sum_i \phi_i(u)\, s_i$$

Finite elements

Sequence of finite element spaces

	Base functions	Continuity across element interfaces	Codomains of the operators
S^0	$\{s_i\ ,\ i \in N\}$	value	
S^1	$\{\mathbf{s}_i\ ,\ i \in E\}$	tangential component	$\text{grad}\ S^0 \subset S^1$
S^2	$\{\mathbf{s}_i\ ,\ i \in F\}$	normal component	$\text{curl}\ S^1 \subset S^2$
S^3	$\{s_i\ ,\ i \in V\}$	discontinuity	$\text{div}\ S^2 \subset S^3$



Conformity

Sequence

$$S^0 \xrightarrow{\text{grad}} S^1 \xrightarrow{\text{curl}} S^2 \xrightarrow{\text{div}} S^3$$

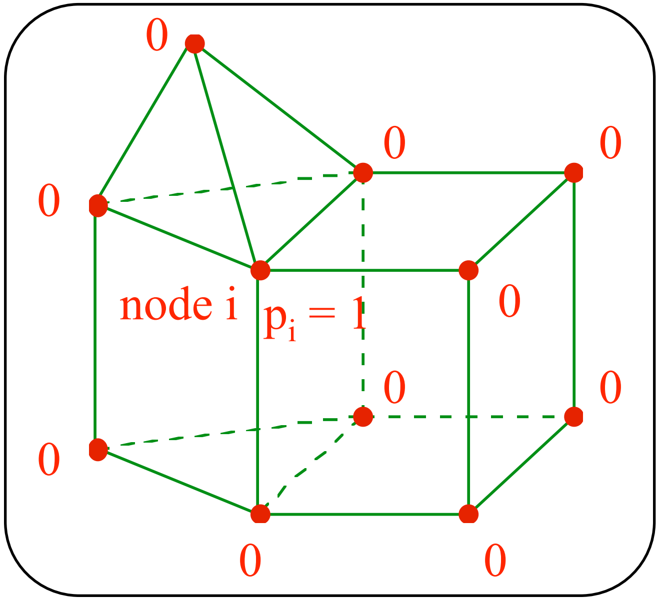
Function spaces S^0 et S^3

For each node $i \in N$ → scalar field

$$s_i(\mathbf{x}) = p_i(\mathbf{x}) \quad \in \; S^0$$

$$p_i = \begin{cases} 1 & \text{at node } i \\ 0 & \text{at all other nodes} \end{cases}$$

p_i continuous in Ω



For each Volume $v \in V$ → scalar field

$$s_v = 1 \,/\, \mathrm{vol}\,(v) \quad \in \; S^3$$

Edge function space S^1

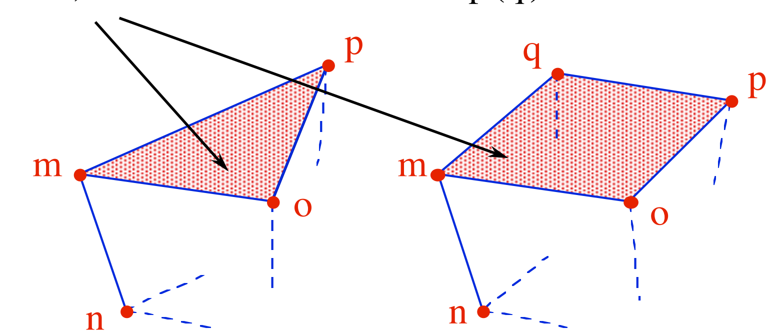
For each edge $e_{ij} = \{i, j\} \in E \rightarrow$ vector field

$$\mathbf{s}_{e_{ij}} = p_j \,\mathrm{grad} \sum_{r \in N_{F,j\bar{i}}} p_r - p_i \,\mathrm{grad} \sum_{r \in N_{F,i\bar{j}}} p_r$$

