Electromagnetic Energy Conversion - ELEC0431
Exercises with solutions

February 9, 2023

Contents

1 Phasors and power in harmonic regime

Theory

Let $\mathbf{F}(\omega)$ be the set of harmonic functions of **angular frequency** ω. All functions in $\mathbf{F}(\omega)$ therefore share the same **working frequency** $f = \frac{\omega}{2\pi}$ and the same **period** $T = \frac{1}{f}$. Let $f(t) \in \mathbf{F}(\omega)$ be one function of this set, with **amplitude** F_{peak} and **phase angle** θ

$$f(t) = F_{\text{peak}} \cos(\omega t + \theta). \tag{1}$$

RMS vs. peak value

The **Root Mean Square** value (RMS value) of $f(t)$, also called its **effective** value, is defined as

$$F_{\text{RMS}} := \sqrt{\frac{1}{T}\int_0^T f(t)^2\, dt}. \tag{2}$$

so that the link between the peak value of $f(t)$ and the RMS value of $f(t)$ is

$$\boxed{F_{\text{peak}} = \sqrt{2}\, F_{\text{RMS}}} \tag{3}$$

as shown by

$$\begin{aligned} F_{\text{RMS}}^2 &= \frac{1}{T}\int_0^T f(t)^2\, dt = \frac{1}{T}\int_0^T F_{\text{peak}}^2 \cos^2(\omega t + \theta)\, dt \\ &= \frac{F_{\text{peak}}^2}{T} \int_0^T \left[\frac{1}{2} + \underbrace{\frac{1}{2}\cos\big(2(\omega t + \theta)\big)}_{=0,\text{ when integrated}}\right] dt = \frac{F_{\text{peak}}^2}{T}\frac{T}{2} \\ &= \frac{F_{\text{peak}}^2}{2}. \end{aligned}$$

Phasors

The identities

$$f(t) = F_{\text{peak}} \cos(\omega t + \theta) = \sqrt{2}\, F_{\text{RMS}} \cos(\omega t + \theta) = \text{Re}\left(\sqrt{2}\, F_{\text{RMS}}\, e^{j(\omega t + \theta)}\right) = \text{Re}\left(\sqrt{2}\, \underbrace{F_{\text{RMS}}\, e^{j\theta}}_{\overline{F}}\, e^{j\omega t}\right) \tag{4}$$

show that any harmonic function of $\mathbf{F}(\omega)$ can be represented unambiguously by the complex number

$$\boxed{\overline{F} = F_{\text{RMS}}\, e^{j\theta}} \tag{5}$$

called **phasor**. The main properties of the equivalence phasor $\Longleftrightarrow$ harmonic function are summarized in the following table :

Time domain		Frequency domain	
$f(t) = F_{\text{peak}} \cos(\omega t + \theta)$	$\Longleftrightarrow$	$\overline{F} = F_{\text{RMS}}\, e^{j\theta}$	(6)
$a\, f(t) + b\, g(t)$	$\Longleftrightarrow$	$a\,\overline{F} + b\,\overline{G}$	(7)
$\frac{df}{dt}(t)$	$\Longleftrightarrow$	$j\,\omega \overline{F}$	(8)
$\int f(t)\, dt$	$\Longleftrightarrow$	$\frac{\overline{F}}{j\,\omega}$	(9)

A fundamental property of the phasor $\Leftrightarrow$ harmonic function equivalence is linearity, i.e., the linear combination of harmonic functions of $\mathbf{F}(\omega)$ is exactly represented by the same linear combination of the corresponding phasors (See eq. 7). Since the Kirchhoff laws in circuits are linear combinations, they can be used to manipulate phasors.

Product of 2 harmonic functions

A complete physical description of a system in harmonic regime requires to consider energy and power quantities, which are products of two harmonic functions. For instance, the instantaneous power in a one-port (see definition below) is determined by the product of the current $i(t)$ flowing through it with the voltage $v(t)$ across it. However, the product of two harmonic functions of the same frequency is **NOT** another function of the same frequency:

$$\begin{aligned} f(t)\ g(t) &= F_{\text{peak}} \cos(\omega t + \theta)\ G_{\text{peak}} \cos(\omega t + \psi) \\ &= 2F_{\text{RMS}} G_{\text{RMS}} \cos(\omega t + \theta)\ \cos(\omega t + \psi) \\ &= F_{\text{RMS}} G_{\text{RMS}} \Big[\underbrace{\cos(\theta - \psi)}_{\text{DC, 0 Hz}} + \underbrace{\cos(2\omega t + \theta + \psi)}_{\text{Double frequency,} 2f} \Big]. \end{aligned}$$

The product results in the sum of a first term constant in time (DC) with a second term oscillating at twice the working frequency, i.e., at $2f$. In general, the **instantaneous power** $p(t)$ is precisely expressed by a product of two harmonic function, so we write

$$p(t) = f(t)\, g(t) = \underbrace{F_{\text{RMS}} G_{\text{RMS}}\ \cos(\theta - \psi)}_{P} + \underbrace{F_{\text{RMS}} G_{\text{RMS}}\ \cos(2\,\omega\, t + \theta + \psi)}_{p_f(t)},$$

and so one can split the **instantaneous power** $p(t)$ into two parts, and call **active power** P the part that is constant in time, and **fluctuating power** $p_f(t)$ the part that oscillates at twice the working frequency.

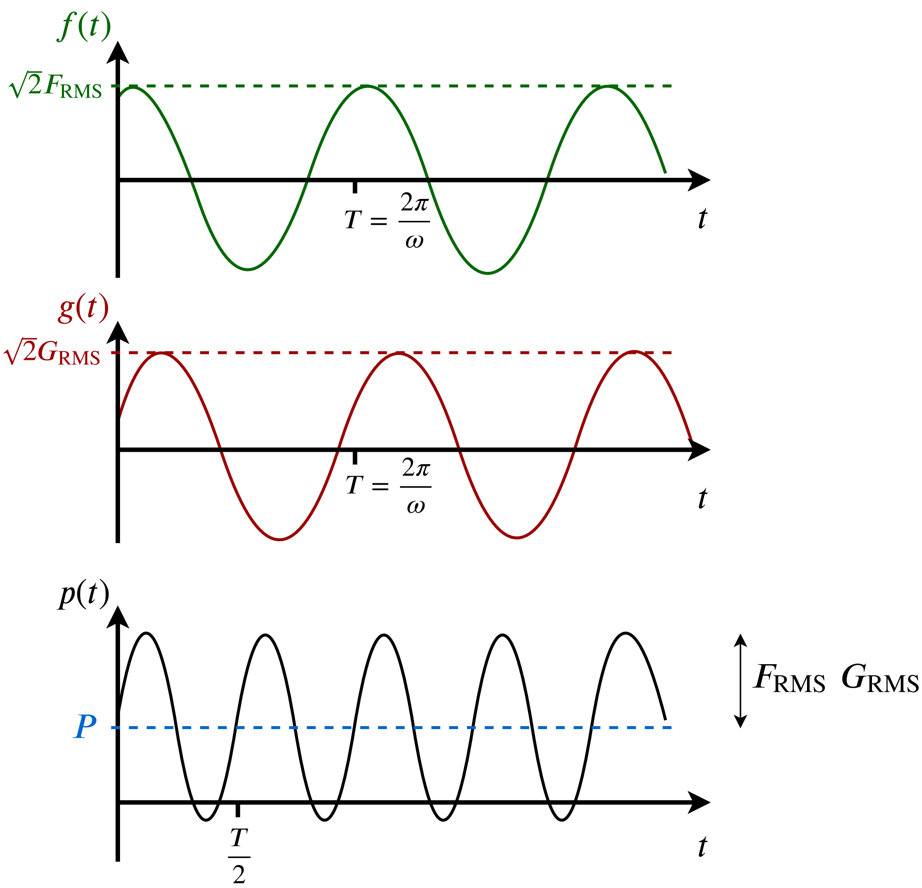


Figure 1: Instantaneous power as the product of two harmonic functions. Note the graphical interpretation of the active power P and of the fluctuating power with double frequency and amplitude $F_{\text{RMS}} G_{\text{RMS}}$.

The complex power S

For a better interpretation of the power quantities in harmonic regime, the instantaneous power is further developed with some algebra, starting over from the last identity above to obtain

$$\begin{aligned} p(t) &= F_{\text{RMS}} G_{\text{RMS}} \left(\cos(\theta-\psi) + \cos(2\omega t + \theta + \psi)\right) \\ &= F_{\text{RMS}}\, G_{\text{RMS}} \Big(\cos(\theta-\psi) + \cos\left(2(\omega\, t + \theta) - (\theta - \psi)\right)\Big) \\ &= F_{\text{RMS}}\, G_{\text{RMS}} \Big(\cos(\theta-\psi) + \cos\left(2(\omega t + \theta)\right)\cos(\theta-\psi) + \sin\left(2(\omega t+\theta)\right)\sin(\theta-\psi)\Big) \\ &= \underbrace{F_{\text{RMS}}\, G_{\text{RMS}}\, \cos(\theta-\psi)}_{P} \underbrace{\Big(1+\cos\left(2(\omega t+\theta)\right)\Big)}_{\geq 0} + \underbrace{F_{\text{RMS}} G_{\text{RMS}}\, \sin(\theta-\psi)}_{Q}\, \sin\left(2\,(\omega t+\theta)\right) \end{aligned}$$

This expression is extremely important. It shows that the instantaneous power $p(t)$ can be decomposed into a flow of energy with always the same direction and an average value in time equal to P (always positive or always negative, according to the sign of P), and a second purely fluctuating flow of energy, exchanged back and forth twice at each period, with an average value in time equal to zero and an amplitude equal to Q. To understand the meaning of these quantities, it is important to note that the amount of energy transfered at each period to a harmonic system is a constant. This constant is the **active power** P that has already been defined above, and it is the main representative thermodynamic quantity that characterizes the energy balance of the harmonic system at hand. One speaks here of the **energy balance averaged in time**, which is the one we are interested in in practice. The presence of the fluctuating flow, however, may have some (usually negative) technical consequences, although it does not directly impact the energy balance averaged in time. It is therefore useful to give it a name for reference. The amplitude Q is therefore called **reactive power**.

At this stage, one makes a very dangerous move. One defines the **complex power** as

$$S = P + jQ = F_{\text{RMS}} G_{\text{RMS}} \cos(\theta-\psi) + jF_{\text{RMS}} G_{\text{RMS}} \sin(\theta-\psi). \tag{10}$$

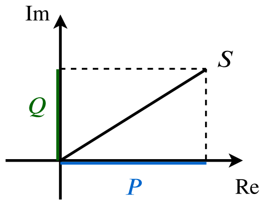


Figure 2: Complex power decomposition into active power and reactive power.

This complex number is ***not*** a phasor, as it does not represent a harmonic function of angular frequency ω (reason why it is noted without upper bar). But its real part P and imaginary part Q provide a complete description of the instantaneous power, since one has seen that

$$p(t) = P\Big(1+\cos\left(2(\omega t+\theta)\right)\Big) + Q\sin\left(2\,(\omega t+\theta)\right). \tag{11}$$

Moreover, the complex power can be obtained by a simple algebraic operation from the phasors $\overline{F}$ and $\overline{G}$ of the initial harmonic functions $f(t)$ and $g(t)$ that were multiplied to form $p(t)$:

$$\begin{aligned} S &= P + jQ \\ &= F_{\text{RMS}} G_{\text{RMS}} \cos(\theta-\psi) + jF_{\text{RMS}} G_{\text{RMS}} \sin(\theta-\psi) \\ &= F_{\text{RMS}} G_{\text{RMS}} e^{j\,(\theta-\psi)} \\ &= F_{\text{RMS}} e^{j\theta}\; G_{\text{RMS}} e^{-j\psi} \\ &= \overline{F}\,\overline{G}^{*} \end{aligned}$$

where $*$ denotes the complex conjugate operator.

Harmonic voltages and currents

Now that the concepts of phasor and power have been explicited in the context of harmonic functions, it is time to apply them to voltages and currents, which are the quantities of interest in this course. In order to shorten notations, F_{RMS} will be simply noted F in the rest of the course (V and I respectively for RMS voltages and currents), and F_{peak} will be noted F_m (V_m and I_m respectively for maximum voltages and currents).

$$\begin{array}{ccc} F_{\text{RMS}} & \Longrightarrow & F \quad (V \text{ and } I) \\ F_{\text{peak}} & \Longrightarrow & F_m \quad (V_m \text{ and } I_m) \end{array}$$

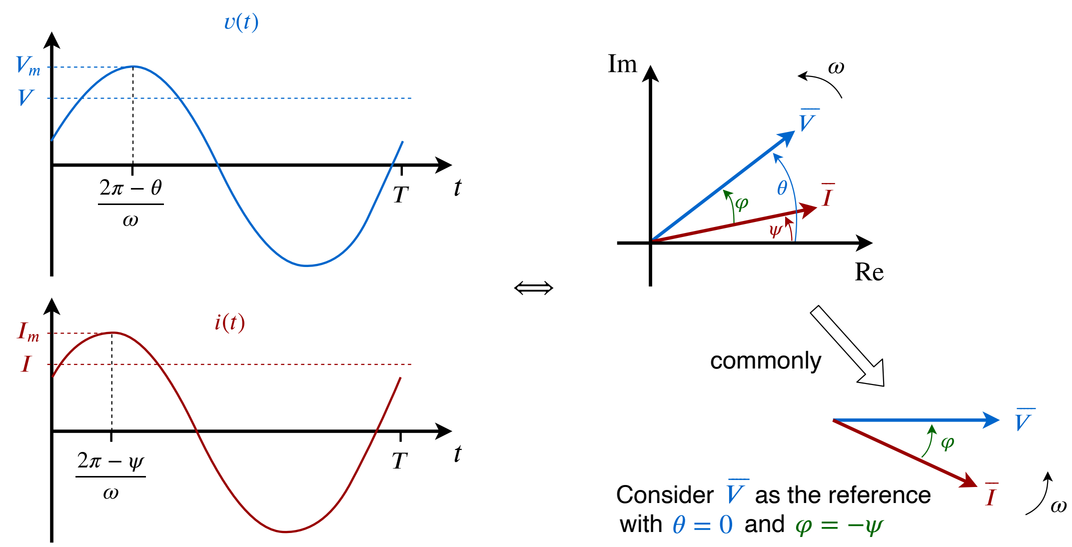
The algebraic and graphical representations for harmonic currents and voltages, and the associated powers, can thus be summarized as follows (Note the caution symbol "!" that reminds that the complex power S is not a phasor) :

$$\begin{array}{rcl} \text{Time domain} & & \text{Frequency domain} \\ \hline v(t) = \sqrt{2}\,V\,\cos(\omega t + \theta) & \Longleftrightarrow & \overline{V} = V\,e^{j\,\theta} \\ i(t) = \sqrt{2}\,I\,\cos(\omega t + \psi) & \Longleftrightarrow & \overline{I} = I\,e^{j\,\psi} \\ p(t) = P\Big(1 + \cos\left(2(\omega t + \theta)\right)\Big) + Q\sin\left(2\,(\omega t + \theta)\right) & \overset{!}{\longleftrightarrow} & S = P + jQ = VI\cos\varphi + jVI\sin\varphi \end{array}$$

where θ is the phase angle of the voltage $v(t)$ and ψ the phase angle of the current $i(t)$. Absolute phase angles are however meaningless. The phase difference between harmonic quantities is what is important. In particular, the phase difference (phase lag) between voltage and current is conventionally noted

$$\boxed{\varphi = \theta - \psi}. \tag{12}$$

It is positive when the voltage is ahead of the current and negative otherwise, as illustrated in the figure below.



Apparent power and power factor

The **apparent power** is, by definition, the norm of the complex power: $|S| = \sqrt{P^2 + Q^2} = V\,I.$ (13)

One can also define the **power factor**, noted PF: $\text{PF} = \cos\varphi = \frac{P}{|S|}.$ (14)

To deliver a given active power P from a source with a given voltage magnitude V, the above identity shows that the needed current

$$I = \frac{|S|}{V} = \frac{\sqrt{P^2 + Q^2}}{V} = \frac{P}{V\,\cos\varphi}. \tag{15}$$

is larger whenever the power factor is smaller or equivalently, the needed current is larger whenever the reactive power Q needed by the system is larger. This confirms the remark made above, that reactive power acts rather like a burden to the transmission of an amount of power from a given source to a load.

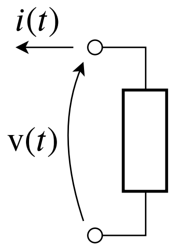
One-port

The simplest part of an electric circuit is called a **one-port** ("dipôle"). Each one-port is associated with a current $i(t)$ and a voltage $v(t)$. As these are signed quantities, a positive reference must be indicated for each of them with an arrow. The current $i(t)$ flowing through the one-port is positive if the positive charges flow in the direction indicated by the arrow. The voltage $v(t)$ across the one-port, on the other hand, is defined as

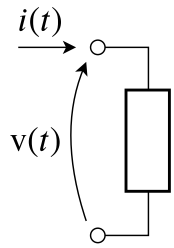
$$v(t) = \text{(Voltage at the head of the arrow)} \; - \; \text{(Voltage at the tail of the arrow).} \tag{16}$$

With this, two different conventions exist for a one-port :

- the **active convention**, also called generator or source (See figure below). In this case, the active one-port ***delivers*** the instantaneous power $p(t) = u(t)i(t)$ and the complex power $S = P + jQ$ through its terminals. This convention is used for voltage sources, current sources, generators, and the secondary side of transformers.
- the **passive convention**, also called receiver, motor or load. (See figure below). In this case, the passive one-port ***receives*** the instantaneous power $p(t) = u(t)i(t)$ and the complex power $S = P + jQ$ through its terminals. This convention is used for R, L, C, lumped parameters, motors and the primary side of transformers.



Active one-port



Passive one-port

In phasor formalism, there exist 4 types of one-ports, distinguished by the linear relationship imposed by the one-port on the phasors quantities $\overline{V}$ and $\overline{I}$:

- **Voltage source** (active) : $\overline{V} = \overline{V}_0$, with $\overline{V}_0$ a given phasor, $\overline{I}$ is determined by the rest of the circuit,
- **Impedance** (passive) : $\overline{V} = Z\overline{I}$, with $Z = R + jX$ a complex number formed with the resistance R and the reactance X of the one-port,
- **Current source** (active) : $\overline{I} = \overline{I}_0$, with $\overline{I}_0$ a given phasor, $\overline{V}$ is determined by the rest of the circuit,
- **Admittance** (passive) : $\overline{I} = Y\overline{V}$, with $Y = G + jB$ a complex number formed, with the conductance G and the susceptance B of the one-port.

The different kinds of power received by impedances and admittances are summarized in the following table, using the passive one-port convention (otherwise, one would have to write $\overline{V} = -Z\overline{I}$ and $\overline{I} = -Y\overline{V}$). SI units are indicated in square brackets. Note that, despite being dimensionally identical, the active, complex and reactive powers are expressed in different units to highlight their important conceptual differences. The active power P is expressed in Watt [W], the complex power S in Volt Ampere [VA], and the reactive power Q in Volt Ampere Reactive [var]. This is another sign that the complex power S and the reactive power Q are not directly relevant to the energy balance (averaged in time) of the harmonic system, but rather two useful and indicative mathematical artifacts.

$$\overline{V} = Z\,\overline{I} = (R + jX)\,\overline{I}$$

Z : impedance	[Ω]
R : resistance	[Ω]
X : reactance	[Ω]

$S = \overline{V}\,\overline{I}^* = \mathrm{Z}\lvert\overline{I}\rvert^2 = (R + jX)I^2$	[VA]
$P = \mathrm{R}\,I^2$	[W]
$Q = \mathrm{X}\,I^2$	[var]

$$\overline{I} = Y\,\overline{V} = (G + jB)\,\overline{V}$$

Y : admittance	[S]
G : conductance	[S]
B : susceptance	[S]

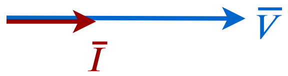
$S = \overline{V}\,\overline{I}^* = \mathrm{Y}^*\lvert\overline{V}\rvert^2 = (G - jB)V^2$	[VA]
$P = \mathrm{G}\,V^2$	[W]
$Q = -\mathrm{B}\,V^2$	[var]

Finally, the phasor representation of the fundamental purely resistive one-port (R), purely inductive one-port (L) and purely capacitive one-port (C) are detailed below. Again, the passive convention is used for such elements. One observes in particular that, according to the chosen convention, inductances receive an always positive reactive power $Q = XI^2 \geq 0$, so that one says that inductances *consume* reactive power, whereas capacitors receive an always negative reactive power $Q = -BV^2 \leq 0$, so that one says that capacitors *produce* reactive power.

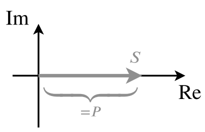
R

$$\overline{V} = R\overline{I}$$

$$\overline{I} = \frac{1}{R}\overline{V}$$



$\overline{V}$ and $\overline{I}$ are in phase, $\varphi = 0$

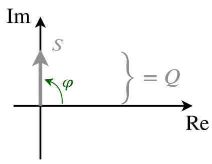


L

$$\overline{V} = j\omega L\,\overline{I}$$

$$\overline{I} = \frac{1}{j\omega L}\overline{V}$$

$\overline{I}$ is lagging $\overline{V}$, $\varphi = \frac{\pi}{2}$

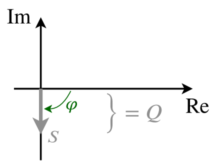


C

$$\overline{V} = \frac{1}{j\omega C}\overline{I}$$

$$\overline{I} = j\omega C\,\overline{V}$$

$\overline{I}$ is leading $\overline{V}$, $\varphi = -\frac{\pi}{2}$



	R	L	C
φ	0	$\frac{\pi}{2}$	$-\frac{\pi}{2}$
$Z = R + jX$	R	$j\omega L$	$\frac{1}{j\omega C}$
R	R	0	0
X	0	ωL	$\frac{-1}{\omega C}$
$Y = G + jB$	$\frac{1}{R}$	$\frac{1}{j\omega L}$	$j\omega C$
G	$\frac{1}{R}$	0	0
B	0	$\frac{-1}{\omega L}$	ωC
$S = P + jQ$	$RI^2 = \frac{V^2}{R}$	$j\omega L\,I^2 = j\frac{V^2}{\omega L}$	$-j\omega C\,V^2 = -j\frac{I^2}{\omega C}$
P	$\mathrm{R}I^2 = \frac{V^2}{\mathrm{R}}$	0	0
Q	0	$\omega\,L\,I^2 = \frac{V^2}{\omega\,L}$	$-\omega\,C\,V^2 = -\frac{I^2}{\omega\,C}$

Exercise 1. Voltage distribution

The circuit of Figure 3 presents a resistive-inductive load powered with an AC generator of sinusoidal voltage (230 V, 50 Hz). Find the voltages accross R and L (magnitude and phase angle) and represent all the voltages and currents on a phasor diagram.

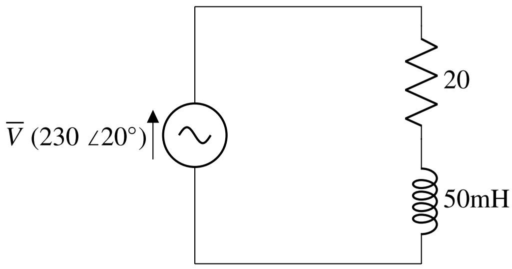
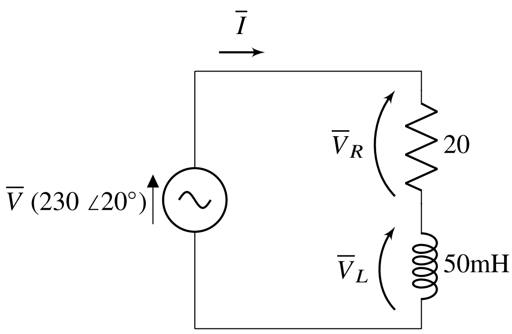


Figure 3: Resistive-inductive circuit.

Solution



First apply the KVL (Kirchhoff Voltage Law) in time domain to this circuit, leading to

$$v(t) = v_{\mathrm{R}}(t) + v_{\mathrm{L}}(t) \tag{17}$$

$$= \mathrm{R}\, i(t) + \mathrm{L}\, \frac{d\, i(t)}{dt} \tag{18}$$

which can be expressed as

$$\mathrm{Re}\left(\overline{V}\, e^{j\,\omega\, t}\right) = \mathrm{Re}\left(\mathrm{R}\,\overline{I}\, e^{j\,\omega\, t}\right) + \mathrm{Re}\left(\mathrm{L}\,\frac{d}{dt}\left(\overline{I}\, e^{j\,\omega\, t}\right)\right) \tag{19}$$

$$\mathrm{Re}\left(\overline{V}\, e^{j\,\omega\, t}\right) = \mathrm{Re}\left(\mathrm{R}\,\overline{I}\, e^{j\,\omega\, t}\right) + \mathrm{Re}\left(j\,\omega\,\mathrm{L}\,\overline{I}\, e^{j\,\omega\, t}\right) \tag{20}$$

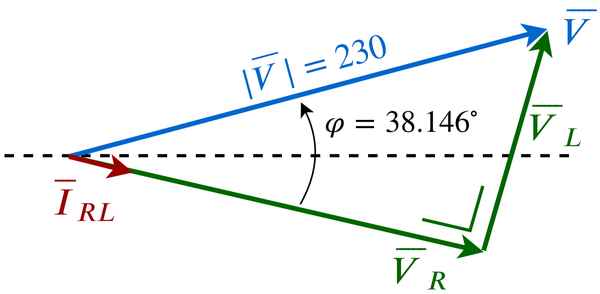
In frequency domain, with phasors as variables, Equation (20) can be rewritten as

$$\overline{V} = \underbrace{\mathrm{R}\,\overline{I}}_{\overline{V}_R} + \underbrace{j\,\mathrm{X}_L\,\overline{I}}_{\overline{V}_L} = (\mathrm{R} + j\ \mathrm{X}_L)\ \overline{I}_{RL} \tag{21}$$

$$\overline{I} = \frac{\overline{V}}{(\mathrm{R} + j\ \mathrm{X}_L)} = \frac{230\angle 20°}{20 + j\ 15.708} = 9.044\angle - 18.146° \tag{22}$$

Once the current circulating in the circuit is determined, the voltages accross the resistance and the inductance can also be known

$$\boxed{\begin{aligned} \overline{V}_R &= \mathrm{R}\ \overline{I} = 180.881\ \angle - 18.146° \quad [\mathrm{V}] \\ \overline{V}_L &= j\ \mathrm{X}_L\ \overline{I} = 142.064\ \angle\ 71.854° \quad [\mathrm{V}] \end{aligned}} \tag{23, 24}$$



The reactive power consumed by the load is

$$Q = V\ I\ \sin(\varphi) = 230 \times 9.044 \times \sin(38.146°) = 1284.82\ \text{[var]} \tag{25}$$

Exercise 2. Reactive power compensation

Your colleague suggests to add a 50 μF capacitor in parallel of the RL load. It is supposed to compensate the reactive power consumed by the inductive load. Is it a good idea ? If so, what would be the exact value of C needed ?

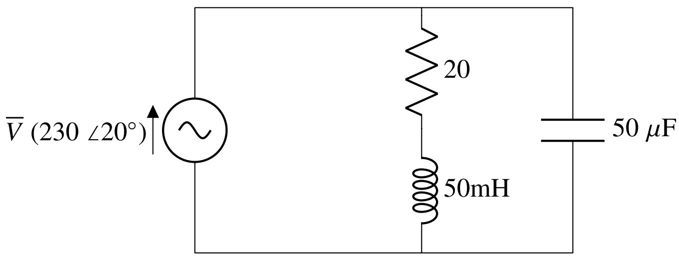
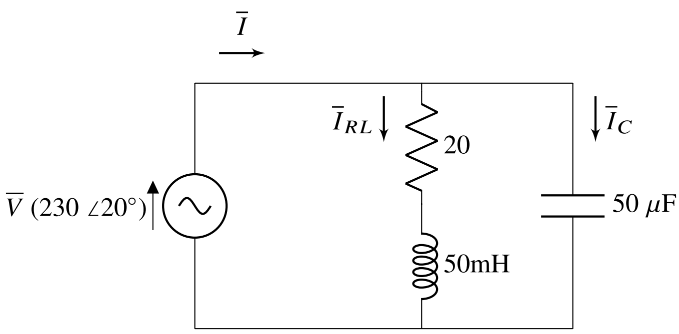


Figure 4: Inductive circuit with compensation capacitor.

Solution



The voltage accross the RL load remains the same ($\overline{V}$= 230 ∠20°) as in Exercise 1.
Then, the current in RL is still $\overline{I}_{RL} = 9.044\angle -18.146°$ [A], as in Exercise 1 and

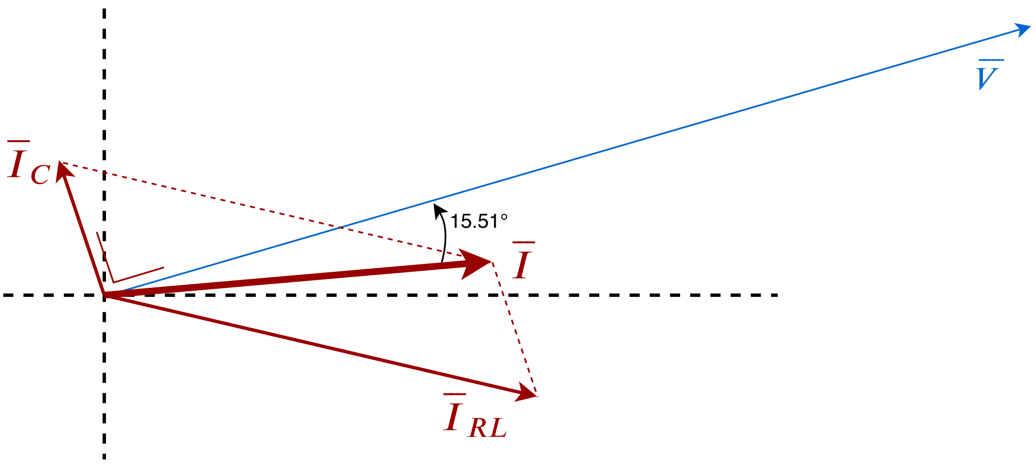
$$\overline{I}_C = \frac{\overline{V}}{Z_C} = Y_C\ \overline{V} = j\ \omega C\ \overline{V} = j\ 2\pi \times 50 \times 5 \times 10^{-5} \times (230\angle 20°) = 3.61\angle 110° \tag{26}$$

Apply the KCL (Kirchhoff Circuit Law) and obtain

$$\overline{I} = \overline{I}_{RL} + \overline{I}_C = 7.381\angle 4.49° \tag{27}$$

The reactive power consumed by the load is

$$\boxed{Q = V\ I\ \sin(\varphi) = 230 \times 7.381 \times \sin(15.51°) = 453.957\ \text{[var]}} \tag{28}$$



The exact value of capacitance to be used must cancel the reactive power consumption of the inductance, such that the reactive power produced by C matches the reactive power consumed by L,

$$Q_C = -Q_{RL} \tag{29}$$

$$-B_C\ V^2 = B_{RL}\ V^2 \tag{30}$$

$$-\omega\ C\ V^2 = -\frac{X}{R^2 + X^2}\ V^2 \tag{31}$$

with $B_C = \omega\mathrm{C}$ and $B_{RL} = -\frac{\omega\, L}{R^2+(\omega\, L)^2}$ because

$$\mathrm{Y}_{RL} = \frac{1}{\mathrm{Z}_{RL}} = \frac{1}{\mathrm{R} + j\ \mathrm{X}} = \frac{\mathrm{R} - j\ \mathrm{X}}{\mathrm{R}^2 + \mathrm{X}^2} \tag{32}$$

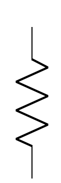
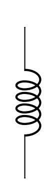
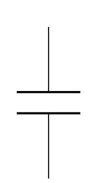
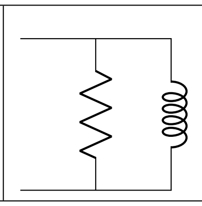
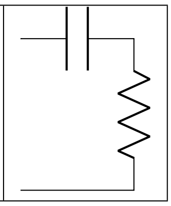
Thus,

$$\boxed{\mathrm{C} = \frac{\mathrm{L}}{\mathrm{R}^2 + (\omega \mathrm{L})^2} = 77.31[\ \mu\mathrm{F}]} \tag{33}$$

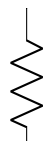
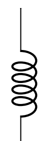
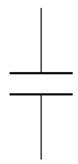
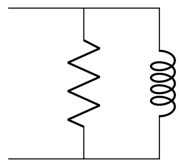
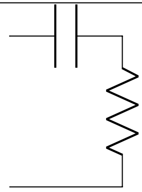
Exercise 3. One-port small quiz

Fill the cells of the table below with the most appropriate answer among:

=0 <0 >0 =1 <1 +∞ -∞

one-port:					
active power consumed					
reactive power produced					
cos ϕ					
tan ϕ					

Solution

one-port:					
active power consumed	> 0	=0	=0	> 0	> 0
reactive power produced	=0	< 0	> 0	< 0	> 0
cos ϕ	=1	=0	=0	< 1	< 1
tan ϕ	=0	+∞	-∞	> 0	< 0

Exercise 4. 2-Ports characterization

It is asked to characterize the 2-ports of Figure 5. In that context, two tests have been performed: a short circuit test and an open circuit test, both at 50 Hz.

- 559 mV and 1.118 A are measured at the access 1 while the access 2 is shorted (short circuit test).
- 5 V and 4.472 A are measured at the access 1 while the access 2 is left open (open circuit test).

1. Determine the value of R and L with the information provided below.

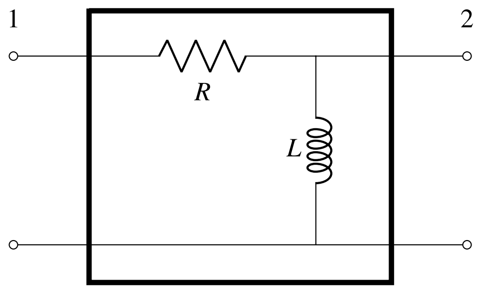


Figure 5: 2-Ports circuit

The 2-ports could be fully caracterized by only one of the two tests if the active power was measured during the tests. The active power can be measured with a wattmeter.

2. Which test would be necessary ?
3. During that test, an active power of 9.99392 W has been measured. Prove that it gives the correct value of R and L.

Solution

1. From the measurements made during the short circuit test, it is possible to determine the value of the resitance.

$$R = \frac{U_{1s}}{I_{1s}} = \frac{0.559}{1.118} = 0.5 \quad [\Omega] \tag{34}$$

And from the measurements made during the open circuit test, it is possible to determine the value of the impedance.

$$|Z| = \frac{U_{1o}}{I_{1o}} = \frac{5}{4.472} = 1.118 \quad [\Omega] \tag{35}$$

Based on the values of the resistance and the impedance, the value of the reactance can be determined as

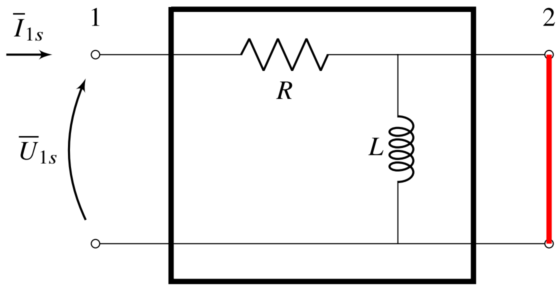


Figure 6: 2-Ports under short circuit test

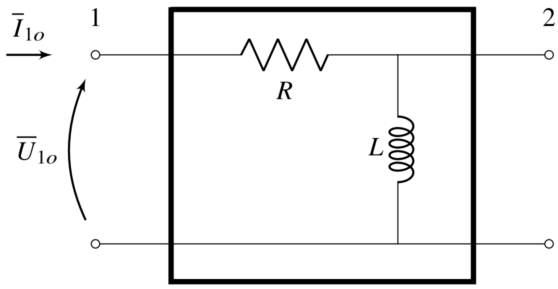


Figure 7: 2-Ports under open circuit test

$$X = \sqrt{|Z|^2 - R^2} = \sqrt{1.118^2 - 0.5^2} = 1 \quad [\Omega] \tag{36}$$

2. The open circuit test would be necessary in this case for the use of the wattmeter. Remark that it depends on the assumption previously made on the organization of the two-ports.
3. If a wattmeter had been used, only one manipulation would have been required.

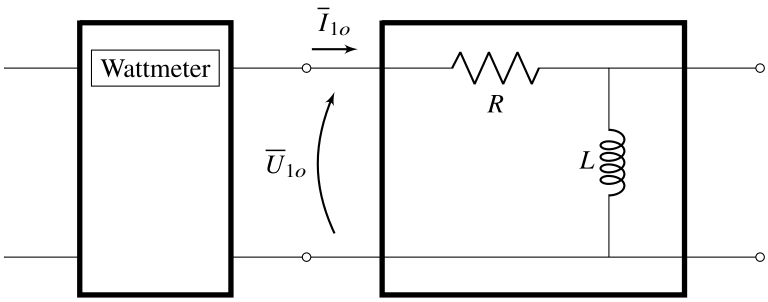


Figure 8: 2-Ports under open circuit test with wattmeter

The active power measured would be $P_{1o} = 9.99392$ [W] and the apparent power $S_{1o} = U_{1o}\, I_{1o} = 5 \times 4.472 = 22.36$ [VA].

From the knowledge of the apparent power and the active power (both measured in open circuit for this test), one can deduce the reactive power

$$Q_{1o} = \sqrt{S_{1o}^2 - P_{1o}^2} = 20.002 \text{ [var]} \tag{37}$$

$$P = R\, I^2 \quad \Longrightarrow \quad R = \frac{P_{1o}}{I_{1o}^2} = \frac{9.99392}{4.472^2} = 0.5 \ [\Omega]$$

$$Q = X\,I^2 \quad \Longrightarrow \qquad X = \frac{Q_{1o}}{I_{1o}^2} = \frac{20.002}{4.472^2} = 1\ [\Omega]$$

Exercise 5. Transformer in open circuit ◇

In open circuit, the transformer can be assimilated to the following circuit

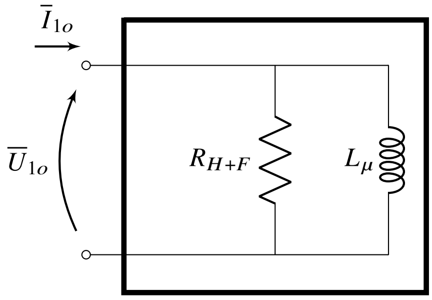


Figure 9: Simplified circuit of a single-phase transformer in open circuit.

Knowing that the input voltage is $\overline{U}_{1o}$ = 230 V (RMS), the frequency is 50 Hz, the resistance R_{H+F} = 1.6 kΩ and the inductance is L_μ = 2 H, compute

1. The active power consumed by the resistance
2. The reactive power consumed by the inductance
3. The apparent power of the total load
4. The power factor
5. The RMS value of the current I_{1o}
6. The phase angle between the voltage and the current (measured from $\overline{I}_{1o}$ to $\overline{U}_{1o}$)

Answers

1. P_{1o} = 33.0625 W
2. Q_{1o} = 84.193 var
3. S_{1o} = 90.452 VA
4. PF = 0.3655
5. I_{1o} = 0.39327 A
6. φ = 68.56 °

2 Three-phase systems

Theory

The **induction phenomenon** is at the center of many AC electic engineering applications. It is implemented to produce, according to the needs, single-phase or poly-phase (prominently three-phase) voltage systems. The induction phenomenon is first presented below in the simplest case : a single-phase generator. Then it is explained how windings and inductors (permanent magnets or electro-magnets) are organised into three-phase systems.

Single-phase generator

The single-phase generator is made of two parts : the **stator** and the **rotor**. The stator is the non-moving part. It is made of an annular magnetic core that carries windings in which alternating (AC) voltages are induced. The rotor is, in this case (see Figure below), a permanent magnet (with a north and a south pole) creating a magnetic field and rotating around a central axis inside the stator magnetic core.

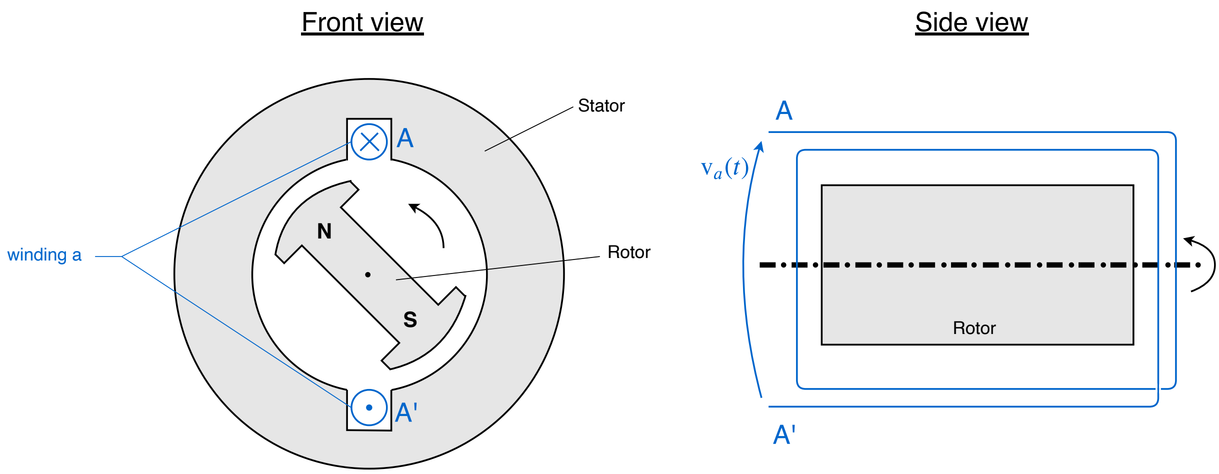
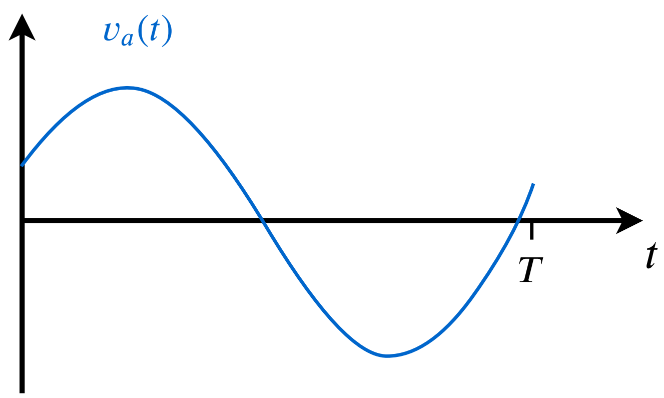


Figure 10: Schematic front and side views of a single-phase synchronous generator.

Due to the rotational motion of the permanent magnet, the stator winding a embraces a time varying magnetic flux $n_a\phi(t)$, where $\phi(t)$ is the magnetic flux embraced by each turn, and n_a the number of turns in the winding. This time-varying flux, in turn, induces a voltage $v_a(t)$ across the accesses of winding a that is given by the Lenz law :

$$v_a(t) = -n_a \underbrace{\frac{d\phi(t)}{dt}}_{\text{varying flux}} \tag{38}$$



In order to obtain a voltage $v_a(t)$ with a frequency $f = 50\,\text{Hz}$, the angular velocity of the rotor must be $N = 60f = 3000\,\text{rpm}$, where the rotational speed of the rotor is expressed in **rounds per minute** (rpm).

Three-phase generator

The purpose of a three-phase generator is to provide not only one voltage $v_a(t)$ with a frequency f, but a system of three harmonic voltages $\{v_a(t), v_b(t), v_c(t)\}$ with all the same frequency f, but shifted in phase by 120° with respect to each other. For this, the stator is now equipped with three independant windings disposed circumferencially along the stator core with a geometrical angular shift of 120°.

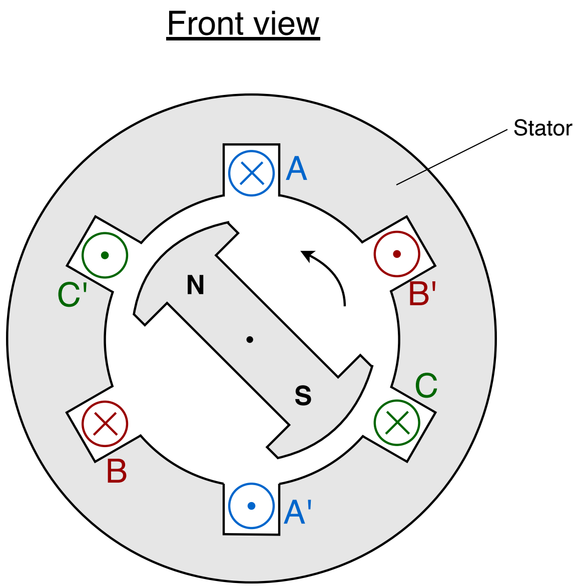


Figure 11: Schematic front view of a three-phase synchronous generator.

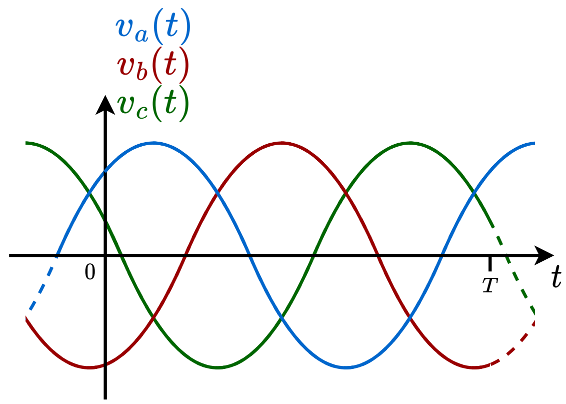


Figure 12: Three-phase signals emerging in the stator windings.

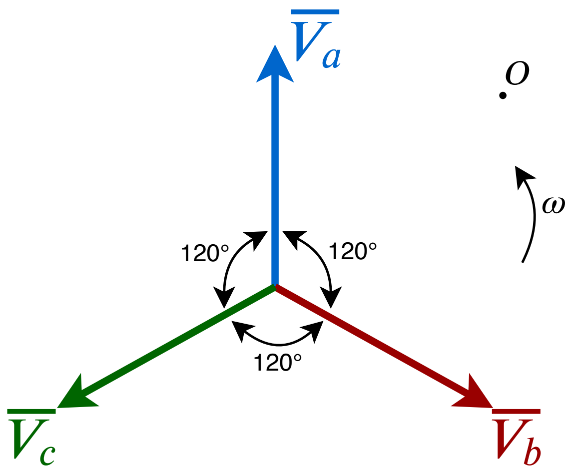


Figure 13: Phasor diagram of the three-phase voltages making a direct sequence.

The three windings of the stator are referred to as **phases**, and respectively noted phase a, phase b and phase c. The voltages induced in the phase windings by the motion of the rotor are depicted in Figure 12. One sees that the geometrical shift of 120° of the windings in space translates, due to the rotation at a constant rotational speed N of the rotor, into a corresponding phase shift in time of 120°.

The induced harmonic voltages also share the same pulsation ω by construction. Its value is proportional to the rotor speed :

$$\omega = 2\pi f \quad , \quad f = \mathrm{p}\frac{N}{60} \tag{39}$$

where the **number of pole pairs** (p) of the machine is here equal to one. As they have the same angular frequency ω, the phase voltages can by represented by three phasors in the complex plane, as depicted in Figure 13. According to the figure, an observer located in O in the complex plane sees the phasors passing in the order a, b, c, and the phasors $\overline{V_a}$, $\overline{V_b}$, $\overline{V_c}$ make a so-called **direct sequence**.

Three-phase circuits

The three voltages produced by the three-phase generator are induced in independent windings that can be connected to an external circuit. For instance, in the figure below, the three voltages are each connected independently to a load (e.g., a resistance).

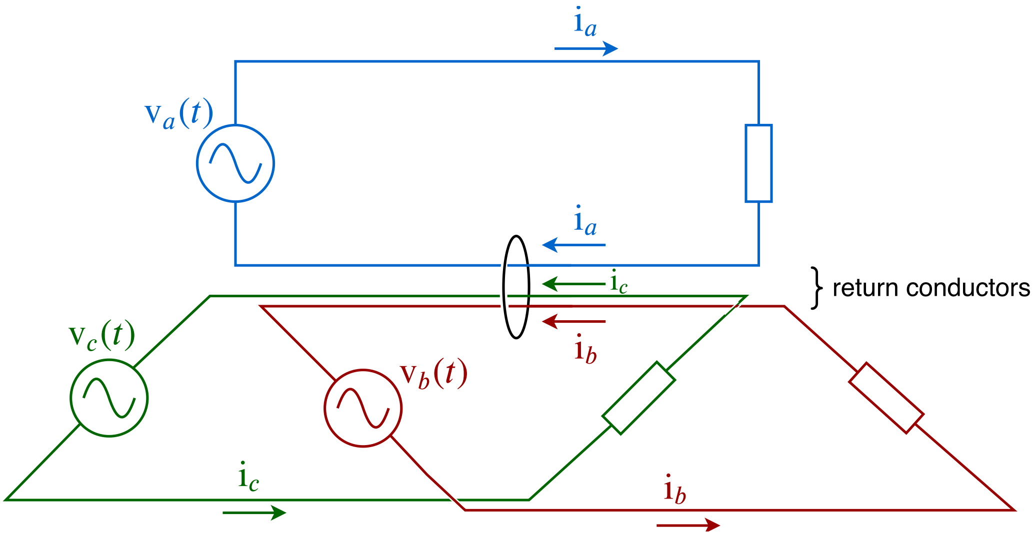


Figure 14: Assembly of 3 single-phase circuits out of phase by 120°

If the three loads are identical, the 3 return conductors carry currents that verify

$$i_a(t) + i_b(t) + i_c(t) = 0 \quad , \quad \forall\, t, \tag{40}$$

so that they can be removed. The circuit then becomes

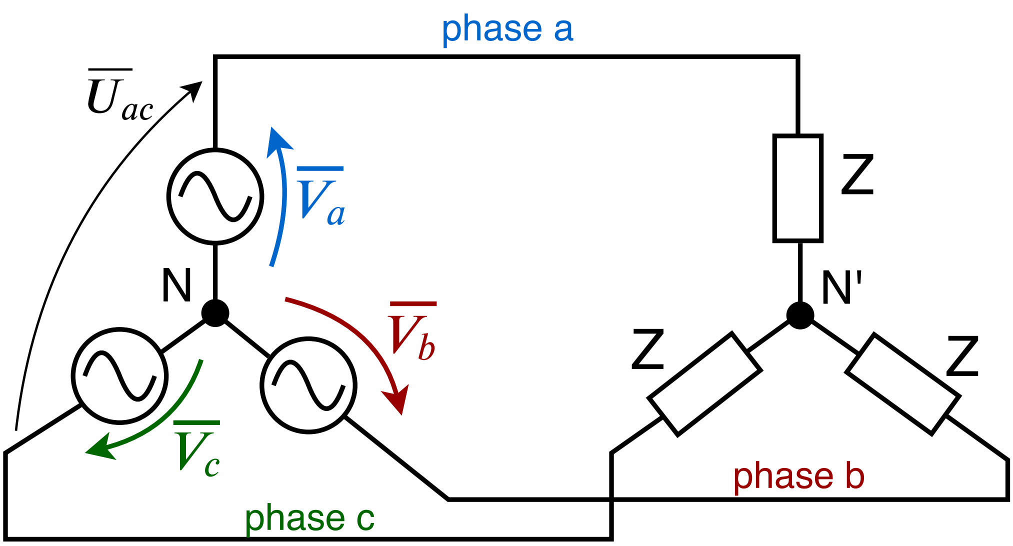


Figure 15: Balanced three-phase circuit.

with a saving of 13% in conductor (Copper) volume (See development below). This saving is clearly economically beneficial in long transmission lines, but also in electrical machines in general. If the voltages and currents have same amplitudes, same frequency and phase differences of 120°, and if the loads Z are identical in each phase, then the three-phase system is said to be **balanced**.

Phase voltages vs. lines voltages

The removal of return conductors has also a less favourable consequence. Without return conductor, one has no longer a convenient access to measure the phase voltages $\{v_a(t), v_b(t), v_c(t)\}$, because the neutral point, noted N in Figure 15, might not be accessible. From this point, one must then differentiate line voltages and phase voltages. In phasor notation, $\overline{V}_a$, $\overline{V}_b$ and $\overline{V}_c$ are the **phase voltages**, measured between a phase and the neutral N (when accessible), whereas $\overline{U}_{ab}$, $\overline{U}_{bc}$ and $\overline{U}_{ca}$ are the **line voltages** (or **phase-to-phase voltages**), measured directly between two phase conductors. The difference in phase and amplitude between phase and line voltages is illustrated, in phasors and algebraically, in the next figure. In particular, one has $U = \sqrt{3}V$, and the phase shift between $\overline{V}_a$ and $\overline{U}_{ab}$, for instance, is 30°.