$\mathbf{s}_e \in S^1$

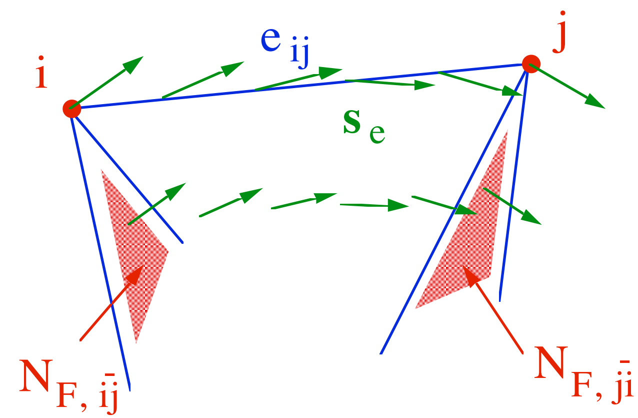
Definition of the set of nodes $N_{F,m\bar{n}}$

$$N_{F,m\bar{n}} = \{ i \in N \,;\; i \in f_{mop(q)} \,,\; o,p,q \neq n \}$$



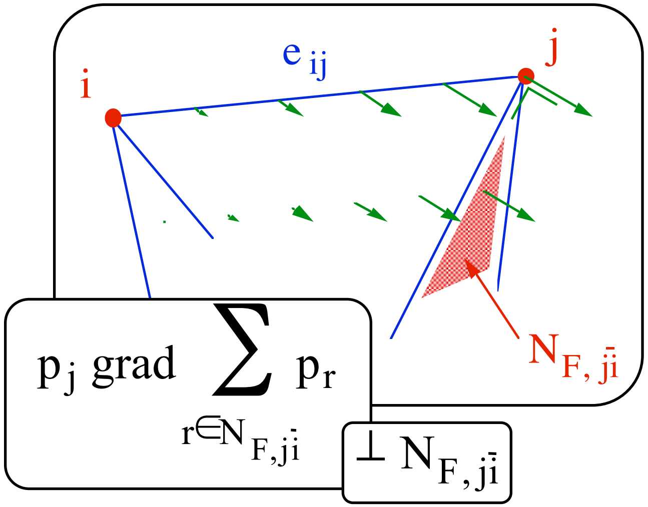
N.B.: In an element : 3 edges/node

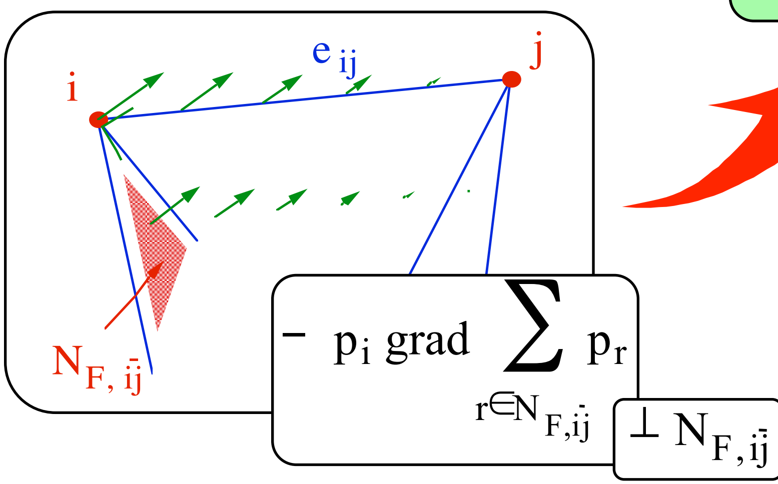
Illustration of the vector field $\mathbf{s}_e$



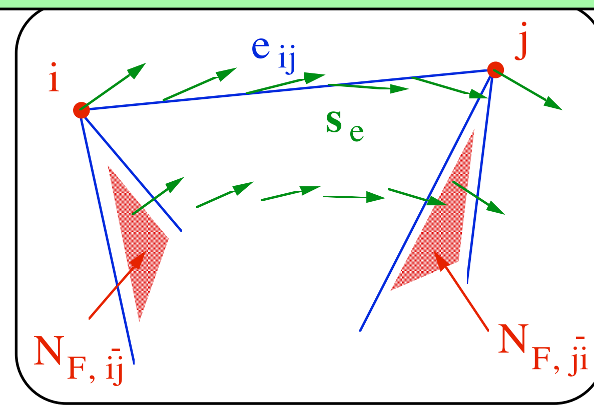
Edge function space S^1

Geometric interpretation of the vector field s_e





$$s_{e_{ij}} = p_j \operatorname{grad} \sum_{r \in N_{F,j\bar{i}}} p_r - p_i \operatorname{grad} \sum_{r \in N_{F,i\bar{j}}} p_r$$



Function space S^2

For each facet $f \in F \;\rightarrow\;$ vector field

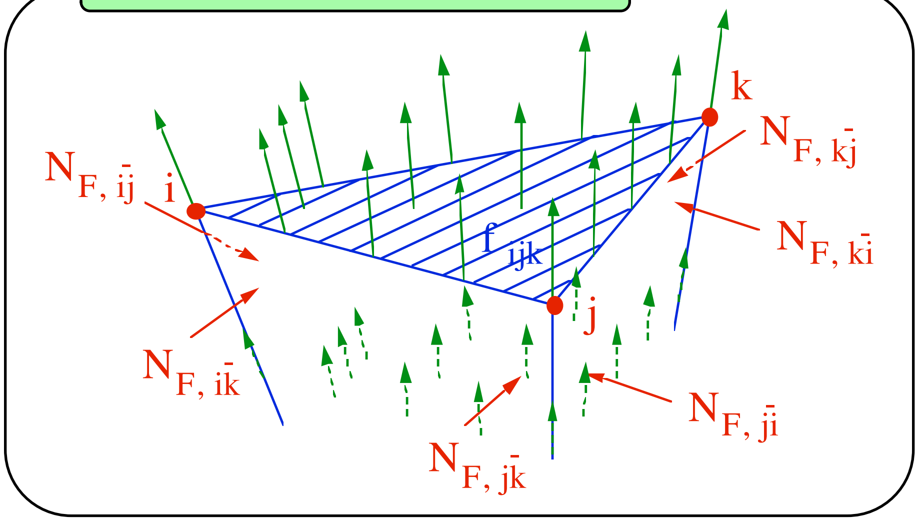
$f = f_{ijk(l)} = \{i, j, k\ (, l)\ \} = \{q_1, q_2, q_3\ (, q_4)\ \}$

$$s_f = a_f \sum_{c=1}^{\#N_f} p_{q_c}\ \mathrm{grad}\left(\sum_{r \in N_{F,q_c \bar{q}_{c+1}}} p_r\right) \wedge \mathrm{grad}\left(\sum_{r \in N_{F,q_c \bar{q}_{c-1}}} p_r\right)$$

$\mathbf{s}_f \in S^2$

$$\#N_f = \begin{cases} 3 \rightarrow a_f = 2 & \blacktriangle \\ 4 \rightarrow a_f = 1 & \blacksquare \end{cases}$$

Illustration of the vector field $\mathbf{s}_f$



Particular subspaces of S^1

Applications

Kernel of the curl operator

$\mathbf{h} \in S^1(\Omega)$; curl $\mathbf{h} = 0$ in $\Omega_c^C \subset \Omega$ $\rightarrow$ $\mathbf{h} \equiv$?

Ampere equation
in a domain Ω_c^C
without current
($\leftrightarrow$ Ω_c)

Gauged subspace

$\mathbf{a} \in S^1(\Omega)$; $\mathbf{b} = \text{curl } \mathbf{a} \in S^2(\Omega)$ $\rightarrow$ $\mathbf{a} \equiv$?