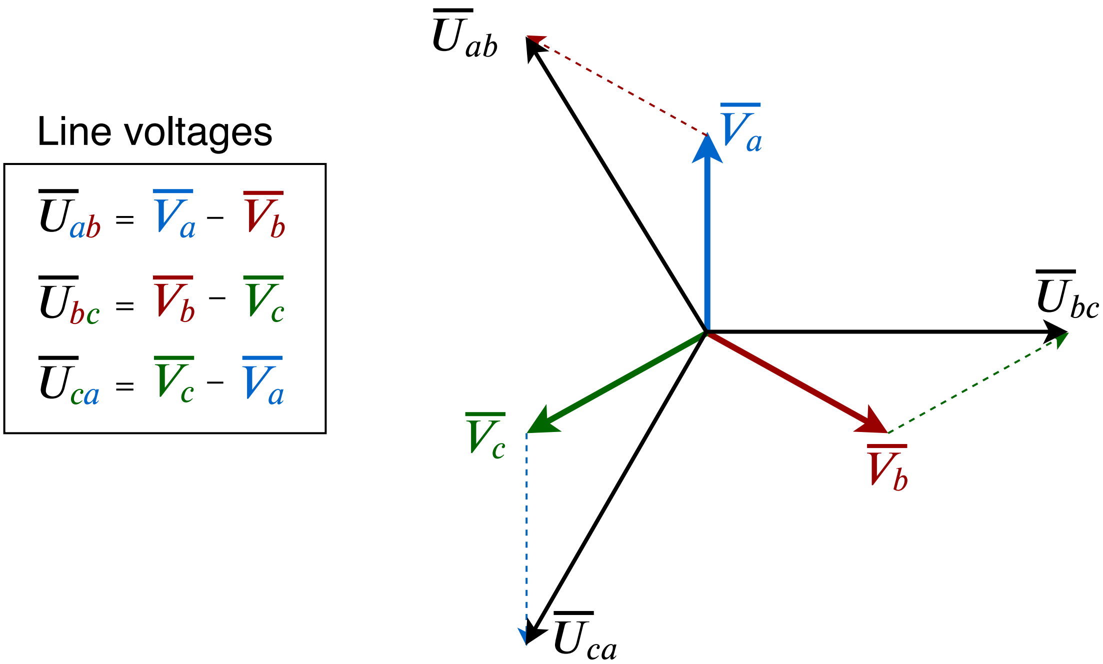
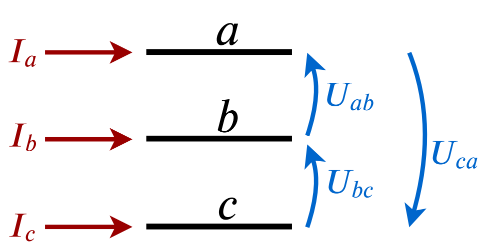


Figure 16: Line and phase voltage phasors in a balanced three-phase system.

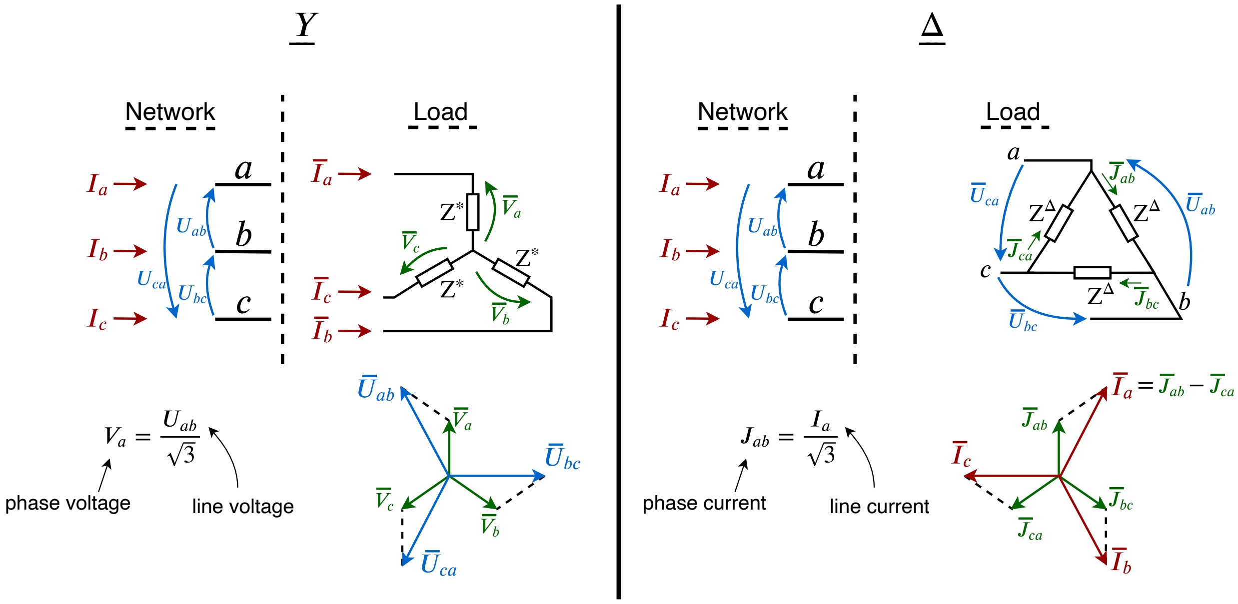


Whenever the three-phase network has not a neutral conductor, one can thus only measure the line voltages $\{\overline{U}_{ab}, \overline{U}_{bc}, \overline{U}_{ca}\}$ and the line currents $\{\overline{I}_a, \overline{I}_b, \overline{I}_c\}$. The three-phase apparent power is then given by

$$\boxed{S = 3\,VI = \sqrt{3}\,UI.} \tag{41}$$

Three-phase loads in Y and Δ connections ("étoile-triangle")

In the balanced circuit presented above, the load is connected in **star** (Y). A balanced circuit can also be obtained if the load is connected in **triangle** (Δ). The circuits, algebraic relations and phasors digrams corresponding to the two types of connection are summarized in the following figure.



Utility of the three-phase transmission lines : save conductor material

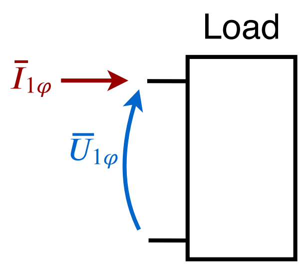
We now show that a balanced three-phase line (without neutral) allows saving a significant volume of conductor material (Cu) with respect to an equivalent single phase line.

Assumptions

1. For heating considerations, the current density in a conductor cannot exceed a given value. The cross-section of each conductor is therefore proportional to the maximal current in the line. One can thus write $A_c = KI$, with K a constant.
2. The number of conductors is $n_{1\varphi} = 2$ for the single-phase line, but only $n_{3\varphi} = 3$ for the three-phase line because return conductors can be removed in the balanced three-phase case.
3. Line voltages are identical : $U_{1\varphi} = U_{3\varphi}$.
4. Maximal transmitted powers are identical : $S_{1\varphi} = S_{3\varphi}$.

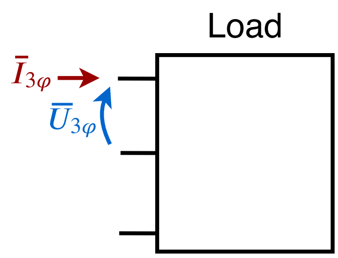
Development

1 Phase



$$S_{1\varphi} = U_{1\varphi}\, I_{1\varphi} \tag{42}$$
$$A_{\text{tot},1\varphi} = n_{1\varphi}\, A_{c,1\varphi} = 2\, K\, I_{1\varphi} \tag{43}$$

3 Phase



$$S_{3\varphi} = \sqrt{3}\, U_{3\varphi}\, I_{3\varphi} \tag{44}$$
$$A_{\text{tot},3\varphi} = n_{3\varphi}\, A_{c,3\varphi} = 3\, K\, I_{3\varphi} \tag{45}$$

Identification of the transmitted (complex) power

$$S_{1\varphi} = S_{3\varphi} \tag{46}$$
$$U_{1\varphi}\, I_{1\varphi} = \sqrt{3}\, U_{3\varphi}\, I_{3\varphi} \tag{47}$$
$$I_{1\varphi} = \sqrt{3}\, I_{3\varphi} \tag{48}$$

shows that the current must be larger by a facor $\sqrt{3}$ in the single-phase conductors. They must therefore be dimensioned with a larger cross-section. One can now evaluate the ratio of the total cross-sections, which is identical to the ratio of the volume of conductor needed :

$$\frac{A_{\text{tot},3\varphi}}{A_{\text{tot},1\varphi}} = \frac{3\,K\,I_{3\varphi}}{2\,K\,I_{1\varphi}} = \frac{\sqrt{3}\,K\,I_{1\varphi}}{2\,K\,I_{1\varphi}} = \frac{\sqrt{3}}{2} \simeq 0.87 \tag{49}$$

and see that transmitting the same amount of power under the same phase voltage is done with a saving of 13% in volume of conductors when using a three-phase line instead a single-phase line.

Exercise 6. Electric heater

Consider an electric heater that dissipates 15 kW of power when connected to a three-phase power system of 208 V. As a first approximation, the heater is modelled as a purely resistive three-phase load.

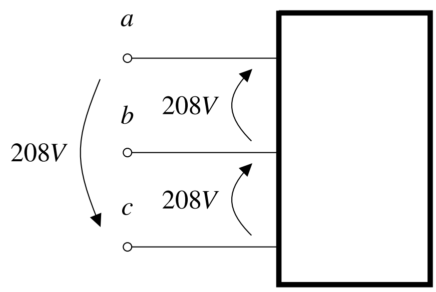
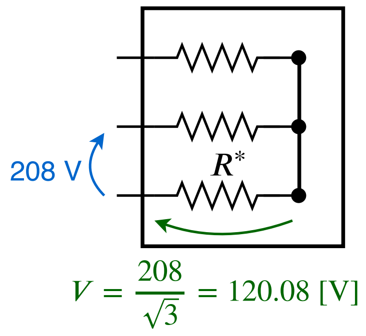


Figure 17: Three-phase Electrical Heater

1. If no additionnal information is provided about the voltage, does the 208 V correspond to the peak or the RMS value ?
2. Compute the line current if the resistive loads are connected in **Y**.
3. If the resistors are connected in **Y**, compute the resistance of each.
4. Compute the line current if the resistive loads are connected in Δ.
5. If the resistors are connected in Δ, compute the resistance of each.

Solution

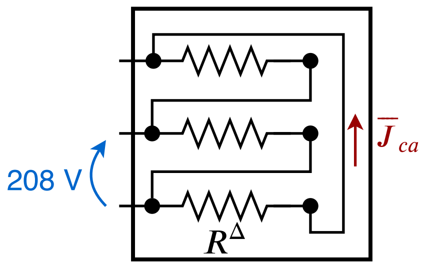
1. The value of 208 V corresponds to an RMS value of the line voltages (phase-to-phase). The peak value is $\sqrt{2} \times 208 = 294.1V$
2. The power of the three-phase system is $P_{3\varphi} = 15$ kW, then, the power of one phase is $P_{1\varphi} = \frac{P_{3\varphi}}{3} = 5$ kW. The line current is
$$I = \frac{P_{1\varphi}}{V} = \frac{5000}{120.08} = 41.635 \text{ [A]} \tag{50}$$



3. If the resistors are connected in star configuration, their value must be
$$R^* = \frac{V_{a,b,c}}{I_{a,b,c}} = \frac{120.08}{41.635} = 2.884 \quad [\Omega] \tag{51}$$
4. Even if the resistors are connected in Δ, in this case, the line currents remain the same because the power consumption and voltages are kept constant : $I = 41.635$ [A]
Remark that the current through the loads is different now : $J = \frac{I}{\sqrt{3}} = 24.038$ [A]

5. If the resistors are known to be connected in Δ, the resistance of each is

$$R^{\Delta} = \frac{U}{J} = \frac{208}{24.038} = 8.6528 \quad [\Omega] \tag{52}$$



Also remark that $Z^{\Delta} = 3\ Z^{*}$ which is the expected results if the two loads consume the same amount of power.

Exercise 7. Line voltages definition based on phase voltages ⋄

From the knowledge of the phase voltages (expressed with phasors),

1. Provide the development that leads to the line voltages
2. Retrieve the expression of the line voltages in phasor, simplify the obtained complex number.

3 Magnetic circuits and transformers

Theory

Magnetic circuits are arrangements of coils, prismatic magnetic cores concentrating the magnetic flux, air gaps and permanent magnets. They can be described in good approximation by circuit equations, very similar to the Kirchhoff laws of electric circuits, which are established in this section.

Let Σ be a smooth oriented surface with normal vector $\vec{n}$. The boundary of Σ is the oriented closed contour C. The consistency of the orientations of the surface Σ and its boundary C is verified with the right hand rule. Now, a direct consequence of Ampere's law is

$$\oint_C \vec{h} \cdot d\vec{L} = \mathcal{F} \tag{53}$$

where the left-hand side is the circulation of the magnetic field $\vec{h}$ along the closed contour C, and the right-hand side $\mathcal{F}$ denotes the magnetomotive force (m.m.f), which is by definition to the algebraic sum of the currents crossing *any* smooth surface bounded by the contour C. The current is counted positively if it crosses the surface in the direction of its normal vector $\vec{n}$, and negatively otherwise. In case of a winding with n turns, the magnetomotive force is $\mathcal{F} = nI$, as all the turns of a winding are conected in series and therefore carry the same current.

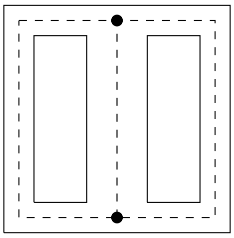


Figure 18: Magnetic circuit with two nodes and two independent loops.

The magnetic circuit depicted in Figure 18 is now considered. The idea of the the magnetic circuit approach is to split the circuit into sections, of length L_k and constant cross-section S_k, where the magnetic field $\vec{h}$ can reasonably be assumed homogeneous (i.e., the same everywhere in the section). A contour C to apply (53) is then the union of the medial axis of N_i successive sections that form the i^{th} closed loop in the circuit. There are two independent closed contours in Figure 18, presented with dashed lines. One also note that the vector $\vec{h}$ is reasonably parallel to the medial axis of each section, and therefore to the contour C. Under these assumptions, the integral in (53) can be replaced in good approximation, for each loop of the circuit, by a sum

$$\mathcal{F}_i = \sum_{k=1}^{N_i} h_k L_k \tag{54}$$

where h_k is the norm of $\vec{h}$ in the k^{th} section, and L_k the length of the section. If one defines $\mathcal{F}_k = h_k L_k$, one has

$$\boxed{\mathcal{F}_i = \sum_{k=1}^{N_i} \mathcal{F}_k} \tag{55}$$

which is the **second Kirchhoff law** for magnetic circuits. In contrast to the voltage law in electric circuits, the $\mathcal{F}_k$'s, which are analogous in magnetic circuits to the voltages in electric circuits, do not sum up to zero but to the magneto-motive force of the loop $\mathcal{F}_i$.

Under the same assumptions as above, each section also carries a magnetic flux ϕ_k defined by

$$\phi_k = b_k S_k \tag{56}$$

where b_k is the norm of $\vec{b}$ in the k^{th} section, and S_k the cross-section of the section. A consequence of the Maxwell law $\operatorname{div} \vec{b} = 0$ is that the sum of the flux crossing any closed surface is zero. Applying this to a closed surface enclosing a node of the magnetic circuit (black dots in Figure 18), one obtains the **first Kirchhoff law** for magnetic circuits

$$\boxed{\sum_{k=1}^{N_j} \phi_k = 0} \tag{57}$$

where N_j is the number of sections incident to the node. The flux is counted positively if it crosses the surface in the direction of its normal vector $\vec{n}$, and negatively otherwise.

It remains to relate b_k and h_k in the different kinds of sections. One has

$$\begin{aligned}
\text{Air gap:} \quad & b_k = \mu_0 h_k \\
\text{Magnetic core:} \quad & b_k = \mu_0 \mu_r h_k \\
\text{Permanent magnet:} \quad & b_k = \mu_0 (h_k - h_c)
\end{aligned}$$

where $\mu_0 = 4\pi \times 10^{-7}$[H/m] is the magnetic permeability of empty space (or air), μ_r the relative permeability of the core material (a number without dimension between 10^3 and 10^5, typically 2000 for iron or steel), and h_c[A/m] the coercive field of the permanent magnet ($h_c = b_r/\mu_0$ where b_r is about 1 Tesla in a SmCo magnet).

Each term of the sum in (54) can now be expressed in terms of the flux ϕ_k

$$\begin{aligned}
\text{Air gap:} \quad & \mathcal{F}_k = \frac{L_k}{\mu_0 S_k} \phi_k = \mathcal{R}_k \phi_k \\
\text{Magnetic core:} \quad & \mathcal{F}_k = \frac{L_k}{\mu_0 \mu_r S_k} \phi_k = \mathcal{R}_k \phi_k \\
\text{Permanent magnet:} \quad & \mathcal{F}_k = \frac{L_k}{\mu_0 S_k} \phi_k - h_c L_k = \mathcal{R}_k \phi_k - \mathcal{F}_k^{\text{PM}}
\end{aligned}$$

where we have defined the **reluctance** of a section

$$\boxed{\mathcal{R}_k = \frac{1}{\mu_k} \frac{L_k}{S_k}} \tag{58}$$

(Note the analogy with Pouillet's law $R = \frac{1}{\sigma} \frac{L}{S}$ to express the resistance of a conductor), and the magneto-motive force due to the permanent magnets

$$\mathcal{F}_k^{\text{PM}} = h_c L_k. \tag{59}$$

Single-phase transformer

A single-phase transformer is a made of a magnetic core carrying two windings called **primary** and **secondary**. Figure 19 indcates positive references for voltage and currents. Note that the passive convention is used at the primary side, and the active convention at the secondary. It has been shown in the theoretical lecture how transformers can be represented by a T-shaped 2-port circuit with a primary access (AB in the Figure) and a secondary access (CD). Various equivalent 2-port circuits are available to represent transformers with different levels of accuracy and sophistication. They are reviewed in this section.

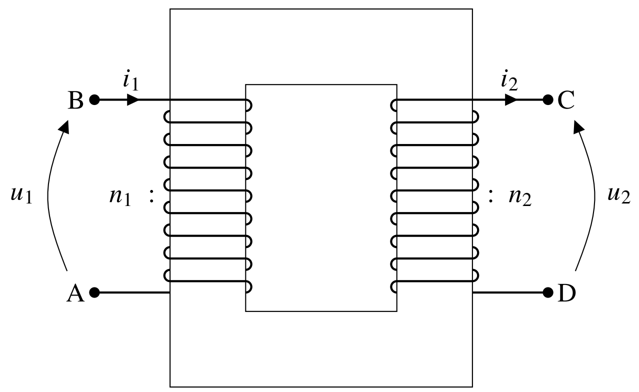


Figure 19: Single-phase transformer.

Ideal transformer

The ideal transformer obeys the very simple equations:

$$\boxed{\begin{aligned} \overline{U}_2 &= n\,\overline{U}_1 \\ \overline{I}_2 &= \frac{\overline{I}_1}{n} \end{aligned}}$$

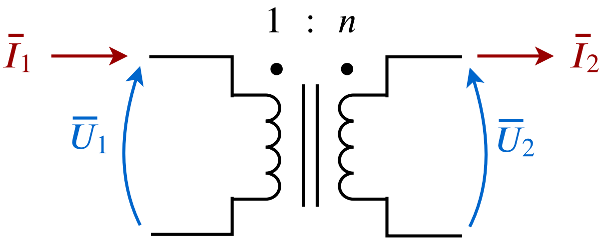


Figure 20: Model of the ideal transformer

where $n = n_2/n_1$ is the **transformation ratio**. Note that the transformation ratio can also be defined $n = n_1/n_2$, in which case, n must be replaced by $1/n$ in the above equations. As the ideal transformer model does not involve any losses, the complex power is conserved, and one can write successively :

$$S_1 = S_2$$
$$\bar{U}_1 \bar{I}_1^* = \bar{U}_2 \bar{I}_2^*$$
$$P_1 + j\, Q_1 = P_2 + j\, Q_2$$

Moving components from side to side of the transformer

The equations governing the ideal transformer being fairly simple, it is often convenient, for the sake of conciseness, to remove it from the circuit. The circuit quantities from one side of the ideal transformer are replaced by equivalent quantities directly connected to the other side of the ideal transformer, which can then be removed from the circuit.

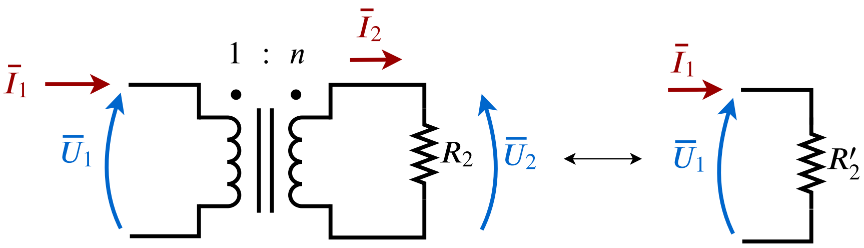


Figure 21: Secondary resistance R_2 and its equivalent R_2' seen from the primary.

For instance, in the figure above, the ideal transformer and the secondary resistance R_2 are replaced by the equivalent resistance R_2' directly connected in the primary circuit. The value of the equivalent resistance R_2' is calculated as follows. One has

$$R_2 = \frac{\bar{U}_2}{\bar{I}_2} \quad , \quad R_2' = \frac{\bar{U}_1}{\bar{I}_1} \tag{60}$$

and by using the equations of the ideal transformer,

$$R_2' = \frac{\frac{\bar{U}_2}{n}}{n\,\bar{I}_2} = \frac{\bar{U}_2}{\bar{I}_2}\,\frac{1}{n^2} \tag{61}$$

to finally obtain

$$\boxed{R_2' = \frac{R_2}{n^2}} \tag{62}$$

which is called the secondary resistance R_2 *seen* from the primary.

Real transformer

A more realistic transformer model, including losses, is depicted in Figure 22. The different circuit elements stand for the following physical phenomenon :

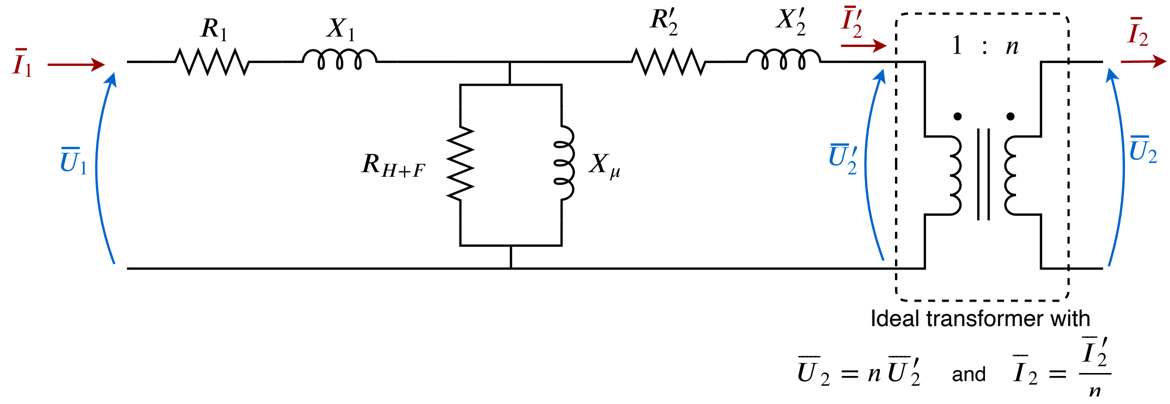


Figure 22: Electric model of a real transformer, including losses.

R_1 : Joule losses in the primary winding

X_1 : Leakage flux of the primary winding

R_2' : Joule losses in the secondary winding, seen from the primary

X_2' : Leakage flux of the secondary winding, seen from the primary

X_μ : Magntic flux exchanged between primary and secondary windings

R_{H+F} : Magnetic losses in the core

In practice, transformers are dimensioned to minimize losses and the undesired effects, so that one has

$$R_1 << R_{H+F} \quad ; \quad R_2' << R_{H+F} \quad ; \quad X_1 << X_\mu \quad ; \quad X_2' << X_\mu \tag{63}$$

These assumptions will be used in the exercises. The next question is to identify the parameters of the equivalent circuit. This can be done in practice with to specific experimental tests : the no-load (open circuit) test and the short circuit test.

No-load test

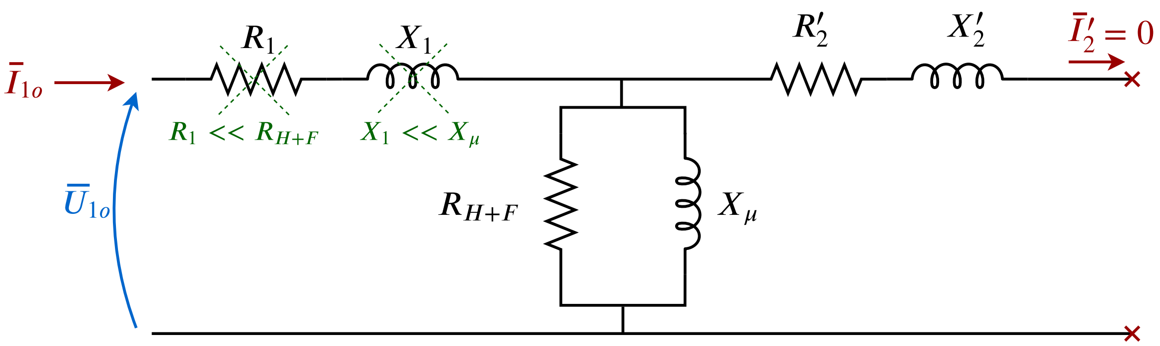


Figure 23: Equivalent circuit during of the real transformer in no-load test.

In the no-load test, the secondary of the transformer is left open, and the nominal voltage is applied at the primary. If one neglects R_1 and X_1 which are small with respect to R_{H+F} and X_μ, the latter can be identified from the measurement of the active power P_{1o} and the reactive power Q_{1o} delivered to the transformer during the no-load test as follows :

$$S_{1o} = Y^* U_{1o}^2 = \left(\frac{1}{R_{H+F}} + j\,\frac{1}{X_\mu}\right) U_{1o}^2$$
$$P_{1o} = G\ U_{1o}^2 = \frac{U_{1o}^2}{R_{H+F}}$$
$$Q_{1o} = -B\, U_{1o}^2 = \frac{U_{1o}^2}{X_\mu}$$

and finally

$$\boxed{\begin{aligned} R_{H+F} &= \frac{U_{1o}^2}{P_{1o}} \quad (64)\\ X_\mu &= \frac{U_{1o}^2}{Q_{1o}} \quad (65)\end{aligned}}$$

Short circuit test

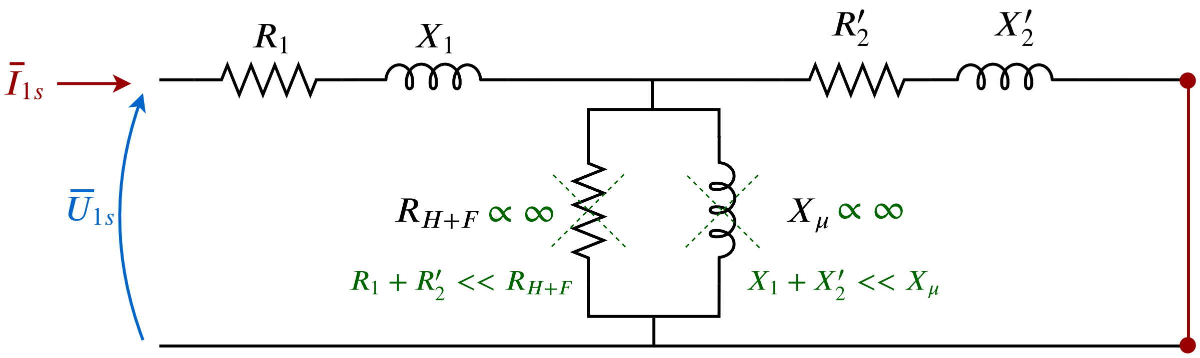


Figure 24: Equivalent circuit during the short circuit test.

In the short circuit test, the secondary winding is short circuited. As the winding resistances and leakage reactances are small impedances, a reduced primary voltage is applied, usually ~ 10% of the nominal voltage. By measuring the active and reactive powers, it is possible with this test to identify $R_1 + R_2'$ and $X_1 + X_2'$ as follows :

$$S_{1s} = Z\, I_{1s}^2 = (R_1 + jX_1 + R_2' + jX_2')\, I_{1s}^2$$
$$P_{1s} = R\, I_{1s}^2 = (R_1 + R_2')\, I_{1s}^2$$
$$Q_{1s} = X\, I_{1s}^2 = (X_1 + X_2')\, I_{1s}^2$$

and finally

$$\boxed{\begin{aligned} R_1 + R_2' &= \frac{P_{1s}}{I_{1s}^2} \quad (66)\\ X_1 + X_2' &= \frac{Q_{1s}}{I_{1s}^2} \quad (67)\end{aligned}}$$

where P_{1s} and Q_{1s} are the active and reactive powers delivered to the transformer during the short-circuit test.

Simplified transformer model

Under the reasonable conditions

$$R_1', R_2' << R_{H+F} \quad (68)$$
$$X_1', X_2' << X_\mu \quad (69)$$

the secondary quantities can be moved to the primary, or vice versa, with a very limited inaccuracy. This yields the simplified equivalent circuits for transformers that are used in practice (See figures). This simplfication is convenient because the short-circuit test does not allow identifying independently primary and secondary quantities (as they appear in the equivalent circuit of Figure 22), but only the sums $R_1 + R_2'$ and $X_1 + X_2'$, as they appear explicitly in the simplified equivalent circuits.

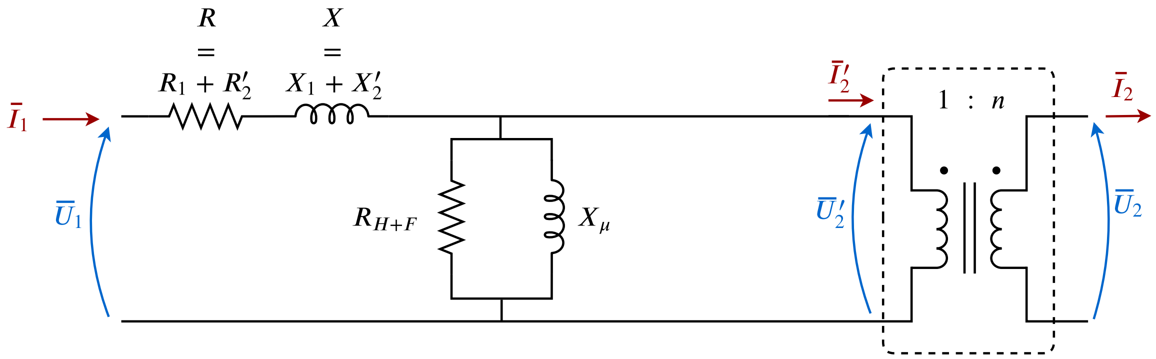


Figure 25: Circuit of transformer simplified to the primary.

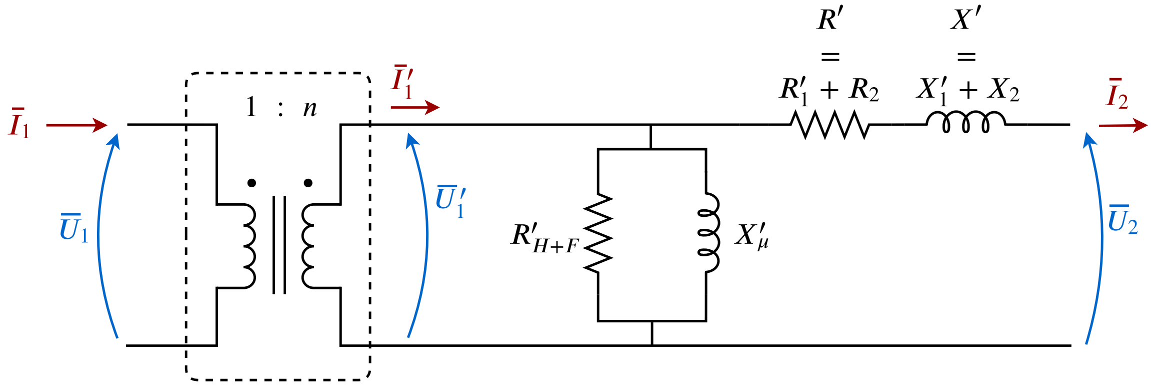


Figure 26: Circuit of transformer simplified to the secondary.

Exercise 8. Reluctance computation

Consider an inductor made of an iron core (as described in Figure 27) and a 60 turns winding, wounded around the central leg.

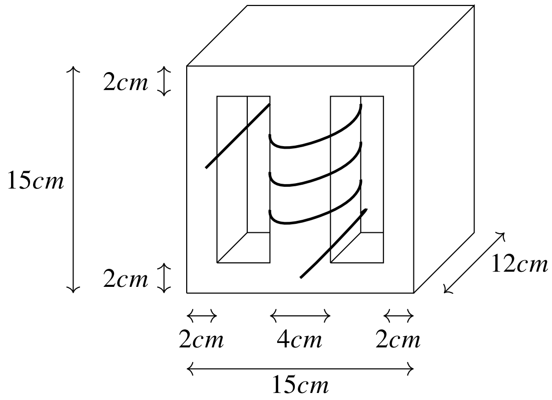
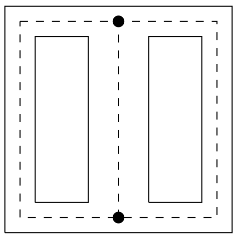


Figure 27: Magnetic circuit of the inductor

1. Draw an equivalent magnetic circuit of the inductor;
2. Compute the total reluctance of this circuit, considering a relative permeability μ_r of 1500 for the iron. Deduce the inductance from it;
3. Do the same computation as in the previous steps, but now considering a constant air gap of $0.1mm$ in each leg;

Solution

The mean path used by the magnetic flux can be represented as



1. Decomposing the reluctance circuit into separated segments : $\mathcal{R}_{side}$ and $\mathcal{R}_{cent}$ which correspond to the side legs and central leg, the magnetic circuit of the inductor can be represented as

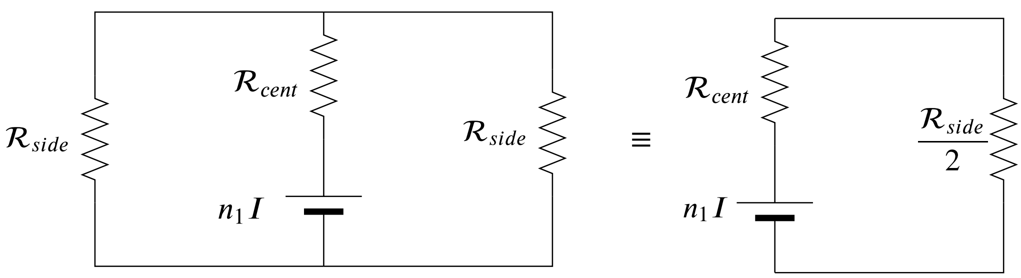


Figure 28: Equivalent magnetic circuit of the inductor.

where $n_1 = 60$ is the number of turns of the inductor.

2.

$$\begin{aligned}
\mathcal{R}_{side} &= \frac{1}{\mu} \cdot \frac{l_{side}}{S_{side}} = \frac{1}{\mu_0 \cdot \mu_r} \cdot \frac{l_{side}}{S_{side}} \\
&= \frac{1}{4\pi \cdot 10^{-7} \cdot 1500} \cdot \frac{2 \cdot (0.075 - \frac{0.02}{2}) + 0.13}{0.02 \cdot 0.12} \\
&= 57\,472 \quad \mathrm{H}^{-1}
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}_{cent} &= \frac{1}{\mu} \cdot \frac{l_{cent}}{S_{cent}} = \frac{1}{\mu_0 \cdot \mu_r} \cdot \frac{l_{cent}}{S_{cent}} \\
&= \frac{1}{4\pi \cdot 10^{-7} \cdot 1500} \cdot \frac{0.13}{0.04 \cdot 0.12} \\
&= 14\,638 \quad \mathrm{H}^{-1}
\end{aligned}$$

Finally, as shown on the equivalent circuit, the total reluctance of the magnetic component is

$$\boxed{\mathcal{R}_{tot} = \mathcal{R}_{cent} + \frac{\mathcal{R}_{side}}{2} = 43\,104 \quad \mathrm{H}^{-1}}$$

Knowing the total reluctance, the related inductance can be deduced:

$$\boxed{L = \frac{n_1^2}{\mathcal{R}_{tot}} = 83.5 \quad \mathrm{mH}} \tag{70}$$

3. The explanation will be the same as before but with a gap of 0.1 mm in each leg added. Remember that in the gap, the relative permeability μ_r is equal to 1 because gaps are filled with air. Furthermore, as the section is two times larger in the central part than on the side part, the reluctance on the central leg will be two times smaller than on the side legs.

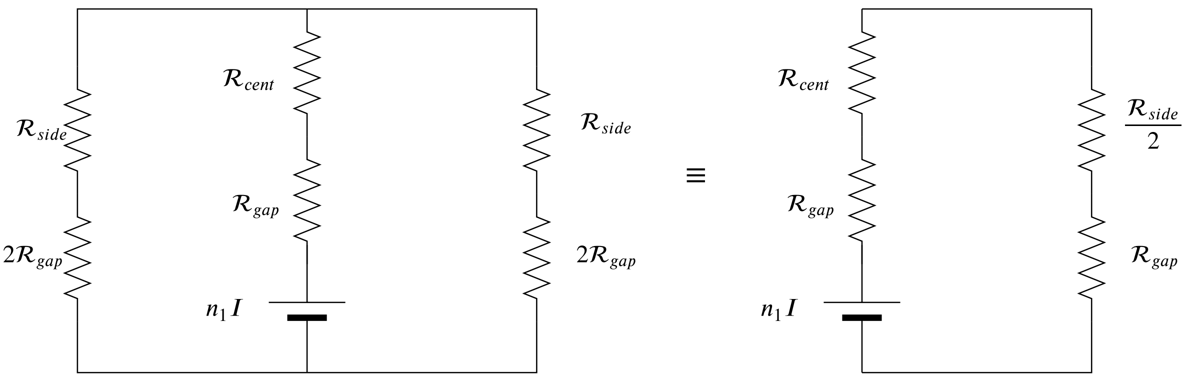


Figure 29: Equivalent magnetic circuit with air gaps.

$$\begin{aligned}
\mathcal{R}_{gap} &= \frac{1}{\mu} \cdot \frac{l_{gap}}{S_{gap}} \\
&= \frac{1}{4\pi \cdot 10^{-7} \cdot 1} \cdot \frac{0.0001}{0.04 \cdot 0.12} \\
&= 16\,579 \quad H^{-1}
\end{aligned}$$

Finally, as shown on the figure,

$$\mathcal{R}_{tot} = \mathcal{R}_{cent} + \frac{\mathcal{R}_{side}}{2} + 2 \cdot \mathcal{R}_{gap} = 76\,262 \quad H^{-1}$$

$$\boxed{L = \frac{n_1^2}{\mathcal{R}_{tot}} = 47.2 \quad mH}$$

Exercise 9. Three-phase transformer

Three-phase power transformers are commonly used to adapt power line voltages and to provide some galvanic insulation between two parts of an electrical grid. The three-phase transformer, described by the normalized scheme in Fig. 30, is connected to a balanced three-phase network of composed voltages u_{AB}, u_{BC}, u_{CA} of Root Mean Square (RMS) voltage U_1 on the primary side, whereas on the secondary side, a three-phase balanced system of composed voltages u_{ab}, u_{bc}, u_{ca} of RMS voltage U_2 is obtained.

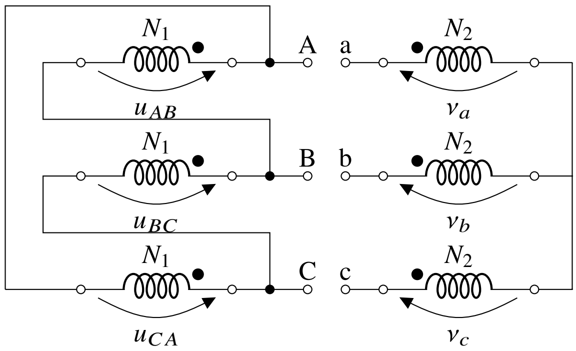


Figure 30: Three-phase transformer.

The line current intensities in the primary and secondary windings are respectively denoted I_1 and I_2. The transformer has the following characteristics:

- Apparent nominal power $S_n = 250\,\mathrm{kVA}$;
- Composed primary winding RMS voltages $U_{1n} = 5.2\,\mathrm{kV}$;
- Nominal frequency $f_n = 50\,\mathrm{Hz}$;

and ferromagnetic losses are neglected. To characterize the transformer two tests have been performed:

- Using open secondary windings, the transformer generates a composed voltage of RMS value $U_{2o} = 400\,\mathrm{V}$ at each secondary winding, for an applied composed nominal voltage of RMS value U_{1n};
- Using short-circuited secondary windings, a composed voltage of RMS value $U_{1s} = 600\,\mathrm{V}$ is applied at each primary winding for a total primary power $P = 7.35\,\mathrm{kW}$, producing line current of RMS intensity $I_{2s} = 350\,\mathrm{A}$.

1. Calculate the transformer ratio n so that it is greater than 1;
2. For the first test condition (open secondary windings), draw a Fresnel diagram including the primary composed voltages u_{AB}, u_{BC}, u_{CA}, the direct secondary voltage v_a, v_b, v_c and the secondary composed voltages u_{ab}, u_{bc}, u_{ca};
3. Express and compute the column ratio $n_c = \frac{N_1}{N_2}$ according to n;
4. Given that the transformer is composed of 3 cores of section $A_c = 5\,\mathrm{dm}^2$, and that the magnetic field amplitude is $B_m = 1.2\,\mathrm{T}$, compute the number of turns N_1 of each primary winding and deduce the value of the number of turns of each winding N_2;
5. Using a simple single-phase equivalent model (leak resistance and inductance moved to the secondary windings), provide the Thévenin's model seen from a secondary winding and calculate the resistance R_{eq} (R') and the reactance X_{eq} (X') of this model;

The nominal regime is now considered by applying the composed nominal voltage U_{1n} at the primary windings and connecting a three-phase balanced load on the secondary side (detailed in Fig. 31). Each branch is composed of a resistor of value $R^* = 554\,\mathrm{m}\Omega$ in series with a coil of value $L^* = 3.05\,\mathrm{mH}$. From now on, the magnetizing branch of the transformer will be neglected for the following calculations.

6. Calculate the power factor $\cos\varphi_2$ of this load;
7. Draw the Fresnel diagram corresponding to the balanced single-phase equivalent model. Deduce the RMS values of the current intensities I_2 and the composed voltages U_2.;

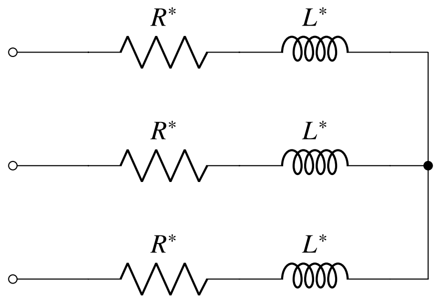


Figure 31: First load connected to the secondary side of the three-phase transformer.

8. Compute the active power P_2 flowing from the transformer to the load;
9. Calculate the transformer efficiency η;
10. Another load is used (Fig. 32), compute the value of the resistance R° and the inductance L° such that this load is equivalent to the one detailed in Fig. 31.

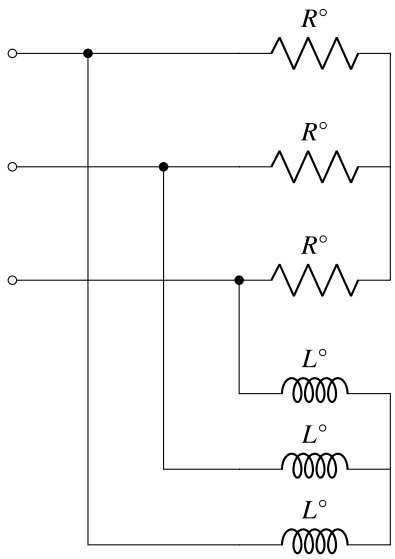


Figure 32: Second load connected to the secondary side of the three-phase transformer.

Solution

1. The transformer ratio is

$$\boxed{n = \frac{U_{AB}}{U_{ab}} = \frac{U_{1n}}{U_{2o}} = \frac{5\,200}{400} = 13}$$

2. The primary windings are connected in Δ (delta) configuration while secondary windings are connected in star.
The three phase transformer is composed of three columns and on each column there are two windings. One

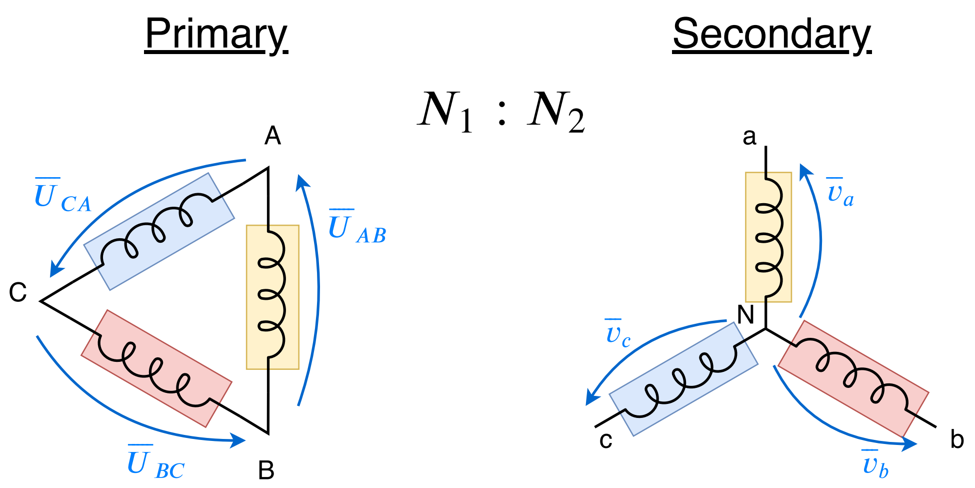


Figure 33: Primary and secondary configuration of the three-phase transformer

winding is the primary and the second is the secondary. In this case, the winding between nodes A and B is on the same column as the winding between nodes a and N such that :

$$\overline{v}_a = \frac{N_2}{N_1} \cdot \overline{U}_{AB}$$

$$\overline{v}_b = \frac{N_2}{N_1} \cdot \overline{U}_{BC}$$

$$\overline{v}_c = \frac{N_2}{N_1} \cdot \overline{U}_{CA}$$

where N_1 and N_2 are respectively the number of turns in the primary and the secondary. Figure 34 shows the

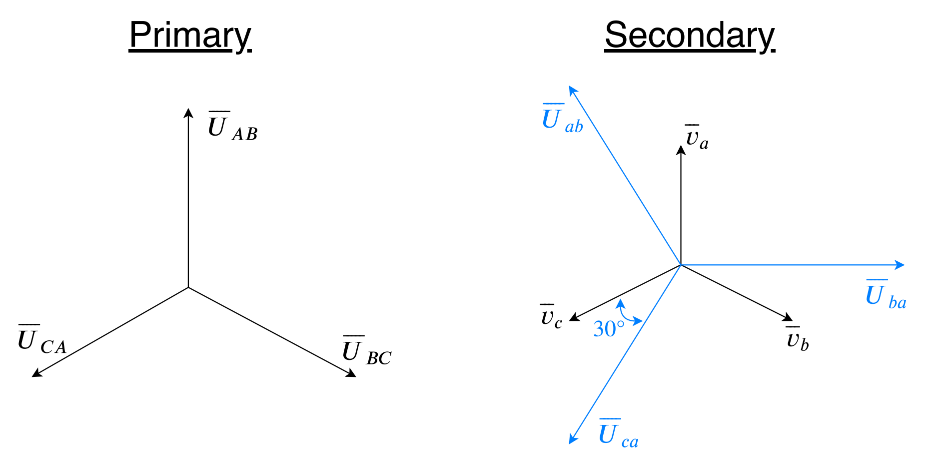


Figure 34: Fresnel diagram of the three-phase transformer.

Fresnel diagram of the three-phase transformer.

Finally,

$$U_{ab} = \frac{N_2}{N_1}\sqrt{3}\cdot U_{AB} = \frac{U_{AB}}{n}$$

3. The column ratio can be obtained from

$$n_c = \frac{N_1}{N_2} = \frac{U_{AB}}{v_a} \tag{71}$$

As

$$n = \frac{U_{AB}}{\sqrt{3}\, v_a}$$

$$\boxed{n_c = \sqrt{3}\, n = 22.517}$$

4. The number of turns N_1 at the primary depends on the value of the magnetic induction that is acceptable in the transformer. The maximum magnetic induction is $B_m = 1.2$ [T] and the cross-section of one leg of the three-phase transformer is $A_c = 0.05\ \text{m}^2$. One can therefore express the maximum magnetic flux in the transformer as

$$\phi_m = B_m\, A_c \tag{72}$$

and according to Lenz law,

$$\begin{aligned} U_1(t) &= N_1\, \frac{d}{dt}(\phi_1(t)) \\ U_{AB}\,\sqrt{2}\,\cos(\omega\, t) &= N_1\, \frac{d}{dt}(\phi_m\,\sin(\omega\, t)) \\ U_{AB}\,\sqrt{2}\,\cos(2\pi\, f\, t) &= N_1\, 2\pi\, f\, \phi_m\,\cos(2\pi\, f\, t) \end{aligned}$$

where U_{AB} is the RMS value of the voltage and B_m is the peak value of the magnetic induction.

$$\boxed{N_1 = \frac{\sqrt{2}\, U_{AB}}{2\pi\, f\, B_m\, A_c} = \frac{\sqrt{2}\cdot 5200}{2\pi\cdot 50\cdot 1.2\cdot 0.05} = 390} \tag{73}$$

Then, as the column ratio n_c is equal to $\frac{N_1}{N_2}$,

$$\boxed{N_2 = \frac{N_1}{n_c} = \frac{390}{22.517} = 17} \tag{74}$$

Remark that

$$B_m = \frac{\sqrt{2}\, U}{2\pi\, f\, N\, A_c} \tag{75}$$

the magnetic induction is directly proportional to the voltage and is inversely proportional to the frequency, number of turns and the cross-section area.

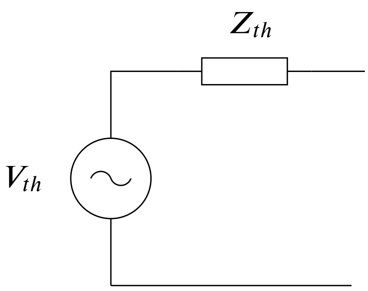


Figure 35: Thevenin equivalent.

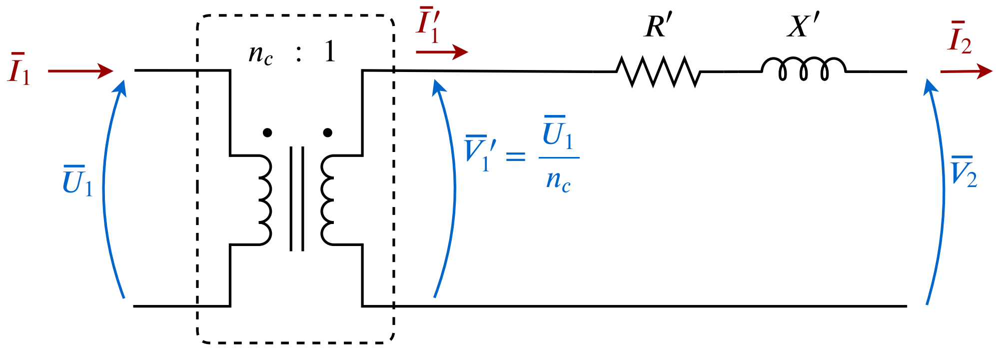


Figure 36: Simplified transformer model for one phase.

5. The Thevenin equivalent circuit is shown in Figure 35. As a reminder, V_{th} is equal to the voltage at no load of the circuit while Z_{th} is the circuit impedance when all voltage sources are shorted.