Gauge $\mathbf{a} \cdot \omega = 0$
to fix $\mathbf{a}$

Gauge condition
on a vector potential

Definition of a
generalized source field $\mathbf{h}_s$
such that curl $\mathbf{h}_s = \mathbf{j}_s$

Kernel of the curl operator

Case of simply connected domains

$$\boxed{H = \{\, \mathbf{h} \in S^1(\Omega)\,;\ \mathrm{curl}\,\mathbf{h} = 0 \text{ in } \Omega_c^{\,C} \,\}}$$

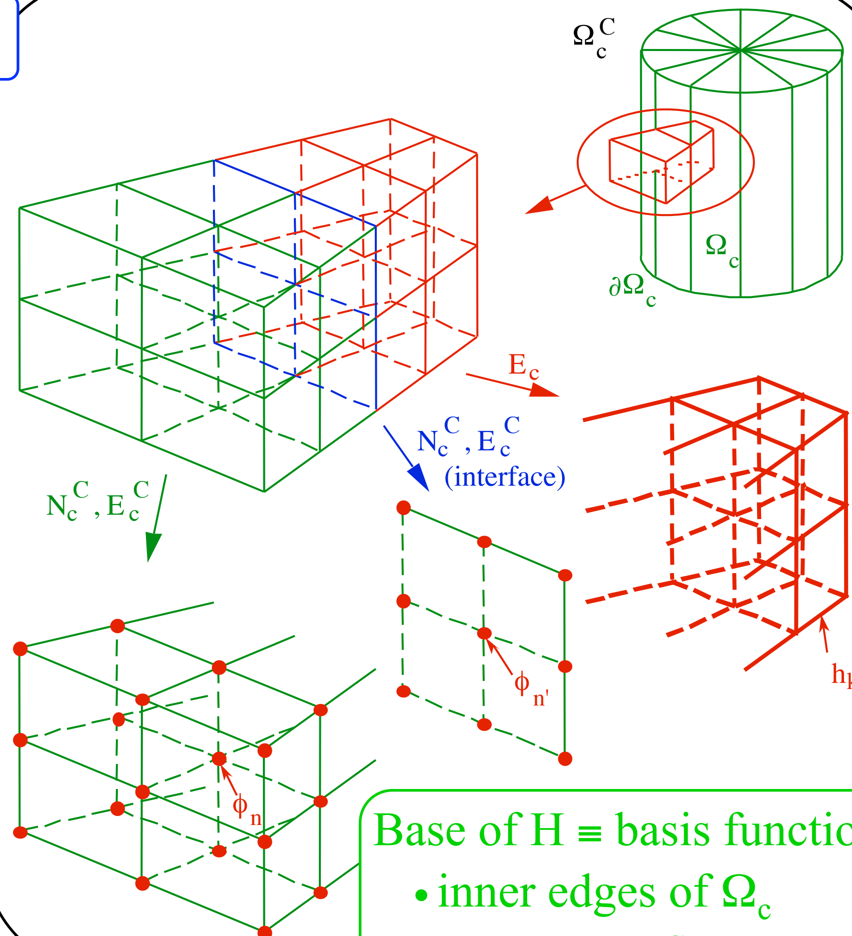
$$\mathbf{h} = -\,\mathrm{grad}\,\phi \text{ in } \Omega_c^{\,C}$$

$$\mathbf{h} = \sum_{e \in E} h_a\, \mathbf{s}_a = \sum_{k \in E_c} h_k\, \mathbf{s}_k + \sum_{l \in E_c^{\,C}} h_l\, \mathbf{s}_l$$

$$h_l = \int_{l_{ab}} \mathbf{h} \,.\, \mathbf{dl} = \int_{l_{ab}} -\mathrm{grad}\,\phi \,.\, \mathbf{dl} = \phi_{a_l} - \phi_{b_l}$$

$$\mathbf{h} = \sum_{k \in E_c} h_k\, \mathbf{s}_k + \sum_{l \in E_c^{\,C}} \left(\phi_{a_l} - \phi_{b_l}\right) \mathbf{s}_l$$

$$\boxed{\mathbf{h} = \sum_{k \in E_c} h_k\, \mathbf{s}_k + \sum_{n \in N_c^{\,C}} \phi_n\, \mathbf{v}_n \quad \text{with} \quad \mathbf{v}_n = \sum_{nj \in E_c^{\,C}} \mathbf{s}_{nj}}$$