   The simplified model of one-phase transformer (with components moved to the secondary) is shown in Figure 36. By definition of the Thevenin equivalent circuit,

$$\boxed{V_{th} = V_2 = \frac{U_2}{\sqrt{3}} = 230.94 \quad \text{V}} \tag{76}$$

$$Z_{th} = R' + jX'$$

   where R' and X' will be computed based on the short circuit measurements :

$$\boxed{R' = \frac{P_{2s}}{I_{2s}^2} = \frac{\frac{7350}{3}}{350^2} = 20 \quad \text{m}\Omega} \tag{77}$$

   and with P_{2s} which is the one phase active power during the short circuit test.

   The same reasoning will be made for the calculation of X' but this time with the reactive power Q_{2s}, consumed during the short circuit test.

$$U_{2s} = \frac{U_{1s}}{n} = \frac{600}{13} = 46.15 \quad \text{V}$$

$$S_{2s} = V_{2S}\ I_{2s} = \frac{U_{2S}}{\sqrt{3}}\ I_{2s} = 9\,326.26 \quad \text{VA}$$

$$\rightarrow Q_{2s} = \sqrt{S_{2s}^2 - P_{2s}^2} = 8\,998.7 \quad \text{var}$$

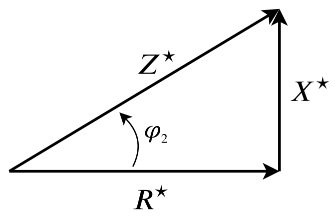
which finally leads to

$$\boxed{X' = \frac{Q_{2s}}{I_{2s}^2} = 73.4 \quad \text{m}\Omega} \tag{78}$$

6. Each branch of the three-phase load is composed of a resistor R^* (554 mΩ) in series with an inductance L^* (3.05 mH).

   First, let us compute the reactance X^* such that :

$$X^* = 2\pi\, f\, L^* = 958.2 \quad \text{m}\Omega \tag{79}$$



$$\varphi_2 = \arctan\left(\frac{X^*}{R^*}\right) = 59.96°$$

$$\boxed{\cos(\varphi_2) = 0.5005} \tag{80}$$

An other solution is :

$$\cos(\varphi_2) = \frac{R^*}{\sqrt{(R^*)^2 + (X^*)^2}}$$

7. In nominal regime, when the three-phase load is connected to the secondary of the transformer, the Fresnel diagram is as following :

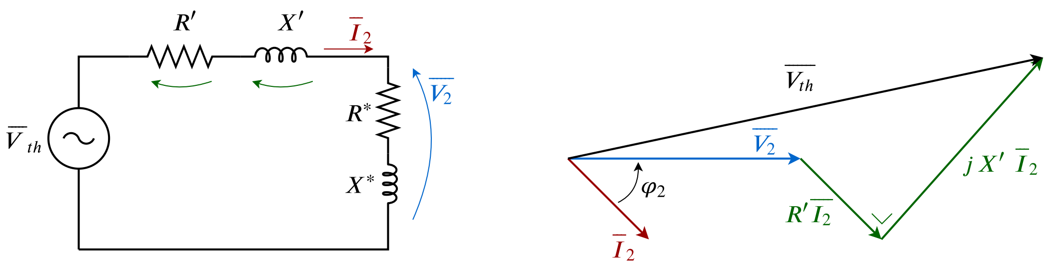


Figure 37: Equivalent circuit and Fresnel diagram during the nominal regime.

From this diagram, the current $\overline{I_2}$ and voltage $\overline{V_2}$ can be calculated as

$$\overline{I_2} = \frac{V_{th}}{Z_{th} + R^* + jX^*}$$

$$\boxed{I_2 = \frac{V_{th}}{\sqrt{(R' + R^*)^2 + (X' + X^\star)^2}} = 195.7 \quad \text{A}} \tag{81}$$

and the voltage is

$$\overline{V_2} = (R^* + jX^*)\,\overline{I_2}$$

$$\boxed{V_2 = \sqrt{(R^*)^2 + (X^*)^2}\; I_2 = 216.58 \quad \text{V}} \tag{82}$$

8. The active power consumed by the load is

$$\boxed{P_2 = V_2\ I_2\ \cos\varphi_2 = 216.58 \cdot 195.7 \cdot 0.5 = 21.219 \quad \text{kW}} \tag{83}$$

9. In order to comput the transformer efficiency, one can first compute the losses in the transformer, noted P_{loss}

$$P_{\text{loss}} = R'\ I_2^2 = 0.02 \cdot 195.7^2 = 765.97 \quad \text{W} \tag{84}$$

Then, the efficiency can be computed as

$$\boxed{\eta = \frac{P_2}{P_2 + P_{\text{loss}}} = \frac{21\,219}{21\,984} = 0.9651\ \%} \tag{85}$$

10. The equivalence between the two load can be made from the power perspective. Indeed, the two loads will be equivalent seen from the transformer as long as they consume the same complex power (thus, the same active and reactive power).

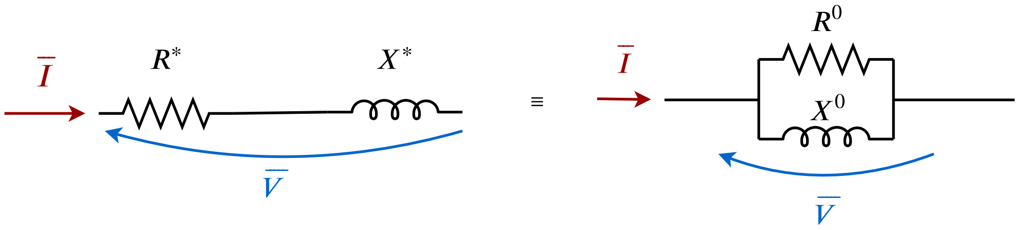


Figure 38: Equivalence between the two loads.

With the following equations,

$$V = \sqrt{R^{*2} + X^{*2}}\ I \tag{86}$$

$$P = R^*\ I^2 = \frac{V^2}{R^0} \tag{87}$$

$$Q = X^*\ I^2 = \frac{V^2}{X^0} \tag{88}$$

one can obtain

$$\boxed{\begin{aligned} R^0 &= \frac{V^2}{I^2} \cdot \frac{1}{R^*} = \frac{R^{*2} + X^{*2}}{R^*} \quad (89) \\ X^0 &= \frac{V^2}{I^2} \cdot \frac{1}{X^*} = \frac{R^{*2} + X^{*2}}{X^*} \quad (90) \\ & \quad (91) \end{aligned}}$$

Exercise 10. Single-phase transformer and auto-transformer

When a galvanic insulation is not required, due to its better efficiency, reduced cost and smaller size, the autotransformer is an interesting alternative to the classical transformer. Autrotransformers are also known to have larger short circuit currents which is not always suitable. Two tests are performed on the transformer, illustrated in Fig. 39:

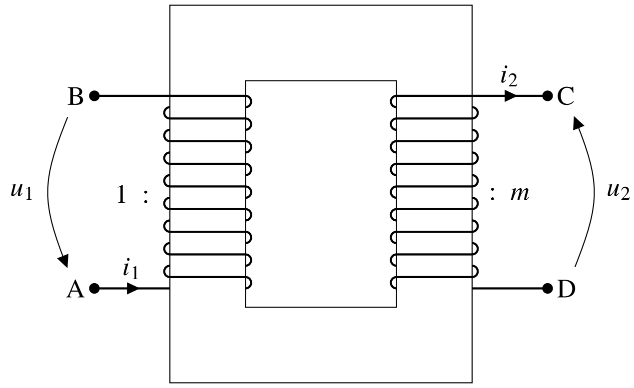


Figure 39: Single-phase transformer.

- Using open secondary winding, the transformer generates a voltage of RMS value $U_{2o} = 100\,\text{V}$ at the secondary winding, for an applied voltage of RMS value $U_{1o} = 20\,\text{V}$ with a drawn current intensity of RMS value $I_{1o} = 3.2\,\text{A}$ and a consumed power $P_{1o} = 8\,\text{W}$;
- Using short-circuited secondary winding, a voltage of RMS value $U_{1s} = 0.8\,\text{V}$ for a total power of $P_{1s} = 24\,\text{W}$ is measured, causing a current flow of RMS value $I_{2s} = 10\,\text{A}$ through the secondary winding.

Considering a simplified equivalent model of the transformer (resistances and inductances gathered and moved to the secondary winding):

1. Calculate the transformer ratio m;
2. Calculate the resistance R'_{H+F} and the magnetizing inductance L'_μ, placed at the secondary of the transformer;
3. Compute the values of the resistance R' and the reactance X' corresponding to the Joule losses and the leakage reactance, placed at the secondary of the transformer.

Using the transformer connected to a load on the secondary side drawing a current of RMS value $I_2 = 12\,\text{A}$ with a power factor $\cos\phi_2 = 0.8$ (the current is lagging the voltage), a RMS voltage of $U_1 = 20\,\text{V}$ is applied to the primary winding.

4. Calculate the RMS voltage U_2 appearing across the secondary winding by using a wise approximation of the voltage dropout ΔU_2 and justify that the approximation is relevant;
5. Deduce the active power P_2 provided to the load;
6. Calculate the RMS current I_1 on the primary side;
7. Compute the transformer efficiency η.

To turn the transformer into an autotransformer, the terminals A and D are connected together, such that the primary winding is located between C and B and the secondary between C and D (Fig. 40).

8. Compute the RMS voltage U'_1 to be applied on the primary winding to reach a RMS voltage value of $U_{2o} = 100\,\text{V}$ at the terminals of the secondary winding;
9. Calculate the RMS current I'_{1o} drawn at the primary winding in the case of an open secondary winding;
10. Neglecting the open circuit magnetomotive force, provide the Thévenin's model of the secondary winding including the electromotive force as well as the impedance components namely R'_s and X'_s;

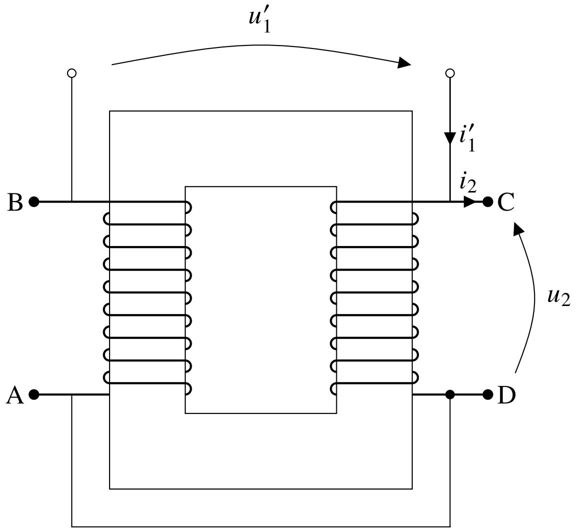


Figure 40: Single-phase autotransformer.

Solution

1. The transformer ratio is

$$\boxed{m = \frac{U_{2o}}{U_{1o}} = \frac{100}{20} = 5}$$

2. The magnetizing reactance X_μ and the magnetic losses resistance R_{H+F} can be deduced from the open circuit measurements : $U_{1o} = 20$V ; $I_{1o} = 3.2$A and $P_{1o} = 8$W. From these values, one can compute the apparent power and then the reactive power necessary for the magnetizing inductance.

$$\begin{aligned} S_{1o} &= U_{1o} \cdot I_{1o} && (92) \\ &= 20 \cdot 3.2 && (93) \\ &= 64 \text{ VA} && (94) \end{aligned}$$

Then, the reactive power can be computed as

$$\begin{aligned} Q_{1o} &= \sqrt{S_{1o}^2 - P_{1o}^2} && (95) \\ &= \sqrt{64^2 - 8^2} && (96) \\ &= 63.498 \text{ var} && (97) \end{aligned}$$

The components seen from the primary can be obtained from

$$\begin{aligned} R_{H+F} &= \frac{U_{1o}^2}{P_{1o}} = \frac{20^2}{8} = 50 \;\; \Omega && (98) \\ X_\mu &= \frac{U_{1o}^2}{Q_{1o}} = \frac{20^2}{64} = 6.299 \;\; \Omega && (99) \\ & && (100) \end{aligned}$$

$$\boxed{\begin{aligned} R'_{H+F} &= R_{H+F}\, m^2 = \frac{U_{1o}^2}{P_{1o}}\, m^2 && (101) \\ X'_\mu &= X_\mu\, m^2 = \frac{U_{1o}^2}{Q_{1o}}\, m^2 && (102) \end{aligned}}$$

$$\boxed{\begin{aligned} R'_{H+F} &= 1250\ \Omega \\ X'_\mu &= 157.485\ \Omega \\ L'_\mu &= 497.359\ \text{mH} \end{aligned}}$$

3. The leakage reactance X' and the joules losses resistance R' can be deduced from the short circuit measurements : $U_{1s} = 0.8$V ; $I_{2s} = 10$A and $P_{1s} = 24$W. From these values, one can compute the apparent power and then the reactive power necessary for the leakage inductance.

   First of all, one must convert the current measured at the secondary into primary value.

$$I_{1s} = m\ I_{2s} = 5 \cdot 10 = 50\ \text{A} \tag{103}$$

   Then, compute the apparent power

$$\begin{aligned} S_{1s} &= U_{1s} \cdot I_{1s} &(104)\\ &= 0.8 \cdot 50 &(105)\\ &= 40\ \text{VA} &(106) \end{aligned}$$

   Then, the reactive power can be computed as

$$\begin{aligned} Q_{1s} &= \sqrt{S_{1s}^2 - P_{1s}^2} &(107)\\ &= \sqrt{40^2 - 24^2} &(108)\\ &= 32\ \text{var} &(109) \end{aligned}$$

   The components seen from the primary can be obtained from

$$R = \frac{P_{1s}}{I_{1s}^2} = \frac{24}{50^2} = 9.6\ \ \text{m}\Omega \tag{110}$$

$$X = \frac{Q_{1s}}{I_{1s}^2} = \frac{32}{50^2} = 12.8\ \ \text{m}\Omega \tag{111}$$

$$R' = R\ m^2 = \frac{P_{1s}}{I_{1s}^2}\ m^2 \tag{112}$$

$$X' = X\ m^2 = \frac{Q_{1s}}{I_{1s}^2}\ m^2 \tag{113}$$

$$\boxed{\begin{aligned} R' &= 240\ \ \text{m}\Omega \\ X' &= 320\ \ \text{m}\Omega \\ L' &= 1.019\ \ \text{mH} \end{aligned}}$$

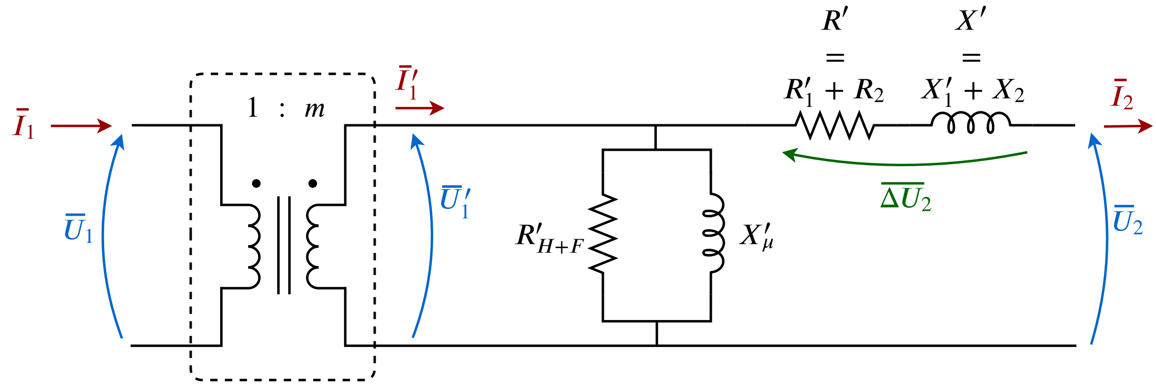
4. The equivalent circuit used in the development is the following

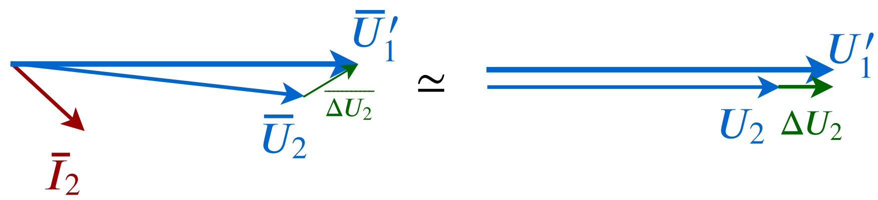
   where $\overline{\Delta U_2}$ is the voltage drop at the secondary due to the current $\overline{I}_2$. Remark that $\overline{\Delta U_2}$ is a phasor, defined by its amplitude and its phase. In this exercise, the approximation consists in considering the voltage drop ΔU_2 as a scalar, meaning that $\overline{U}_1'$ and $\overline{U}_2$ are in phase. This approximation remains correct if the current $\overline{I}_2$ is small enough. Remark that here the phase angle of the current allows to make the approximation, which is not always the case !

   The approximation can be summarized on a phasor diagram as

   The voltage drop ΔU_2 is obtained from

$$\Delta U_2 = Z'\ I_2 = \sqrt{R'^2 + X'^2}\ I_2 = \sqrt{0.24^2 + 0.32^2} \cdot 12 = 4.8\ \ \text{V} \tag{114}$$





Finally, the RMS value of the voltage at the secondary is

$$\boxed{U_2 = U'_1 - \Delta U_2 = 100 - 4.8 = 95.2 \text{ V}} \tag{115}$$

5. The active power can be directly deduced from the voltage, the current and the knowledge of the power factor.

$$\boxed{P_2 = U_2\ I_2\ \cos(\phi_2) = 95.2 \cdot 12 \cdot 0.8 = 913.92 \text{ W}} \tag{116}$$

6. The RMS primary current is the sum of the magnetizing current $\overline{I}_\mu$ and the secondary current $\overline{I}_2$,

$$\overline{I}'_1 = \overline{I}_\mu + \overline{I}_2 \tag{117}$$

where

$$\overline{I}_\mu = Y_\mu\ \overline{U}'_1 = \left(\frac{1}{R'_{H+F}} - j\ \frac{1}{X'_\mu}\right) \overline{U}'_1 \tag{118}$$

and

$$\overline{I}_2 = \frac{\overline{U}'_1 - \overline{U}_2}{R' + j\ X'} \simeq \frac{\Delta U_2}{R' + j\ X'} \tag{119}$$

Finally,

$$\boxed{\overline{I}'_1 = \left(\frac{1}{R'_{H+F}} - j\ \frac{1}{X'_\mu}\right) \overline{U}'_1 + \frac{\Delta U_2}{R' + j\ X'} = 7.28 - j\ 10.24 = 12.564\angle - 54.59°} \tag{120}$$

Thus, I'_1 is 12.564 A and

$$\boxed{I_1 = m\ I'_1 = 5 \cdot 12.564 = 62.82 \text{ A}} \tag{121}$$

7. The efficiency can be defined based on the transformer losses and output power, such that

$$\eta = \frac{P_2}{P_1} = \frac{P_2}{P_2 + P_{\text{loss}}} \tag{122}$$

where

$$P_{\text{loss}} = R' I_2^2 + \frac{U_1'^2}{R_{H+F'}} = 0.24 \cdot 12^2 + \frac{100^2}{1250} = 42.56 \text{ W} \tag{123}$$

leading to

$$\boxed{\eta = \frac{P_2}{P_2 + P_{\text{loss}}} = \frac{913.92}{913.92 + 42.56} = 95.55\%} \tag{124}$$

Remark that here, computing the input power P_1 under the approximation made previously (ΔU_2 scalar) would not provide the correct answer. The exact value of P_1 must be computed without the approximation made on the secondary voltage drop.

Answers

8. $U_1' = 80$ V
9. $I_{1o}' = 0.8$ A
10. $R_s' = 15$ m Ω and $X_s' = 20$ mΩ

Exercise 11. Single-phase transformer ◇

In Belgium, most of the railways are powered using Direct Current (DC) 3 kV voltage. High speed train lines are however supplied with Alternating Current (AC) 25 kV 50 Hz (single-phase) voltage, requiring the use of high power single phase transformers. In this exercise, an input transformer of a locomotive is considered. It is modeled as a single-phase transformer, as detailed in Fig. 41. A nominal RMS voltage of $U_{1n} = 25$ kV with nominal frequency $f = 50$ Hz is supplied to the primary winding with an apparent power $S_n = 5.6$ MVA.

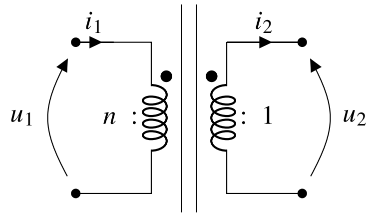


Figure 41: Four secondaries single-phase transformer.

To characterize the transformer two tests have been performed:

- Using open secondary winding with the nominal voltage applied to the primary, the transformer generates a voltage $U_{2o} = 1.36$ kV at the secondary winding, for a current drawn at the primary $I_{1o} = 1.25$ A, and an active consumed power $P_{1o} = 6.8$ kW;
- Using short-circuited secondary winding, the transformer consumes an active power $P_{1s} = 25$ kW, considering that a reduced voltage of 10.1 % of U_{1n} was applied to the primary winding to maintain the secondary winding current to its nominal value I_{2n}.

1. Calculate the transformer ratio n;
2. Determine the nominal RMS secondary current I_{2n} and primary current I_{1n};

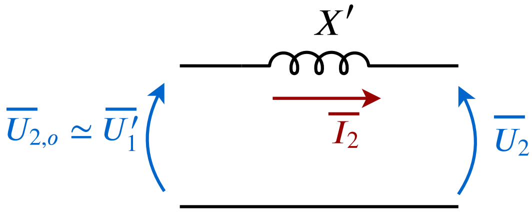
3. Compute the power factor $\cos\varphi_{1o}$ for the first test (open secondary winding) and deduce the phase shift φ_{1o} of the current at the primary winding with respect to the primary winding voltage;
4. Give the reactive power Q_{1o} for the first test (open secondary winding);
5. Considering a simplified circuit of the transformer model, calculate the resistance R_{H+F} (related to the core losses) and the magnetizing inductance L_μ, seen from the primary. One can also calculate R'_{H+F} and L'_μ, the same components seen from the secondary winding;
6. Compute the RMS current intensity I_{2s} in the secondary winding for the second test (shorted secondary winding), compute the primary winding voltage U_{1s}, and calculate the values of the resistance R and of the inductance L of the primary winding in the equivalent model. One can also calculate R' and L', seen from the secondary winding;
7. Considering that L' is chosen large enough to provide sufficient smoothing at the input of single-phase rectifiers, compare the values of R' and $X' = L'\omega$ (for ω the angular frequency corresponding to f) and propose a simplified version of the equivalent model of the transformer.

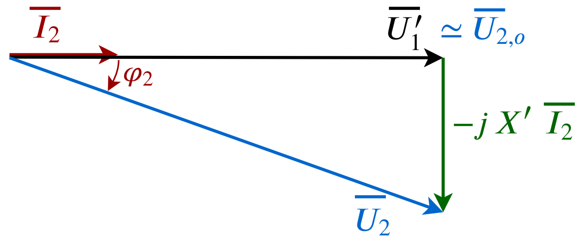
The nominal regime is now considered by applying the nominal voltage U_{1n} at the primary winding and connecting a load at the secondary winding, drawing a RMS current $I_2 = 4.097$ kA with a power factor $\cos\varphi_2$, the current being ahead on the voltage. The current i_2 in the secondary winding is aimed to be in phase with the voltage u_{2o} of the considered secondary winding.

8. Build the corresponding Fresnel diagram, clearly identifying the load voltage u_2;
9. Compute the phase shift φ_2 of the current i_2 with respect to u_2, and deduce the load power factor $\cos\varphi_2$;
10. Compute the RMS voltage value U_2 appearing at the secondary winding;
11. Compute the reactive power Q_2 drawn by the load at the secondary winding, and the reactive power Q_1 at the primary winding;
12. Compute the active power P_2 drawn by the load at the secondary winding, and the active power P_1 at the primary winding;
13. Compute the transformer efficiency η.

Answers

1. $n = 18.382$
2. $I_{2n} = 4\ 117.647$ A and $I_{1n} = 224$ A
3. $\cos\varphi_{1o} = 0.2176$ and $\varphi_{1o} = 77.43$ °
4. $Q_{1o} = 30\ 501{,}19$ var
5. $R_{H+F} = 91\ 911.765\ \Omega$; $R'_{H+F} = 272\ \Omega$; $L_\mu = 65.2$ H ; $L'_\mu = 193$ mH
6. $I_{2s} = 4\ 117.647$ A ; $U_{1s} = 2\ 525$ V ; $R = 498.25$ mΩ ; $R' = 1.474$ mΩ ; $L = 35.85$ mH ;$L' = 106.08\ \mu$H;
7. $X' » R'$





8. Neglect the series resistance in the model.
9. $\varphi_2 = 5.73298°$ and $\cos\varphi_2 = 0.994998$
10. $U_2 = \frac{U_{2,o}}{\cos\varphi_2} = 1\,366.84$ V
11. $Q_2 = -559\,392$ var and $Q_1 = 30\,924$ var
12. $P_2 = 5\,599\,943$ W and $P_1 = 5\,631\,553$ W
13. $\eta = 99.439$ %

Exercise 12. Single-Phase transfomer parameters ◇

Consider a single-phase transformer (50 Hz) to be characterized experimentally. The equivalent circuit corresponds to

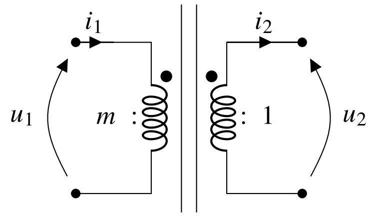


Figure 42: Single-phase transformer.

Using an open secondary winding, the RMS value of the applied voltage is U_{1o} = 120 V at the primary. The current drawn at the primary corresponds to a RMS value of I_{1o} = 1 A and a consumed active power of P_{1o} = 40 W. Using a short-circuited secondary winding, the primary RMS voltage is U_{1s} = 16 V for an active power consumption of P_{1s} = 65 W. During the short-circuit test, the secondary RMS current is I_{2s} = 72 A. Considering the transformer ratio m = 12

1. Calculate the reactive power consumed during the short-circuit test
2. Calculate the total leakage inductance seen from the secondary
3. Calculate the total leakage inductance seen from the primary
4. Calculate the resistance (accounting for the total Joule losses) seen from the secondary
5. Calculate the magnetizing inductance seen from the primary
6. Calculate the magnetizing inductance seen from the secondary
7. Calculate the resistance (accounting for the ferromagnetic losses) seen from the primary

Answers

1. Q_{1s} = 70.647 var
2. L'_s = 43.38 μH
3. L_s = 6.247 mH
4. R'_s = 12.54 mΩ
5. L_μ =405.1 mH
6. L'_μ = 2.813 mH
7. R_{H+F} = 360 Ω

Exercise 13. Three-phase transformer ◇

Consider a three-phase transformer (50 Hz) in a Star-Delta configuration (the primary is connected in star and the secondary in delta). In normal working operation, the primary phase-to-neutral voltage corresponds to 325.27 V (peak), the secondary phase-to-phase voltage corresponds to 1 kV (RMS).