Base of H ≡ basis functions of
- inner edges of Ω_c
- nodes of $\Omega_c^{\,C}$, with those of $\partial\Omega_c$

Kernel of the curl operator

Case of multiply connected domains

$$H = \{ \mathbf{h} \in S^1(\Omega)\,;\, \mathrm{curl}\,\mathbf{h} = 0 \text{ in } \Omega_c^{\,C} \}$$

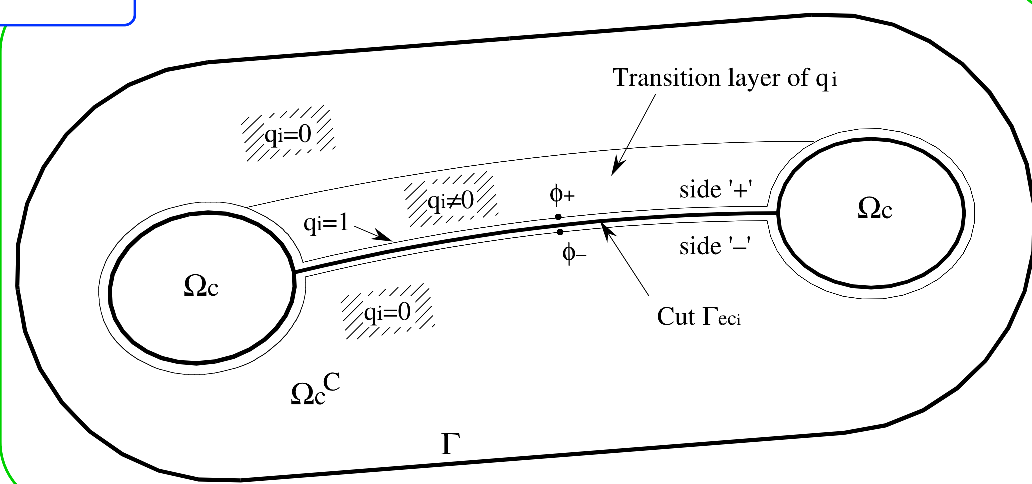
$\mathbf{h} = -\,\mathrm{grad}\,\phi$ in $\Omega_c^{\,C}$ (cuts)

$$\phi = \phi^{cont} + \phi^{disc}$$

$$\phi^+ - \phi^- = \phi\,|_{\Gamma^+_{eci}} - \phi\,|_{\Gamma^-_{eci}} = I_i$$

$$\phi^{disc} = \sum_{i \in C} I_i\, q_i$$

discontinuity of ϕ^{disc}



q_i
- defined in $\Omega_c^{\,C}$
- unit discontinuity across Γ_{eci}
- continuous in a transition layer
- zero out of this layer

$$\mathbf{h} = \sum_{k \in A_c} h_k\, \mathbf{s}_k + \sum_{n \in N_c^{\,C}} \phi^{cont}{}_n\, \mathbf{v}_n + \sum_{i \in C} I_i\, \mathbf{c}_i$$

with $\displaystyle \mathbf{c}_i = \sum \mathbf{s}_{nj}$

edges of $\Omega_c^{\,C}$ ← $nj \in A_c^{\,C}$

starting from a node of the cut ← $n \in N_{ec_i}$

and located on side '+' ← $j \in N_c^{\,C+}$

but not on the cut ← $j \notin N_{ec_i}$

Basis of H ≡ basis functions of
- inner edges of Ω_c
- nodes of $\Omega_c^{\,C}$
- cuts of C