Using an open secondary winding, the RMS value of the applied voltage is V_{1o} = 230 V at the primary. The current drawn at the primary corresponds to a RMS value of I_{1o} = 5 A and the three-phase active power of P_{1o} = 310 W is consumed. Using a short-circuited secondary winding, the primary RMS voltage is V_{1s} = 23 V for a three-phase active power consumption of P_{1s} = 180 W. During the short-circuit test, the secondary RMS current is I_{2s} = 0.2 A.

1. Calculate the column ratio (from primary to secondary) of the transformer
2. Calculate the transformer ratio (as the ratio of the phase voltages from primary to secondary)
3. Calculate the equivalent resistance (accounting for ferromagnetic losses) as seen from the primary, between one phase and the neutral
4. Calculate the equivalent resistance (accounting for ferromagneetic losses) as seen from the primary, between two phases

Answers

1. $n_c = 4.3478$
2. $n = 2.51$
3. $R_{H+F} = 511.935\ \Omega$ for one-phase-to-neutral
4. $R_{H+F} = 1535{,}806\ \Omega$ for phase-to-phase

Exercise 14. Transformer and magnetic induction ◇

Consider a single-phase transformer (50 Hz) with a 110 V primary (nominal RMS phase-to-phase voltage) and a 220 V secondary (nominal RMS phase-to-phase voltage). The primary is made of 42 turns and the secondary is made of 84 turns. Knowing that the core cross-section is $A_c = 0.03\ m^2$,

1. Compute the RMS value of the magnetic flux density (denoted B).
2. What is the minimum number of turns required at the primary so that the maximum magnetic flux density (B_m) does not exceed 0.56 T ?

Answers

1. $B = 278$ mT
2. $N_1 = 30$

Exercise 15. Alternator and synchronous condenser◇

A synchronous condenser is a DC excited synchronous motor, whose rotating shaft is not connected to any mechanical load. By controlling its field (or the excitation current), using a voltage regulator, the condenser is able to generate or absorb reactive power as needed to adjust the voltage on the power grids, or to improve the power factor.

- Synchronous speed: 428 RPM for 14 poles,
- Star-shape coupling with a voltage $V_n = 8.95\,\text{kV}$ between phase and neutral,
- Nominal intensity $I_n = 6.33\,\text{kA}$,
- Apparent nominal power $S_n = 170\,\text{MVA}$,
- Nominal synchronous reactance $X_s = 1.2\,\Omega$ at the nominal frequency $f_n = 50\,\text{Hz}$.

The machine is used as an alternator, providing active power $P = 100\,\text{MW}$ and reactive power $Q = 50\,\text{Mvar}$ to the power grid.

1. Calculate $\cos\varphi$, φ and the line current intensity I.

The machine is now turned into a freely spinning motor, keeping the excitation current constant and assuming $P \approx 0\,\text{W}$,

2. Calculate the reactive power $Q < 0$ provided by the motor.

Going back in the alternator mode, the machine is working at a constant power $P_n = 100\,\text{MW}$ and exchanges reactive power from $Q_{\min} = -100\,\text{Mvar}$ to $Q_{\max} = 100\,\text{Mvar}$,

3. Between $Q_{\min}$ and $Q_{\max}$, plot I with respect to the electromotive force E.

Answers

1. $\cos\varphi$ = 0.8944 ; φ = 26.565° ; I = 4.16 kA ; (δ_{int} = 21.77°)
2. Q = -69.14 Mvar
3. graph

Exercise 16. Three-phase turbo-alternator

Turbo-alternators are alternators coupled to turbines allowing to convert the mechanical power of a moving fluid (steam or liquid) to electrical power. In this exercise the turbo-alternator has the following nominal characteristics:

- Power $P_n = 600\,\text{MW}$,
- Frequency $f_n = 50\,\text{Hz}$,
- Speed of rotation $\dot{\theta}_n = 3000\,\text{RPM}$,
- Power factor $\cos\varphi_n = 0.9$,
- RMS value of the composed voltages $U_n = 20\,\text{kV}$,
- Ferromagnetic losses $p_f = 543\,\text{kW}$,
- Mechanical losses $p_m = 1.35\,\text{MW}$,
- Rotor resistance $R_e = 0.17\,\Omega$,
- Excitation system efficiency $\eta_e = 0.92$,
- Stator phase resistance $R = 2.3\,\text{m}\Omega$.

To characterize the turbo-alternator three tests have been performed:

- Using open stator windings, at the nominal speed of rotation $\dot{\theta}_n$, the RMS direct voltage values have been measured with respect to the RMS current intensity I_e flowing through the inductor (Table 1);
- Using short-circuited stator windings, at the nominal speed of rotation $\dot{\theta}_n$, using an excitation current of RMS value $I_e = 1.18\,\text{kA}$ has allowed a current flow in each phase winding of the stator reaching the half of the RMS nominal value;
- Using an inductive load, an excitation current of RMS value $I_e = 2.085\,\text{kA}$ has allowed a current flow in each phase winding of the stator reaching the half of the RMS nominal value. Also, the output voltage was measured as half the nominal voltage.

Table 1: Alternator open circuit stator windings test measurements. (Voltages measured between two phases)

I_e [A]	E_v [kV]
400	5.2
700	9.1
963	11.5
1200	13
1450	14
1900	15

1. Calculate the nominal RMS intensity I_n of the stator currents;
2. Compute the total losses and the turbo-alternator efficiency at the nominal operating point, knowing the RMS excitation current value is $I_e = 3.2\,\text{kA}$;
3. Calculate the needed mechanical power for each of the considered test;
4. Calculate the (unsaturated) synchronous reactance X_s of the turbo-alternator.

Using Behn-Eschenburg diagram with the experimental measurements

5. Neglecting resistive losses in the rotor, plot Behn-Eschenburg diagram for the nominal operating point;
6. Compute the RMS value E_v of the synchronous electromotive force;
7. Draw the internal lag δ_{int} angle and give its value;

Solution

1. In a three-phase circuit, the relation between the apparent power and the nominal current is :

$$S_n = \sqrt{3}U_n I_n = P_n \cos\varphi_n + j\,Q_n \sin\varphi_n \tag{125}$$

Therefore,

$$\boxed{I_n = \frac{|S_n|}{\sqrt{3}U_n} = \frac{\frac{P_n}{\cos\varphi_n}}{\sqrt{3}U_n} = \frac{\frac{600\cdot 10^6}{0.9}}{\sqrt{3}\cdot 20\cdot 10^3} = 19\,245\text{ A}} \tag{126}$$

2. The power balance of the synchronous machine can be expressed as :

$$P_{\text{mec}} = P_n + \underbrace{p_m + p_f + p_{\text{excitation}} + p_{\text{joule}}}_{\text{LOSSES}} \tag{127}$$

with

- P_{mec} = mechanical power provided by the turbine
- P_n = useful three-phase electrical power
- p_m = mechanical losses

- p_f = ferromagnetic losses
- $p_{\text{excitation}}$ = losses in excitation system
- p_{joule} = Joule losses in the stator

The electrical power available at the rotor, p_{rotor}, is only a fraction of the power provided to the excitation system, $p_{\text{excitation}}$:

$$p_{\text{rotor}} = \eta_e \; p_{\text{excitation}} \Rightarrow p_{\text{excitation}} = \frac{p_{\text{rotor}}}{\eta_e} = \frac{R_e I_e^2}{\eta_e} \tag{128}$$

One can compute

$$p_{\text{losses}} = p_m + p_f + \frac{R_e I_e^2}{\eta_e} + 3RI_n^2 = 1.35 + 0.543 + \frac{0.17 \cdot 3.2^2}{0.92} + 3 \cdot 0.0023 \cdot (19\,245)^2 \tag{129}$$

$$\boxed{p_{\text{losses}} = 6.34 \text{ MW}} \tag{130}$$

Once one gets p_{losses}, one can compute the machine efficiency as :

$$\boxed{\eta = \frac{P_{out}}{P_{in}} = \frac{P_n}{P_n + P_{\text{losses}}} = \frac{600}{600 + 6.34} = 98.95\%} \tag{131}$$

3. According to the situation, all losses listed above are not necessarily encountered. In the open circuit test for example, there are no stator losses. So, we will distinguish the different cases.

   Open circuit-test:

   The losses correspond to $p_{\text{excit}} + p_m + p_f$.

I_e	p_f	p_m	$p_{\text{excit}} = \frac{R_e I_e^2}{\eta_e}$	P_{tot} [MW]
400	1.35	0.543	0.0296	1.923
700	1.35	0.543	0.0905	1.984
1200	1.35	0.543	0.2661	2.159
1900	1.35	0.543	0.667	2.56

Table 2: Mechanical power for open-circuit tests

   Short circuit test:

   The losses correspond to the rotor losses + stator losses $+p_m + p_f$

$$P_{sc} = p_f + p_m + \frac{R_e I_e^2}{\eta_e} + 3R\left(\frac{I_n}{2}\right)^2 = 2.79 \text{ [MW]}$$

   Inductive load test:

$$P_{ind} = p_f + p_m + \frac{R_e I_e^2}{\eta_e} + 3R\left(\frac{I_n}{2}\right)^2 = 3.34 \text{ [MW]}$$

4. As can be seen in Figure 43, the Behn-Eschenburg approximation is good for the linear part of the relation (*i.e.* for small values of I_e). After, a saturation phenomenon appears. In order to have a good approximation of the factor k_e, one uses the test for small I_e.

   Using the two first open circuit tests ($I_e = 400\ A - E_v = 5.2\ kV$ and $I_e = 700\ A - E_v = 9.1\ kV$), the relation $k_e = E_v / I_e$ leads to

$$\boxed{k_e = 13}$$

   If you compute k_e using higher value of I_e, you will no longer get this value of 13 precisely.
   Once k_e is known, E_v for the short-circuit test can be found ($I_e = 1.18$ kA):

$$E_v = k_e I_e \Leftrightarrow E_v = 13 \cdot 1.18 \cdot 10^3 = 15.34 \text{ kV} \tag{132}$$

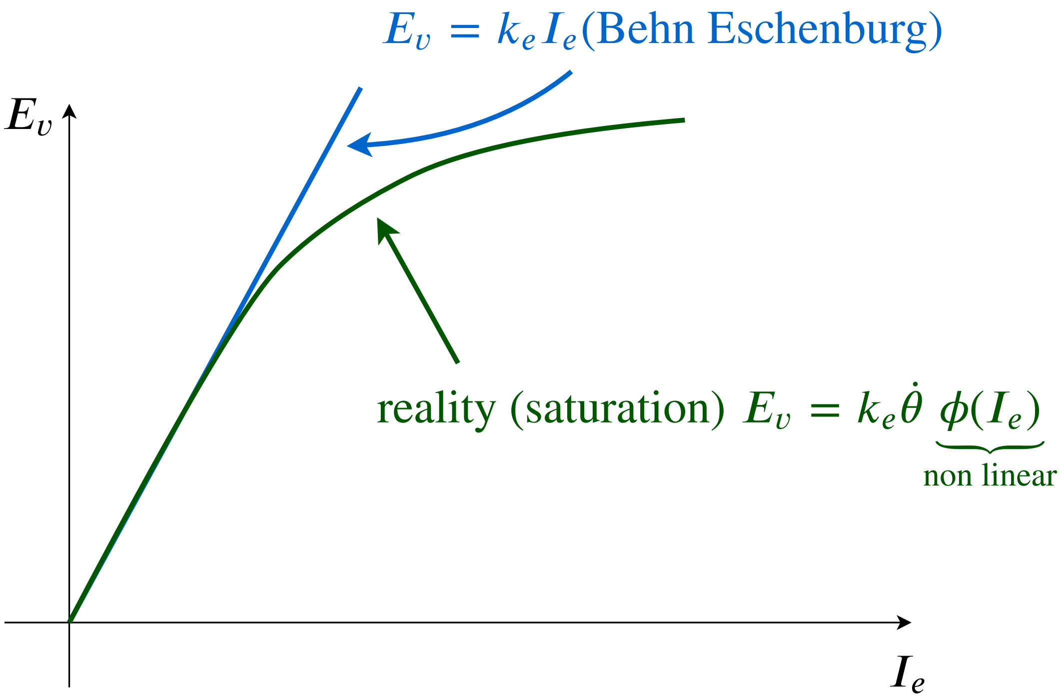


Figure 43: $E_v - I_e$ relation to determine the synchronous reactance X_s.

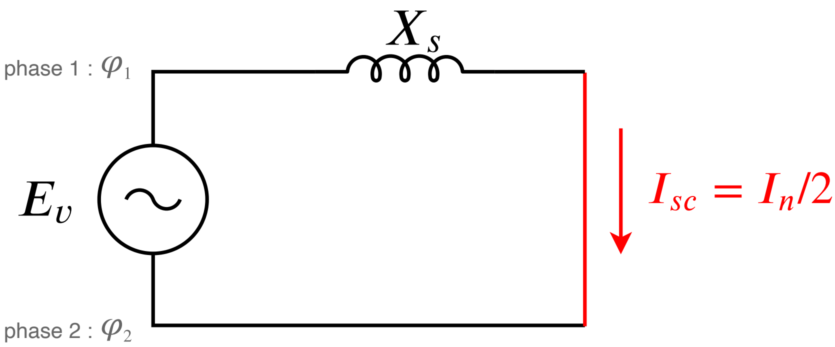


Figure 44: Short-circuit test.

The short-circuit test is drawn in Fig.44. From that, we get:

$$X_s = \frac{E_v}{I_{sc}} = \frac{15.34}{9.6225} = 1.59148\Omega \quad (133)$$

$$L_s = \frac{X_s}{\omega} = 5.07 \text{ mH} \quad (134)$$

5. The Behn-Eschenburg diagram for the nominal point is as following

6. The total load current is

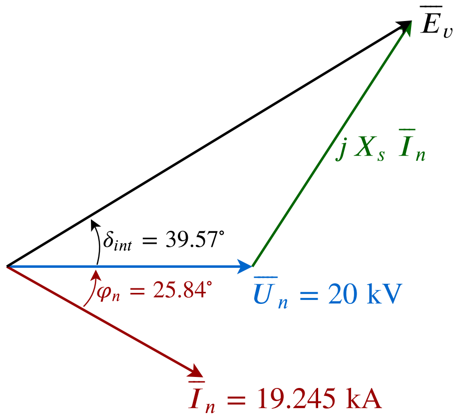


Figure 45: Behn-Eschenburg diagram.

$$\overline{I}_n = \left(\frac{S}{\sqrt{3}\,\overline{U}_n}\right)^* \quad (135)$$
$$= \frac{P_n - j\,Q_n}{\sqrt{3}\,U_n} \quad (136)$$
$$= \frac{600 - j\,290.593}{\sqrt{3}\cdot 20} \quad (137)$$
$$= 19.245\angle -25.84^\circ \text{ kA} \quad (138)$$
$$\quad (139)$$

and the volage drop inside the stator windings is

$$j\,X_s\,\overline{I}_n = j\cdot 1.59148\cdot 19.245\angle -25.84^\circ = 30.628\angle 64.158^\circ \quad (140)$$

Finally, the internal emf E_v is

$$\boxed{\overline{E}_v = \overline{U}_n + j\,X_s\,\overline{I}_n = 43.268\angle 39.57^\circ} \quad (141)$$

and the RMS value is $E_v = 43.268$ kV.

7. The internal load angle is measured between the output voltage $\overline{U}_n$ and the no-load emf $\overline{E}_v$. It has been computed for the previous sub-question and is

$$\boxed{\delta_{int} = 39.57^\circ} \quad (142)$$

Exercise 17. Constant air gap alternator

Several alternators are used on airplanes which, coupled to the reactors, feed all the necessary onboard electrical grids. Those alternators are characterized by the higher frequency of the generated voltage and currents compared to alternators coupled to 50 Hz or 60 Hz electrical grids. Moreover, due the the variable speed of the airplane reactors, the delivered frequency is not constant. The considered constant air gap three-phase alternator and its rotor winding are coupled following a star shape. Magnetic leakage, saturation, hysteresis and Eddy currents will be neglected.

For a rotating speed of the alternator shaft of $\dot{\theta} = 11\,100$ RPM, the frequency of the delivered voltages and currents is $f = 370$ Hz for a nominal apparent power $S_n = 150$ kVA and a direct voltage of RMS value $V_n = 115$ V. The rotation speed of the reactor $\dot{\theta}_e$ varies from 4160 RPM to 9000 RPM. The alternator is therefore coupled to the reactor through a gear box of ratio $k_m = \frac{\dot{\theta}}{\dot{\theta}_e} = 2.67$.

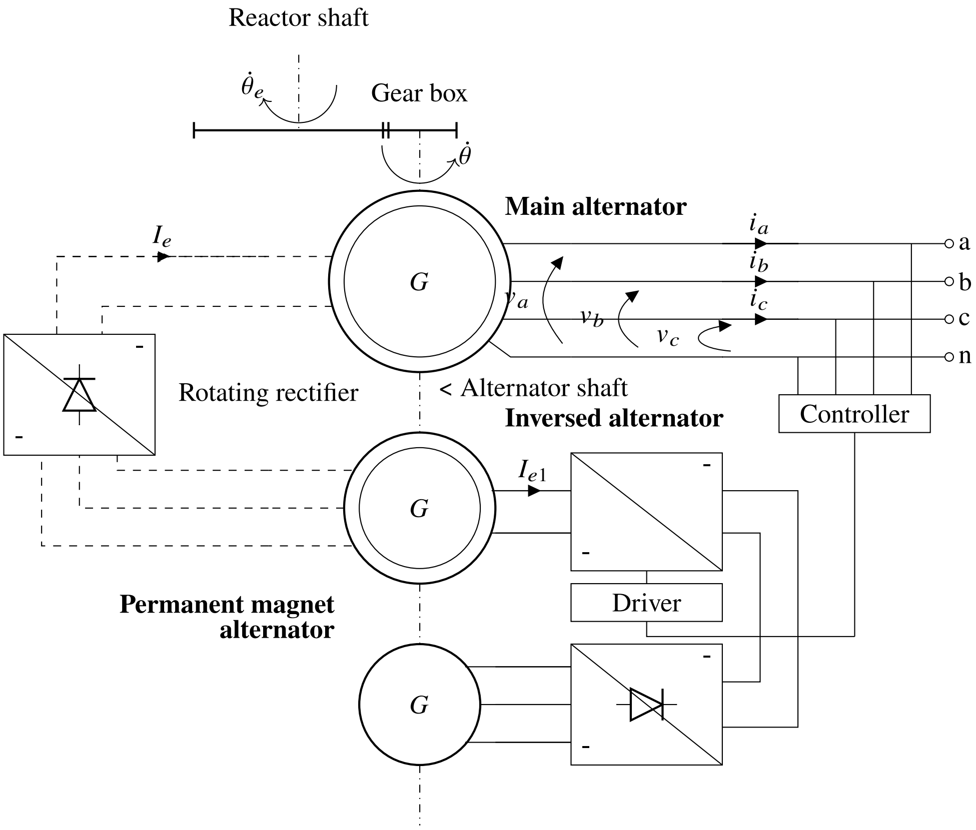


Figure 46: Excitation system of the alternator.

The excitation of the alternator is controlled such that the output voltage of the alternator is 115 V (direct voltage or 200 V for the composed voltage). This excitation consists of an inversed alternator coupled with a permanent magnet alternator (Fig. 46).

1. Explain how the excitation system works. What are the main advantages of such a system?
2. Express the frequency of the generated voltages and currents f with respect to the rotation speed of the reactor n_e, the gear box ratio k_m and the number of pairs of poles of the alternator p;
3. Deduce the number of pair of poles, as well as the minimal and maximal values $f_{\min}$, $f_{\max}$ of the generated voltages and currents;
4. For an airplane, justify the relevance of a system working at a variable frequency in the targeted range;
5. Calculate the nominal RMS current I_{sn} of the line currents of the alternator;
6. The flux generated by a pole is:
$$\phi(t) = \Phi_m \cos\left(p\left(\dot{\theta}t - \theta_0\right)\right),$$
where Φ_m is the flux amplitude, p the number of pairs of poles, $\dot{\theta}$ the speed of rotation, t the time variable and θ_0 the initial angular position of the rotor. Express the electromotive force e_s induced in a single turn of the rotor with respect to Φ_m, f, t and θ_0. Deduce the RMS value E_s of $e_s(t)$ with respect to Φ_m and f;
7. The RMS value E of the induced electromotive force in a phase is $E = k_b N_s E_s$ where $k_b = 0.850$ is the coil factor and $N_s = 16$ is the number of turn per phase. The magnetic circuit is built using laminations allowing to reach a maximal magnetic field corresponding to a flux amplitude $\Phi_{m0} = 6.84$ mWb and a current $I_{e0} = 2.95$ A. Express the RMS value E of the electromotive force induced in each phase with respect to k_b, N_s, Φ_{m0}, I_{e0}, I_e and f. Plot E with respect to I_e in the range between $f_{\min}$ and $f_{\max}$ and conclude.

The stator of the machine is composed of three-phase windings whose phases are noted a, b and c, while the rotor is composed of a inductor winding f (Fig. 47). Each phase has an impedance composed of a resistance R_s, a self inductance λ (also noted L_s) and a mutual inductance λ_m (also noted M_s) with respect to each other phase.

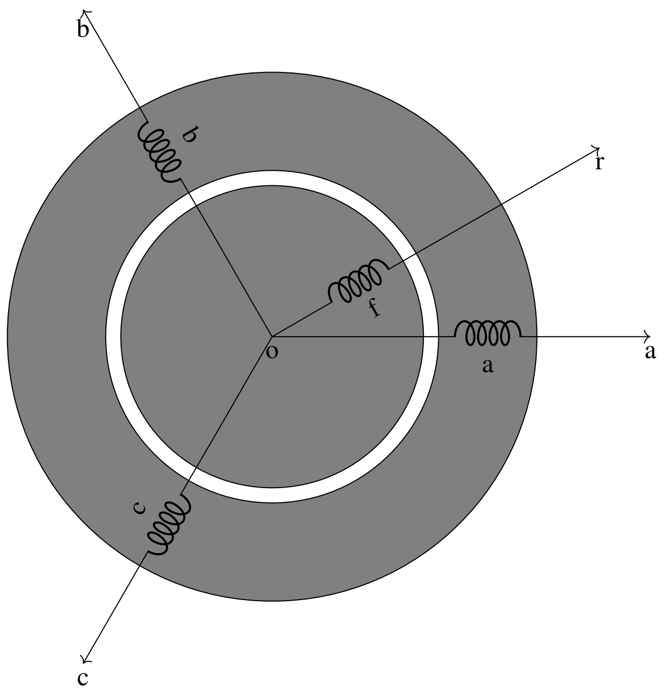


Figure 47: Electrical representation of the alternator windings.

The mutual inductances between each phase and the inductor phase have a sinusoidal pulsation with respect to the rotation angle θ:

$$\lambda_{m,af} = \lambda_{m,sf}\ \cos(p\theta)$$
$$\lambda_{m,bf} = \lambda_{m,sf}\ \cos(p\theta - \frac{2\pi}{3})$$
$$\lambda_{m,cf} = \lambda_{m,sf}\ \cos(p\theta + \frac{2\pi}{3})$$

8. Express the total fluxes Ψ_a, Ψ_b and Ψ_c crossing the phase windings a, b and c with respect to the flowing current intensities i_a, i_b and i_c, the excitation current intensity I_e, the self inductance λ, the stator mutual inductance λ_m, the mutual inductance between the stator and the rotor $\lambda_{m,sf}$ and the angle $p\theta$;
9. Express the voltages v_a, v_b and v_c across the phase windings a, b, c with respect to i_a, i_b and i_c, the total flux derivatives Ψ_a, Ψ_b and Ψ_c and R_s;
10. Show that the direct voltages of the stator can be written:

$$v_a = e_a - R_s i_a - \lambda_f\ \frac{di_a}{dt}$$
$$v_b = e_b - R_s i_b - \lambda_f\ \frac{di_b}{dt}$$
$$v_c = e_c - R_s i_c - \lambda_f\ \frac{di_c}{dt}$$

Express the electromotive forces e_a, e_b and e_c with respect to $\lambda_{m,sf}$, I_e, ω, t, p and θ_0, and their common RMS value E with respect to $\lambda_{m,sf}$, I_e and ω. Explain the significance of λ_f and reexpress it in terms of λ and λ_m.

The single-phase equivalent model of Behn-Eschenburg is now considered with $R_s = 0.4\,\text{m}\Omega$. To characterize the alternator two tests have been performed:

- Using open stator windings, at the speed of rotation $\dot{\theta} = 11\,100\,\text{RPM}$, the RMS direct voltage values have been measured with respect to the RMS current intensity I_e flowing through the inductor (Table 3);

Table 3: Alternator open and short-circuited stator windings test measurements.

I_e [A]	E [V]	I_s [A]
0.4	21.2	94.8
0.8	42.2	190
1.2	63.6	284
1.6	84.8	379
2	106	474
2.4	122	569
3	137	670
3.6	143	770
4.2	145	860
4.8	147	948
5.4	148	1040

- Using short-circuited stator windings, at the speed of rotation $\dot{\theta} = 11\,100$ RPM, the RMS current intensity I_c have been measured with respect to the RMS current intensity I_e flowing through the inductor (Table 3).

11. Knowing that $E = \alpha\,\omega I_e$, compute the value of the coefficient α for $I_e = 0.4\,, 3.0\,, 5.4$ A;
12. Plot the open stator windings curve, E with respect to I_e, for $f_{\min} = 370$ Hz and $f_{\max} = 770$ Hz;
13. Calculate the synchronous reactance X_s for the linear part of the curve;
14. Plot the short-circuited stator windings curve, I_s with respect to I_e, for $f_{\min} = 370$ Hz and $f_{\max} = 770$ Hz;
15. The alternator is connected to a star-shaped load composed of 3 resistors of value $R_L = 0.5\,\Omega$ working at a frequency $f = 500$ Hz for an excitation current $I_e = 2$ A.
    (a) Calculate the stator RMS current and voltage values I_s and V_s;
    (b) Sketch the Behn-Eschenburg diagram;
    (c) Explain how I_s and V_s vary when the frequency increases;
16. Working at constant I_e, a balanced inductive load is now considered with a corresponding impedance $Z_c = R_c + \jmath\omega L_c$ for each phase.
    (a) Sketch the Behn-Eschenburg diagram for a power factor $\cos\phi = 0.75$;
    (b) Express the stator RMS voltage V_s with respect to α, ω, R_c, L_c, L_s and I_e;
    (c) Express the resistive torque C_r with respect to p, α, ω, R_c, L_c, L_s and I_e;
    (d) Knowing that $R_c = 0.5\,\Omega$ and $L_c = 150\,\mu$H, compute I_s, V_s and C_r for $f_{\min}$ and $f_{\max}$ for $I_e = 0.4$ A, 3.0 A and 5.4 A;
    (e) How does the frequency variation influence the load power factor?