Gauged subspace of S^1

Gauged space in Ω

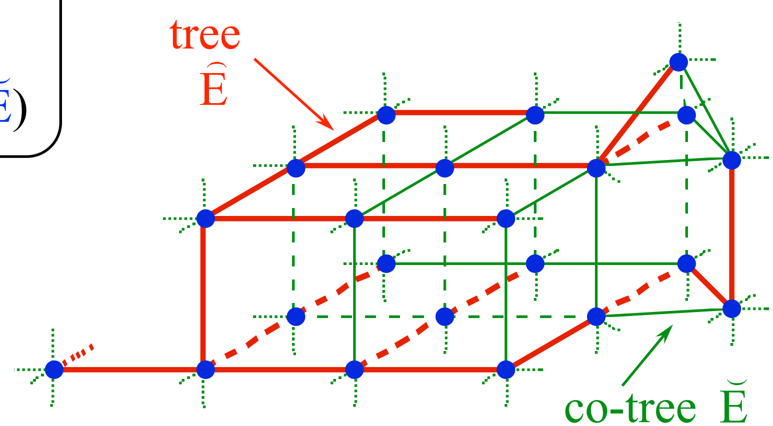
$\boxed{\mathbf{b} = \text{curl}\,\mathbf{a}}$ with $\mathbf{a} = \sum_{e \in E} a_e\, \mathbf{s}_e \in S^1(\Omega)$, $\mathbf{b} = \sum_{f \in F} b_f\, \mathbf{s}_f \in S^2(\Omega)$

$b_f = \sum_{e \in E} i(e,f)\, a_e$, $f \in F$ ➡ matrix form: $\boxed{b_f} = \boxed{\mathbf{C}_{FE}}\ \boxed{a_e}$

Face-edge incidence matrix

Tree ≡ set of edges connecting
(in Ω) all the nodes of Ω without
forming any loop ($\widehat{E}$)

Co-tree ≡ complementary set of the tree ($\breve{E}$)



Gauged space of $S^1(\Omega)$

$\breve{S}^1(\Omega) = \{\mathbf{a} \in S^1(\Omega)\ ;\ a_j = 0\ ,\ \forall\, j \in \breve{E}\}$

$\mathbf{a} = \sum_{i \in \breve{E}} a_i\, \mathbf{s}_i \in \breve{S}^1(\Omega)$

Basis of $\breve{S}^1(\Omega)$ ≡ co-tree edge basis functions
(explicit gauge definition)

Mesh of electromagnetic devices

- **Electromagnetic fields extend to infinity (unbounded domain)**
  - **Approximate boundary conditions:**
    - **zero fields at finite distance**
  - **Rigorous boundary conditions:**
    - **"infinite" finite elements (geometrical transformations)**
    - **boundary elements (FEM-BEM coupling)**
- **Electromagnetic fields are confined (bounded domain)**
  - **Rigorous boundary conditions**

Mesh of electromagnetic devices

- **Electromagnetic fields enter the materials up to a distance depending of physical characteristics and constraints**

  - **Skin depth δ (δ<< if ω, σ, μ >>)**

$$\delta = \sqrt{\frac{2}{\omega \, \sigma \, \mu}}$$

  - **mesh fine enough near surfaces (material boundaries)**
  - **use of surface elements when $\delta \to 0$**

Mesh of electromagnetic devices

- **Types of elements**
  - **2D : triangles, quadrangles**
  - **3D : tetrahedra, hexahedra, prisms, pyramids**
  - **Coupling of volume and surface elements**
    - **boundary conditions**
    - **thin plates**
    - **interfaces between regions**
    - **cuts (for making domains simply connected)**
  - **Special elements (air gaps between moving pieces, ...)**