Solution

1. Physical explanations and advantages

   The three synchronous alternators are fixed to the same shaft.

   The permanent magnet alternator is the smallest of the three machines (regarding power) and produces some three-phase current alternating current as it rotates.
   The current produced is then rectified (it becomes DC current) and used for the excitation of the second machine.

   The second machine (here the inversed alternator) is used to increase the power between the small permanent magnet alternator and the main alternator.

   It is called an inversed alternator because the fixed part contains the excitation winding (DC current) (normally in the rotor for a conventionnal machine) and the moving part contains the three-phase windings (normally in the stator for a conventionnal machine).

   The three-phase currents are then rectified and used for the excitation of the main alternator.
   Remark that the rectifier is also rotating along the inversed alternator.

   The main alternator rotor is then excited with the excitation current I_e and the mechanical power applied on the shaft is transfered into three-phase electrical power in the conductors (a, b, c, n).

   Advantages :

   The permanent magnet machine allows an autonomous start (no excitation current is required for this machine). Also, this machine is brushless, meaning that no spark are produced when it is rotating. This is an important point for the design of safe aircraft. A brushed DC generator could not be used instead of the permanent magnet alternator because it would create sparks.

   The electrical connections of the inversed alternator are simpler and do not require any connecting ring either for the excitation winding or the three-phase windings.

2. In order to express the frequency f in terms of $\dot{\theta}_e$, k_m and p, one can write

   $$\dot{\theta} = k_m \, \dot{\theta}_e \tag{143}$$

   where $\dot{\theta}$ is the synchronous speed of the machine ; $k_m = 2.67$ is the gearbox ratio ; $\dot{\theta}_e$ is the aircraft reactor speed.

   Then,

   $$f = p\,\dot{\theta} = p\,k_m\,\dot{\theta}_e \qquad (\dot{\theta} \text{ and } \dot{\theta}_e \text{ expressed in turn/s}) \tag{144}$$

   $$f = p\,\frac{\dot{\theta}}{60} = \frac{p\,k_m\,\dot{\theta}_e}{60} \qquad (\dot{\theta} \text{ and } \dot{\theta}_e \text{ expressed in rpm}) \tag{145}$$

   $$f = \frac{p\,\dot{\theta}}{2\pi} = \frac{p\,k_m\,\dot{\theta}_e}{2\pi} \qquad (\dot{\theta} \text{ and } \dot{\theta}_e \text{ expressed in rad/s}) \tag{146}$$

3. The alternator frequency is 370 Hz when its rotation speed is 11 100 rpm. 11 100 rpm = 185 turn/s. Then, $p = \frac{f}{\dot{\theta}} = \frac{370}{185} = 2$ and the number of pair of poles $p = 2$.

$\dot{\theta}_{e,min} = 4160$ rpm and $\dot{\theta}_{e,max} = 9000$ rpm. Then,

$$f_{min} = \frac{p\ k_m\ \dot{\theta}_{e,min}}{60} = 370.24 \text{ Hz} \tag{147}$$

$$f_{max} = \frac{p\ k_m\ \dot{\theta}_{e,max}}{60} = 801 \text{ Hz} \tag{148}$$

4.
   - The high frequency (higher than the industrial 50 Hz) enables the use of smaller components (L and C).
   - The mechanic is easier, there is only one gearbox with one ratio.
   - The rotation speed of the reactor can vary even if there are only two pairs of poles.

5. The nominal current I_{sn} is

$$\boxed{I_{sn} = \frac{S_n}{3\,V} = \frac{150\,000}{3\cdot 115} = 434.783 \text{ A}} \tag{149}$$

6. Considering $\dot{\theta}$ in rad/s, the expression of the magnetic flux is provided as

$$\phi(t) = \Phi_m\ \cos\left(p(\dot{\theta}\,t - \theta_0)\right) \tag{150}$$

Then, the variation of the flux is linked to the electromotive force by Lenz law

$$e_s(t) = -\frac{d\phi(t)}{dt} \tag{151}$$

And Lenz law can be further developed as following :

$$e_s(t) = -\frac{d\phi(t)}{dt} \tag{152}$$

$$= p\,\dot{\theta}\ \Phi_m\ \sin\left(p(\dot{\theta}\,t - \theta_0)\right) \tag{153}$$

$$= 2\pi f\ \Phi_m\ \sin\left(p(\dot{\theta}\,t - \theta_0)\right) \tag{154}$$

$$= E_m\ \sin\left(p(\dot{\theta}\,t - \theta_0)\right) \tag{155}$$

and the RMS value of the voltage (or electromotive force) induced in a single turn of the stator can be expressed as

$$\boxed{E_s = \frac{E_m}{\sqrt{2}} = \sqrt{2}\pi\, f\, \Phi_m} \tag{156}$$

7. The total RMS voltage available at the accesses of a coil is E. It is proportionnal to the number of turn N_s and the emf induced in each turn, denoted E_s. Finally, a factor smaller than 1 is applied to the expression to take the global effects of the non idealities. It is noted k_b, the coil factor.

The coil factor k_b is a global coefficient that takes some non idealities into account, such as : the leakage flux between the rotor and stator, the angular section of the turns of a phase.

$$E = k_b\ N_s\ E_s \tag{157}$$

with

$$E_s = \sqrt{2}\ \pi\ f\ \Phi_m \tag{158}$$

Also, in the linear range, the magnetic flux induced in the machine is directly proportionnal to the excitation current I_e

$$\Phi_m = \frac{\Phi_{m0}}{I_{e0}} I_e \tag{159}$$

Then,

$$E = k_b\ N_s\ \sqrt{2}\ \pi\ f\ \frac{\Phi_{m0}}{I_{e0}}\ I_e \tag{160}$$

$$= 0.85\ 16\ \sqrt{2}\ \pi\ 370\ \frac{0.00684}{2.95}\ 1 = 51.84\,V \quad (\text{for } f_{min}\text{=370 Hz}) \tag{161}$$

$$= 0.85\ 16\ \sqrt{2}\ \pi\ 800\ \frac{0.00684}{2.95}\ 1 = 112.08\,V \quad (\text{for } f_{max}\text{=800 Hz}) \tag{162}$$

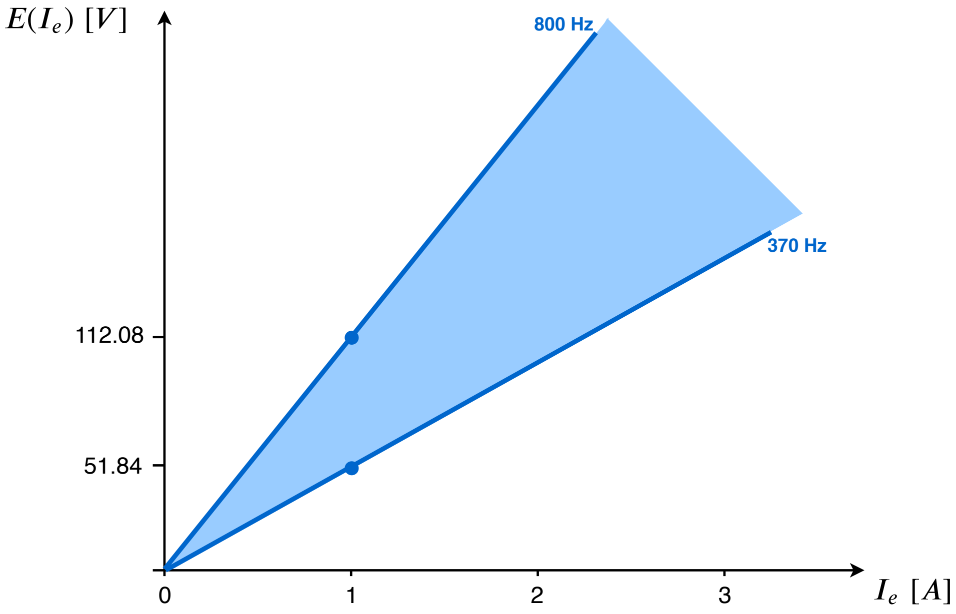


Figure 48: Emf at a stator winding as a function of the excitation current.

8. With the assumption of linear behavior, the total flux Ψ of any winding can be linked to the current i flowing through this winding by defining the inductance λ such that

$$\Psi = \lambda\, i \tag{163}$$

Also, if another winding creates a magnetic flux caught by the first winding, its contribution can be added to the previous expression by introducing the mutual inductance λ_m

$$\Psi = \lambda\, i + \lambda_m\, i_{\text{bis}} \tag{164}$$

and where i_{bis} corresponds to the current carried by the second winding.

The general equations (163) and (164) can be used to express the total fluxes $\Psi_{a,b,c}$ seen by the windings a, b and c of the stator :

$$\begin{aligned}
\Psi_a &= \lambda\, i_a + \lambda_m\, i_b + \lambda_m\, i_c + \lambda_{m,af}\; I_e && (165)\\
&= \lambda\, i_a + \lambda_m\, i_b + \lambda_m\, i_c + \lambda_{m,sf}\; \cos(p\,\theta)\; I_e && (166)\\
& && (167)\\
\Psi_b &= \lambda_m\, i_a + \lambda\, i_b + \lambda_m\, i_c + \lambda_{m,bf}\; I_e && (168)\\
&= \lambda_m\, i_a + \lambda\, i_b + \lambda_m\, i_c + \lambda_{m,sf}\; \cos\left(p\,\theta - p\,\frac{2\pi}{3}\right)\; I_e && (169)\\
& && (170)\\
\Psi_c &= \lambda_m\, i_a + \lambda_m\, i_b + \lambda\, i_c + \lambda_{m,cf}\; I_e && (171)\\
&= \lambda_m\, i_a + \lambda_m\, i_b + \lambda\, i_c + \lambda_{m,sf}\; \cos\left(p\,\theta - p\,\frac{4\pi}{3}\right)\; I_e && (172)
\end{aligned}$$

where λ is the self-inductance of the winding, λ_m is the mutual inductance between two stator windings and $\lambda_{m,sf}$ is the maximum mutual inductance between a stator winding and the field winding (*i.e.* the rotor).

Remark that the mutual inductance between a stator winding and the field winding depends on the rotor angular position θ. Indeed, the mutual inductance $\lambda_{m,af}$ between the winding a and the rotor will be maximized if the windings are aligned (when $\theta = 0$), zero if perpendicular (when $\theta = \frac{\pi}{2}$ or $\theta = -\frac{\pi}{2}$) and minimized if the windings are aligned in the opposite direction ($\theta = \pi$). The reasoning holds true for phases b and c except that they are each out of phase by $\frac{2\pi}{3}$.

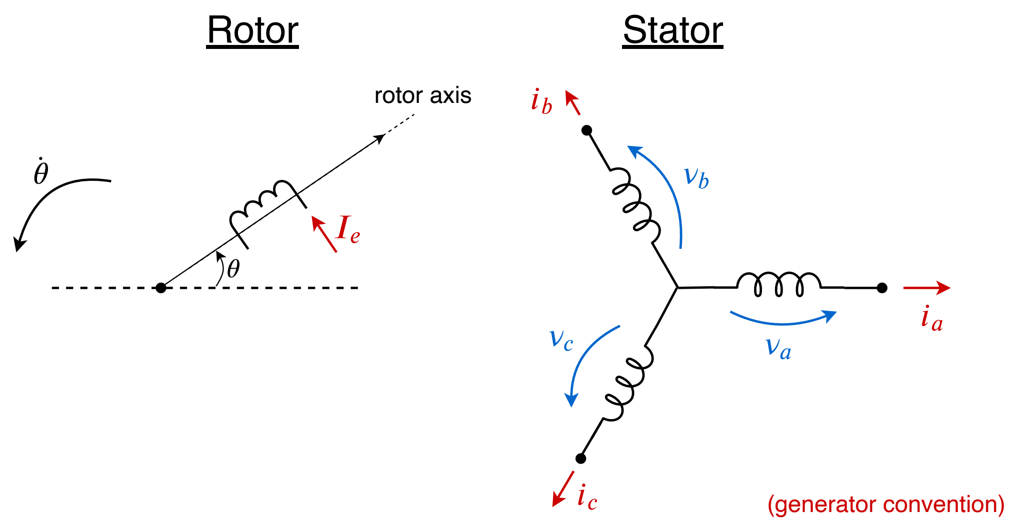


Figure 49: Rotor and stator windings. The stator resistance R_s is not represented but well considered in the model.

Do not forget that θ is a function of time, in fact it is a shortcut for $\theta = \dot{\theta}\, t - \theta_0$. Also remember that the currents $i_{a,b,c}$ are also functions of time.

9. The voltages available at each phase are respectively v_a, v_b and v_c. These voltages are all made of two components : the first component originates from Lenz law applied to the total flux of each winding ($\Psi_{a,b,c}$) and the second component comes from Ohm's law which expresses the voltage drop accross the stator resistance R_s.

$$v_a = -\frac{d\Psi_a}{dt} - R_s\, i_a \quad ; \quad v_b = -\frac{d\Psi_b}{dt} - R_s\, i_b \quad ; \quad v_c = -\frac{d\Psi_c}{dt} - R_s\, i_c \tag{173}$$

With R_s the resistance of each phase of the stator and $\Psi_{a,b,c}$ the flux linkage of each phase.

10. Considering (for simplicity) that $\theta_0 = 0$, one can express the flux linkage of phase a as

$$\Psi_a(t) = \lambda\, i_a(t) + \lambda_m\, i_b(t) + \lambda_m\, i_c(t) + \lambda_{m,sf}\ \cos(p\,\dot{\theta}\,t)\ I_e \tag{174}$$
$$= \lambda\, i_a(t) + \lambda_m\, \big(i_b(t) + i_c(t)\big) + \lambda_{m,sf}\ \cos(p\,\dot{\theta}\,t)\ I_e \tag{175}$$

With $\qquad i_b(t) + i_c(t) = -i_a(t) \qquad \Longleftrightarrow \qquad i_a(t) + i_b(t) + i_c(t) = 0$

Then,

$$\Psi_a(t) = (\lambda - \lambda_m)\, i_a(t) + \lambda_{m,sf}\ \cos(p\,\dot{\theta}\,t)\ I_e \tag{176}$$
$$= \quad \lambda_f \quad i_a(t) + \lambda_{m,sf}\ \cos(p\,\dot{\theta}\,t)\ I_e \tag{177}$$

where λ is the self-inductance of a winding and often called the synchronous inductance. Remark that λ and L_s is the same parameter, only the notation can differ. λ_m is the mutual inductance between two windings. Finally, $\lambda_f = \lambda - \lambda_m$ is the leakage inductance of a stator winding, it is also called the cyclic inductance. Remark that λ_f and L_f is the same parameter, only the notation can differ.

From that expression, the time derivative of the magnetic flux linkage Ψ_a can be developed as

$$-\frac{d\Psi_a(t)}{dt} = -\lambda_f\, \frac{di_a(t)}{dt} + p\,\dot{\theta}\,\lambda_{m,sf}\ \sin(p\,\dot{\theta}\,t)\ I_e \tag{178}$$
$$-\frac{d\Psi_a(t)}{dt} = -\lambda_f\, \frac{di_a(t)}{dt} + e_a(t) \tag{179}$$

is obtained by defining, λ_f : the leakeage inductance (or cyclic inductance) and e_a : the internal emf of phase a.

Finally, one can get,

$$v_a = -\frac{d\Psi_a(t)}{dt} - R_s\, i_a(t) = e_a(t) - R_s\, i_a(t) - \lambda_f \frac{di_a(t)}{dt} \tag{180}$$

The same derivation remains valid for phases b and c.

Equation (180) can be expressed with phasors as

$$\overline{U} = \overline{E}_r - R_s\,\overline{I} - j\,X_f\,\overline{I} \tag{181}$$

with $\overline{E}_r$ that is the internal emf of the machine.

11. The α parameter can be obtained from the no-load voltage. But first, one must compute the angular pulsation ω.

$$\omega = 2\pi f = 2\pi\left(\frac{2\times 11100}{60}\right) = 2324.78 \text{ rad/s} \tag{182}$$

$$\alpha = \frac{E}{\omega\, I_e} \tag{183}$$

I_e	E	α
0.4	21.2	0.02279787
3	137	0.01964345
5.4	148	0.011789256

Table 4: α parameter computation.

12. The voltage E can be plotted with respect to the excitation current I_e for the two frequencies f_{min} = 370 Hz and f_{max} = 770 Hz.

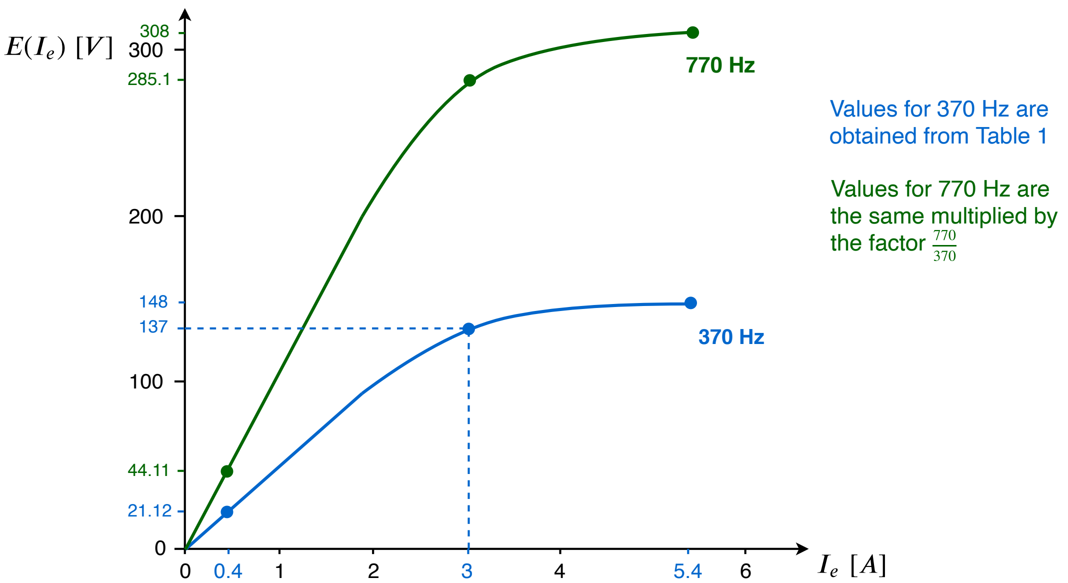


Figure 50: Emf as a function of the excitation current in the non linear case.

13. The next step is to calculate the synchronous reactance X_s for the linear part of the curve. The curve is linear for $I_e \in [0; 2]$. One can take the values corresponding to $I_e = 1.6$ A in order to minimize the measurement error and still remain in the linear part.

    For $I_e = 1.6$ A, the no load voltage is $E = 84.8$ V and the short circuit current is $I_s = 379$ A.

$$Z_s = R_s + j\,X_s \tag{184}$$

$$Z_s = \frac{E}{I_s} = \frac{84.8}{379} = 0.2237\Omega \tag{185}$$

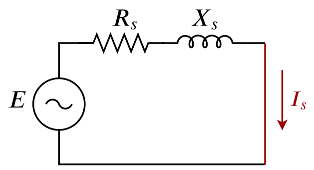


Figure 51: Short-circuit test.

Taking the winding resistance $R_s = 0.4$ mΩ into account.

$$\boxed{X_s = \sqrt{Z_s^2 - R_s^2} = 0.223699\ \Omega} \tag{186}$$

Then, $L_s = \lambda = 96.22\ \mu$H and it is clear that the resistance R_s can be neglected in the electrical model of the machine.

14. It is asked to plot the short-circuit current I_s with respect to the excitation current I_e for two frequencies $f_{min} = 370$ Hz and $f_{max} = 770$ Hz.

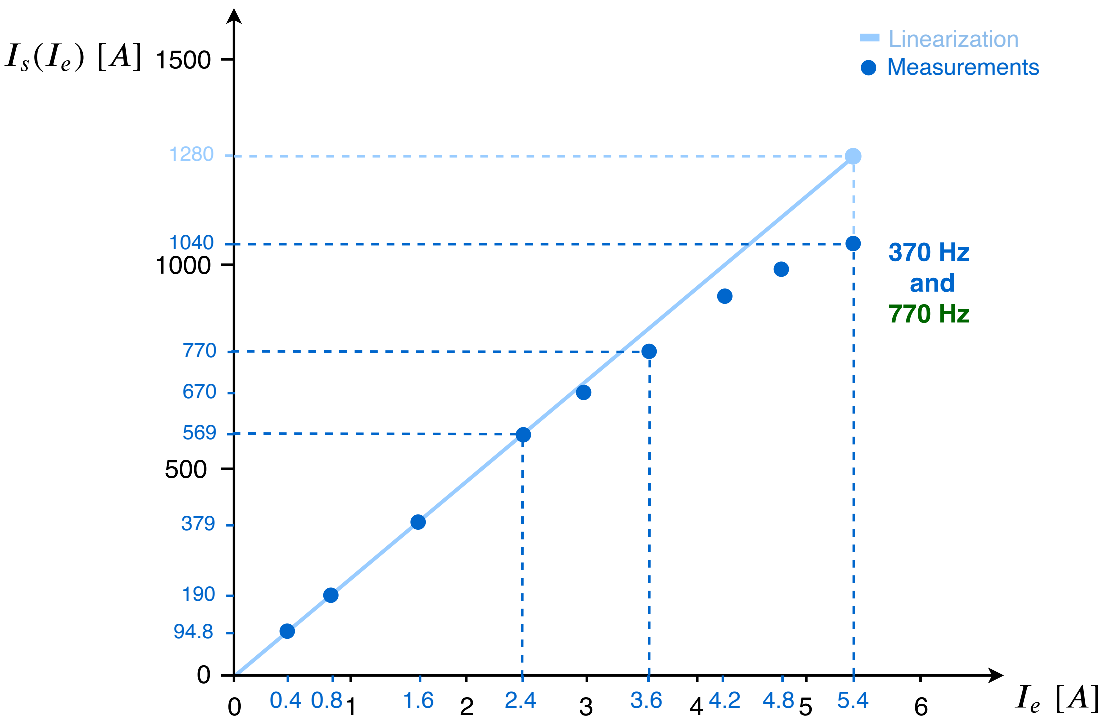


Figure 52: Short circuit current as a function of the excitation current.

As,

$$E = \alpha\,\omega\,I_e \qquad \text{(obtained from question 11)} \tag{187}$$

and

$$I_s = \frac{E}{X_s} \qquad \text{(obtained from question 13)} \tag{188}$$

One can deduce that :

$$\boxed{I_s = \frac{E}{X_s} = \frac{\alpha\,\omega\,I_e}{\omega\,L_s} = \frac{\alpha}{L_s}\,I_e} \tag{189}$$

Therefore, the short circuit current I_s is proportionnal to the excitation current I_e (less true out of the linear zone, but considered linear as α does not vary much) and the short circuit current I_s does not depend of the frequency.

15. Resisitve load at 500 Hz with $I_e = 2$ A

The first table of the statement provides an emf of 106 V for $I_e = 2$ A and a frequency of 370 Hz. At 500 Hz, the emf becomes E = 106 $\frac{500}{370}$ = 143.24 V.

(a)

$$\overline{I} = \frac{\overline{E}}{R_L + j\,X_s} \tag{190}$$

$$I = \frac{E}{\sqrt{R_L^2 + X_s^2}} = \frac{143.24}{\sqrt{0.5^2 + (2\pi \cdot 500 \cdot 96.22 \times 10^{-6})^2}} = 245.17 \text{ A} \tag{191}$$

$$V = R_L\;I = 0.5\;245.17 = 122.59 \text{ V} \tag{192}$$

(b)

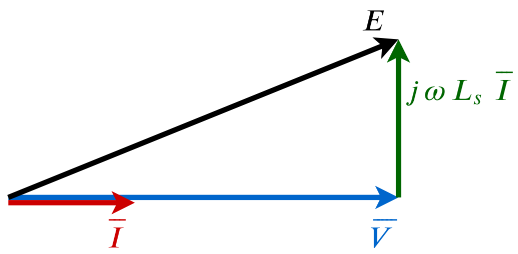


Figure 53: Phasor diagram for the purely resistive load.

(c) The load current is proportionnal to the frequency. Then, if the frequency increases, both the output voltage and current increase.

16. Resistive-inductive load

(a)

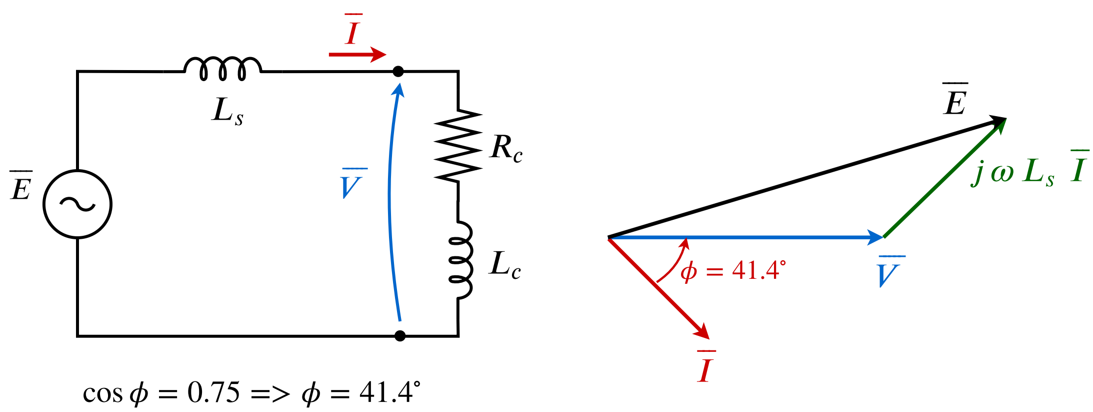


Figure 54: Phasor diagram for the resistive-inductive load.

(b)
Use

$$E = \omega\,\alpha\,I_e \tag{193}$$

$$I = \frac{E}{\sqrt{R_c^2 + \Big(\omega(L_c + L_s)\Big)^2}} \tag{194}$$

$$V = \sqrt{R_c^2 + (\omega\,L_c)^2}\;I \tag{195}$$

To expresss

$$I = \frac{\omega\,\alpha\,I_e}{\sqrt{R_c^2 + \Big(\omega(L_c + L_s)\Big)^2}} \tag{196}$$

$$V = \sqrt{R_c^2 + (\omega\,L_c)^2}\;\frac{\omega\,\alpha\,I_e}{\sqrt{R_c^2 + \Big(\omega(L_c + L_s)\Big)^2}} \tag{197}$$

(c)
With

$$P_{3\phi} = C_r\,\dot{\theta} \tag{198}$$

$$\dot{\theta} = \frac{\omega}{p} \tag{199}$$

$$P_{3\phi} = 3\;P_{1\phi} \tag{200}$$

$$P_{1\phi} = R_c\;I^2 = R_c\;\frac{(\omega\,\alpha\,I_e)^2}{R_c^2 + \Big(\omega(L_c + L_s)\Big)^2} \tag{201}$$

One finally obtain

$$\boxed{C_r = 3\,p\;R_c\;\frac{\omega\,(\alpha\,I_e)^2}{R_c^2 + \Big(\omega(L_c + L_s)\Big)^2}} \tag{202}$$

(d)
$R_c = 0.5\ \Omega$, $L_c = 150\ \mu$H and $L_s = 96.224\ \mu$H.

	I_e	α	E	I	V	C_r
$f_{min} = 370$ Hz	0.4	0.02300512	21.2	27.89	17	1
$\omega_{min} = 2324.8$ rad/s	3	0.01982202	137	180.5	109.88	41.92
	5.4	0.01189643	148	194.73	118.7	48.93
$f_{max} = 800$ Hz	0.4	0.02300512	45.84	34.34	31	0.7
$\omega_{min} = 5026.5$ rad/s	3	0.01982202	296.22	221.9	200	29.39
	5.4	0.01189643	320	239.7	216.8	34.3

Table 5: α parameter computation.

(e)
A higher frequency means a lower power factor :

$$\boxed{PF = \frac{R_c}{\sqrt{R_c^2 + \omega^2\,L_c^2}}} \tag{203}$$

Exercise 18. Three-phase alternator ◇

A three-phase alternator coupled in star provides a current I_n = 200 A under a phase-to-phase voltage U_n = 400 V at 50 Hz. The actual power factor is $\cos\varphi = 0.866$ with an inductive load. The resistance between one phase of the stator and the neutral was measured as 30 mΩ and the total of the collective losses (ferromagnetic and windage) and the Joule losses in the rotor amount to 6 kW. The synchronous reactance between one phase and the neutral is X_s = 750 mΩ.

1. Compute the useful power of the alternator.
2. Compute the Joule losses in the stator.
3. Compute the efficiency of the alternator in this configuration.
4. Draw the phasor diagram for one phase of the stator during this situation.
5. Compute the norm of the internal emf under Behn-Eschenburg assumption.
6. Provide the internal load angle δ_{int}.
7. Considering that the load is purely resisitive and that the collective losses and rotor Joule losses are kept constant, compute the efficiency of the alternator in this case.

Answers

1. P_n = 120 kW
2. p_{js} = 3.6 kW
3. η = 92.6 %
4. graph
5. E_v = 335.15 V
6. δ_{int} = 22.25°
7. η' = 93.2 %

4 AC asynchronous machines

Exercise 19. Asynchronous motor for washer cleaner

A star-shaped asynchronous motor of a high-pressure washer cleaner has the following nominal characteristics:

- Power $P_n = 5.5\,\mathrm{kW}$,
- RMS composed voltage value $U_{sn} = 400\,\mathrm{V}$,
- Frequency $f_n = 50\,\mathrm{Hz}$,
- RMS line current intensities $I_{sn} = 11\,\mathrm{A}$,
- Speed of rotation $\dot{\theta}_n = 1460\,\mathrm{RPM}$.

Assume that the stator reactance X_s is equal to the stator resistance R_s. Using a single-phase equivalent model of the asynchronous motor when needed,

1. Calculate the synchronous speed of rotation $\dot{\theta}_s$, the number of pair of poles of the motor and the nominal slip g_n;
2. Determine the value of the stator resistance R_s given that a current of RMS value $I_0 = 10\,\mathrm{A}$ flows when a voltage of RMS value $U_0 = 20.6\,\mathrm{V}$ is applied;
3. At the nominal operating point, without mechanical load, the motor draws a current of RMS value $I_{so} = 3.07\,\mathrm{A}$ for an active power $P_{so} = 245\,\mathrm{W}$. Calculate the overall losses and calculate the resistance modelling ferromagnetic losses R_{HF} and the magnetizing inductance L_μ, asumming that mechanical losses equal ferromagnetic losses;

Consider that the machine operates at nominal speed and produces the nominal mechanical power P_n to the mechanical load. The nominal mechanical power can also be denotted P_{mec}.

4. At the nominal operating point, calculate the transmitted power from the stator to the rotor and the Joules losses in the stator p_{js} and deduce the total consumed power P;
5. Calculate the rotoric resistance R_r' and the leak inductance L_r' seen from the stator;
6. At the nominal operating point, calculate the mechanical torque Γ_{un} and the electromagnetic torque Γ_n, the power factor $\cos\phi_n$ and the efficiency η_n;
7. Compute the RMS value I_s of the line currents, and the power factor $\cos\varphi$ at a rotation speed of $0\,\mathrm{RPM}$.

Solution

1. The synchronous speed $\dot{\theta}_s$ is close to the nominal rotation speed $\dot{\theta}_n$ = 1 460 rpm. Then, $\dot{\theta}_s$ = 1 500 rpm.

$$\dot{\theta}_s = \frac{1500}{60} = 25s^{-1} \tag{204}$$

$$p = \frac{f}{\dot{\theta}_s} = \frac{50}{25} = 2 \tag{205}$$

The slip corresponds to the relative difference between the synchronous speed $\dot{\theta}_s$ and the rotation speed $\dot{\theta}_n$ such that

$$\boxed{g_n = \frac{\dot{\theta}_s - \dot{\theta}_n}{\dot{\theta}_s} = \frac{1500 - 1460}{1500} = 2,67\%} \tag{206}$$

Recall of the power balance for an asynchronous motor

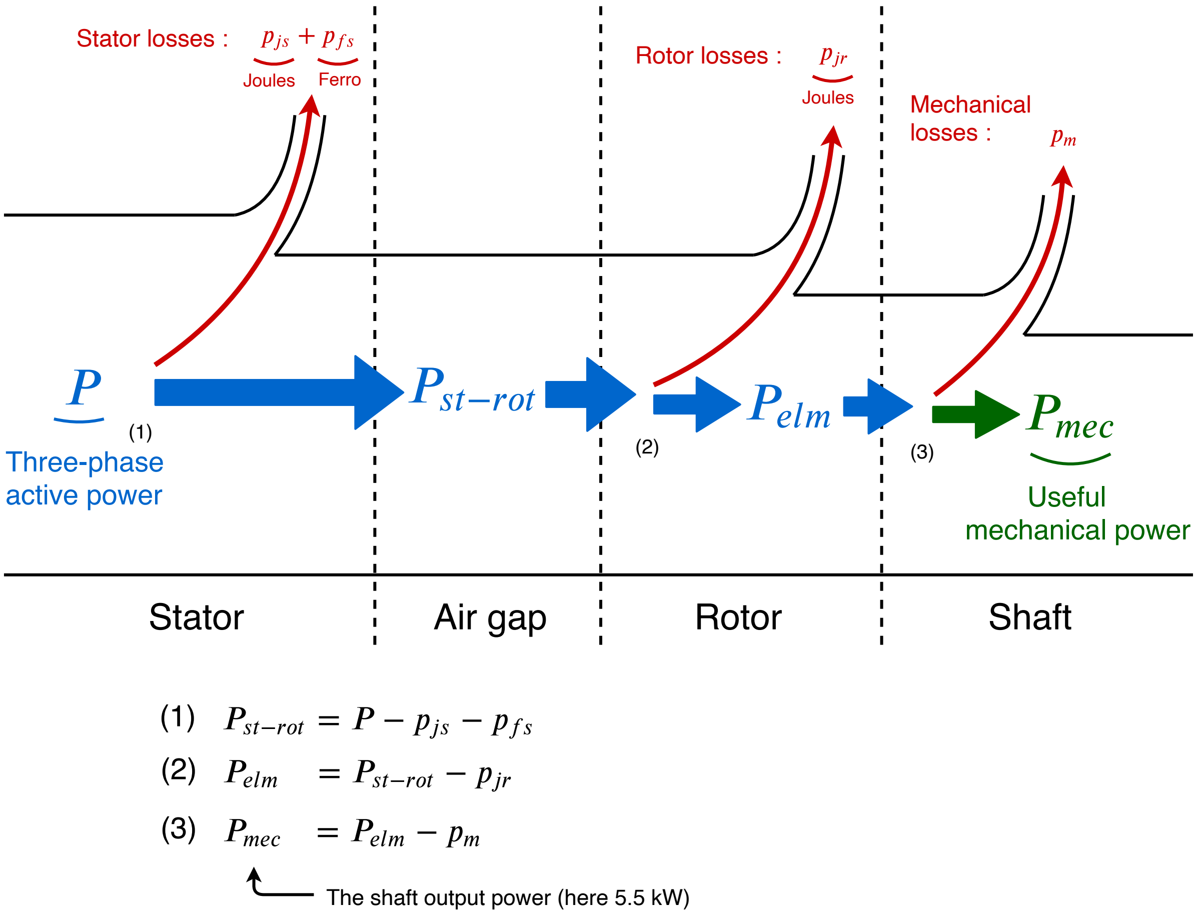


Figure 55: Power balance of the asynchronous motor.

The ferromagnetic losses in the rotor can ben neglected since the frequency of the rotoric currents is much smaller than the grid frequency (*i.e.* the frequency of the statoric currents) : $g_n f << f$ and $p_{fr} \simeq 0$ (good approximation for small slip).

Understanding $\frac{R_2'}{g}$

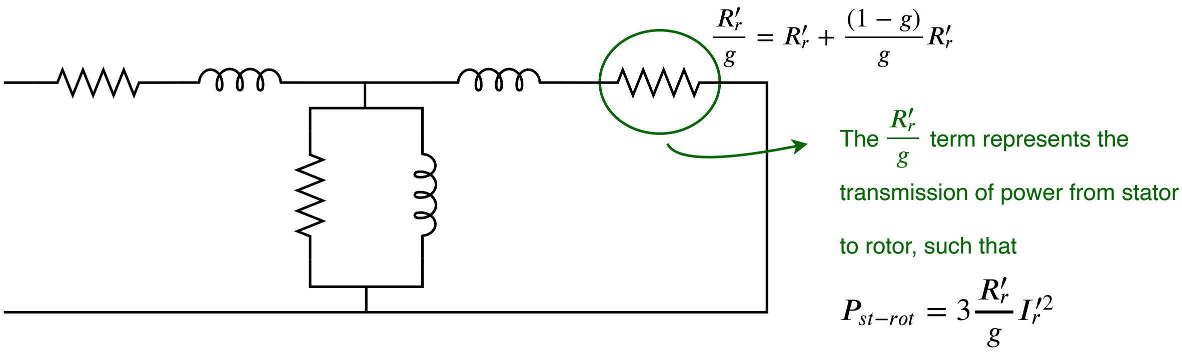


Figure 56: Power transmission between the stator and the rotor of the asynchronous motor.

$$P_{st-rot} = 3\,\frac{R_r'}{g}\,I_r'^2 \tag{207}$$

$$p_{jr} = 3\,R_r'\,I_r'^2 \tag{208}$$

$$P_{elm} = P_{st-rot} - p_{jr} = 3\,\frac{R_r'}{g}\,I_r'^2\,(1-g) \tag{209}$$

$$\tag{210}$$

Which leads to

$$\boxed{P_{elm} = (1-g)\,P_{st-rot}} \tag{211}$$

$$\boxed{p_{jr} = g\,P_{st-rot}} \tag{212}$$

2. The stator resistance R_s has been obtained with a DC test. The DC voltage was applied accross two phases of the machine and therefore, the current crossed two phases.

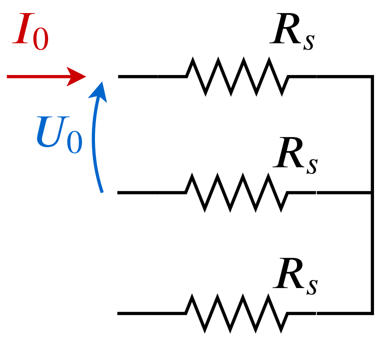


Figure 57: Stator model (star-shaped).

$$R_s = \frac{U_0}{2\,I_0} = \frac{20.6}{2 \cdot 10} = 1.03\ \Omega$$

3. In order to compute R_{HF} and X_μ, one must first redraw the equivalent circuit of the synchronous machine in operation.

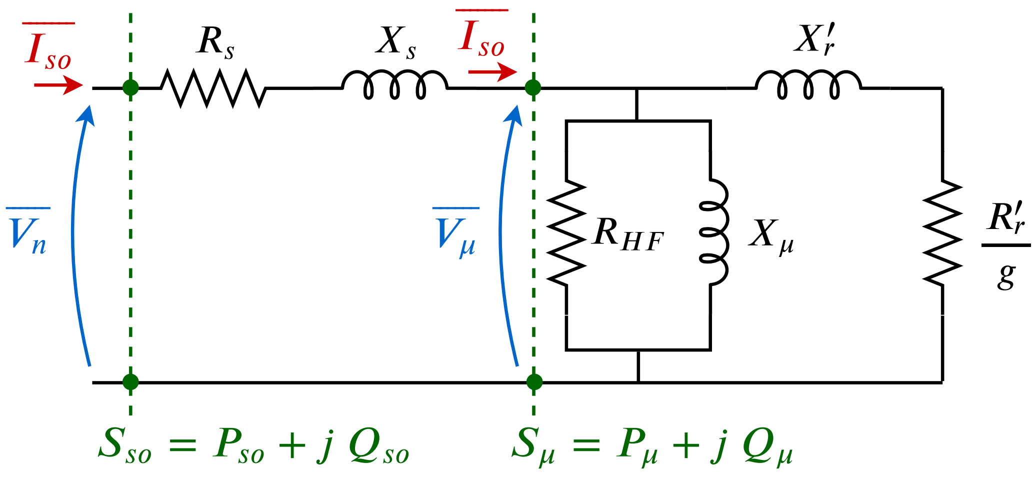


Figure 58: Equivalent circuit of the asynchronous motor running at nominal speed, without mechanical load.

$$I_{so} = 3.07 \text{ A} \tag{213}$$

$$V_n = \frac{U_n}{\sqrt{3}} = \frac{400}{\sqrt{3}} = 230.94 \text{ V} \tag{214}$$

$$P_{so} = \frac{245}{3} = 81.667 \text{ W} \tag{215}$$

$$S_{so} = V_n\, I_{so} = 708.986 \text{ VA} \tag{216}$$

$$Q_{so} = \sqrt{S_{so}^2 - P_{so}^2} = 704.27\text{var} \tag{217}$$

$$P_\mu = P_{so} - R_s\, I_{so}^2 = 81.667 - 1.03 \cdot 3.07^2 = 71.96 \text{ W} \tag{218}$$

$$Q_\mu = Q_{so} - X_s\, I_{so}^2 = 704.27 - 1.03 \cdot 3.07^2 = 694.56 \text{ var} \tag{219}$$

$$S_\mu = \sqrt{P_\mu^2 + Q_\mu^2} = 698.28 \text{ VA} \tag{220}$$

$$V_\mu = \frac{S_\mu}{I_{so}} = 227.45 \text{ V (relatively close to } V_n\text{)} \tag{221}$$

At the rotor, the losses can be decomposed as

$$\frac{R'_r}{g} I_r'^2 = \underbrace{R'_r\, I_r'^2}_{\text{rotor Joule losses}} + \underbrace{\frac{(1-g)\, R'_r}{g}\, I_r'^2}_{\text{mechanical losses}} \tag{222}$$

Knowing that the ferromagnetic losses equal the mechanical losses,
then,

$$\underbrace{\frac{V_\mu^2}{R_{HF}}}_{\text{ferromagnetic losses}} \underset{=}{+} \underbrace{\frac{(1-g)\, R'_r}{g}\, I_r'^2}_{\text{mechanical losses}} = P_\mu \tag{223}$$

$$\rightarrow\ 2\,\frac{V_\mu^2}{R_{HF}} = P_\mu \tag{224}$$

$$\boxed{R_{HF} = 2\,\frac{V_\mu^2}{P_\mu} = 2\,\frac{227.45^2}{71.96} = 1437.88\Omega} \tag{225}$$

$$\boxed{X_\mu = \frac{V_\mu^2}{Q_\mu} = \frac{227.45^2}{694.56} = 74.48\Omega} \tag{226}$$

The losses seen from the magnetizing (μ) branch are $P_\mu = 71.96$ W in the equivalent circuit of one phase. Half of those losses account for the mechanical losses $p_{m,1\phi} = 35.98$ W and the other half account for the ferromagnetic losses $p_{f,1\phi} = 35.98$ W. The total mechanical and ferromagnetic losses of the three-phase motor correspond to

$$p_m = 3\ p_{m,1\phi} = 3 \cdot 35.98 = 107.94 \text{ W} \quad (227)$$
$$p_f = 3\ p_{f,1\phi} = 3 \cdot 35.98 = 107.94 \text{ W} \quad (228)$$

4. One wants to determine P_{st-rot}, p_{js}, P at the nominal operating point of the motor. The mechanical losses are considered independent of the rotation speed. Then, $p_m(1460 \text{ rpm}) = p_m(1500 \text{ rpm}) \simeq 108$ W.

   From the equation (3) of the power balance

$$P_{mec} = P_{elm} - p_m \quad (229)$$
$$P_{elm} = P_{mec} + p_m = 5\,500 + 108 = 5\,608 \text{ W} \quad (230)$$

   From (211)

$$P_{elm} = P_{st-rot}\,(1-g) \quad (231)$$
$$\boxed{P_{st-rot} = \frac{P_{elm}}{1-g} = \frac{5\,608}{1-0.02667} = 5\,761.6 \text{ W}} \quad (232)$$

   The Joule losses in the stator are

$$\boxed{p_{js} = 3\,R_s\,I_{sn}^2 = 3 \cdot 1.03 \cdot 11^2 = 373.89 \text{ W}} \quad (233)$$

   The ferromagnetic losses are kept constant

$$\boxed{p_f = 108 \text{ W}} \quad (234)$$

   Finally, the three-phase active power can be computed as

$$\boxed{P = P_{st-rot} + p_{js} + p_f = 5761.6 + 373.4 + 108 = 6\,243 \text{ W}} \quad (235)$$

5. The parameters R'_r and L'_r (seen from the stator) can be obtained from the study of the machine in operation.

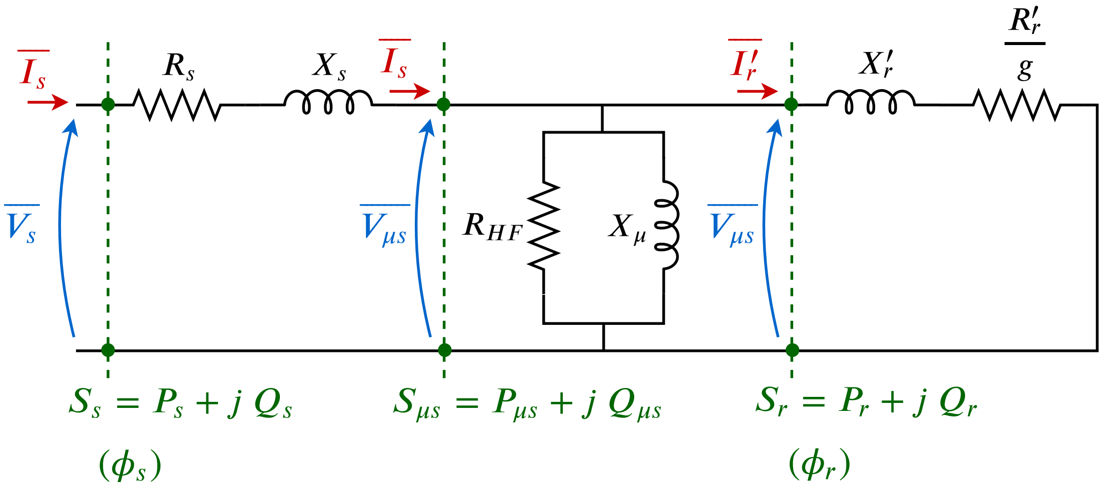


Figure 59: Equivalent circuit of the asynchronous motor in operation.

In order to determine R'_r and X'_r, find P_r, Q_r and I'_r and deduce

$$R'_r = g\frac{P_r}{I'^2_r} \tag{236}$$

$$X'_r = \frac{Q_r}{I'^2_r} \tag{237}$$

Then,

$$P_s = \frac{P}{3} = 2\,081 \text{ W} \tag{238}$$

$$S_s = V_s\,I_s = 230.94 \cdot 11 = 2\,540.34 \text{ VA} \tag{239}$$

$$Q_s = \sqrt{S_s^2 - P_s^2} = 1\,457 \text{ var} \tag{240}$$

$$\phi_s = \arccos\left(\frac{P_s}{S_s}\right) = 35° \tag{241}$$

Accross the magnetizing branch,

$$P_{\mu s} = P_s - R_s\,I_s^2 = 2\,081 - 1.03 \cdot 11^2 = 1\,956.37 \text{ W} \tag{242}$$

$$Q_{\mu s} = Q_s - X_s\,I_s^2 = 1\,457 - 1.03 \cdot 11^2 = 1\,332.37 \text{ W} \tag{243}$$

$$S_{\mu s} = \sqrt{P_\mu^2 + Q_\mu^2} = 2\,366.9 \text{ VA} \tag{244}$$

$$V_{\mu s} = \frac{S_{\mu s}}{I_s} = \frac{2\,366.9}{11} = 215.2 \text{ V} \tag{245}$$

Then, on the rotor side of the equivalent circuit,

$$P_r = P_{\mu s} - \frac{V_{\mu s}^2}{R_{HF}} = 1\,956.37 - \frac{215.2^2}{1\,437.88} = 1\,923.2 \text{ W} \tag{246}$$

$$Q_r = Q_{\mu s} - \frac{V_{\mu s}^2}{X_\mu} = 1\,332,37 - \frac{215.2^2}{74.99} = 714.8 \text{ var} \tag{247}$$

$$S_r = \sqrt{P_r^2 + Q_r^2} = 2\,051.7\text{VA} \tag{248}$$

$$\phi_r = \arccos\left(\frac{P_r}{S_r}\right) = 20.43° \tag{249}$$

$$I'_r = \frac{S_r}{V_{\mu s}} = \frac{2\,052.24}{215.2} = 9.537 \text{ A} \tag{250}$$

Finally,

$$\boxed{R'_r = g\,\frac{P_r}{I_r'^2} = 0.02667\,\frac{1\,923.2}{9,537^2} = 0.564\ \Omega} \tag{251}$$

$$X'_r = \frac{Q_r}{I_r'^2} = \frac{714.8}{9.537^2} = 7.86\ \Omega \tag{252}$$

$$\boxed{L'_r = 25 \text{ mH}} \tag{253}$$

6. Now, one must determine C_{mec}, C_{elm}, $\cos\phi_n$ and η_n for the nominal operating point.

$$P_{mec} = C_{mec,n}\,\dot{\theta}_n \tag{254}$$

Then, the output torque is given by

$$C_{mec,n} = \frac{P_{mec}}{\dot{\theta}_n} = \frac{5\,500}{1\,460 \cdot \frac{2\pi}{60}} = 35.97 \text{ Nm} \tag{255}$$

The electromagnetic torque is

$$C_{elm,n} = \frac{P_{elm}}{\dot{\theta}_n} = \frac{5\,608}{1\,460 \cdot \frac{2\pi}{60}} = 36.7 \text{ Nm} \tag{256}$$

The $\cos\phi_n$ can be computed by two ways,

$$\cos\phi_n = \frac{P}{S_n} = \frac{6\,243}{\sqrt{3}\cdot 400 \cdot 11} = 0.818 \tag{257}$$

or

$$\phi_s = \phi_n \Rightarrow \cos\phi_n = 0.818 \tag{258}$$

Finally, the efficiency during nominal operation is

$$\boxed{\eta_n = \frac{P_{mec}}{P} = \frac{5\,500}{6\,242} = 88.1\%} \tag{259}$$

7. <u>I_s and $\cos\phi$ for $\dot{\theta} = 0$ rpm</u>

At 0 rpm, the motor is stalled, the slip becomes $g = 1$, $\frac{R'_r}{g} = R'_r$ and the equivalent circuit of the asynchronous motor becomes

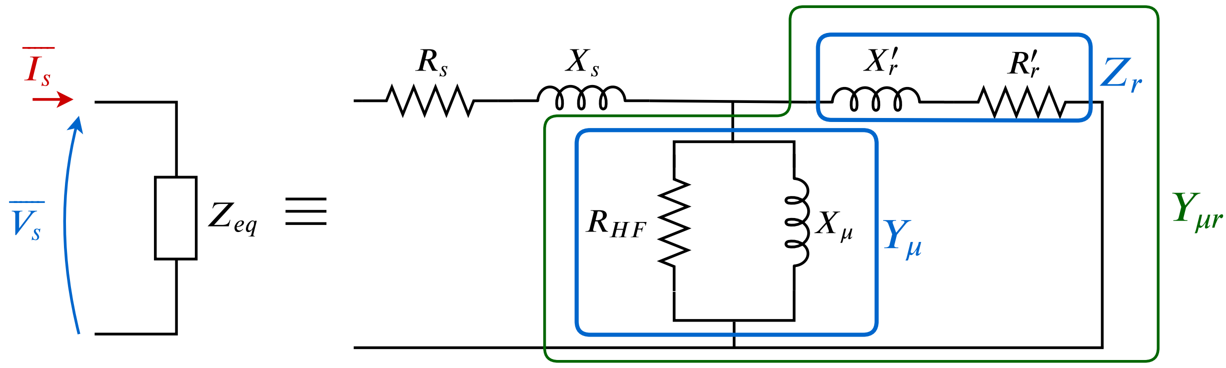


Figure 60: Equivalent circuit of the asynchronous motor when stalled.

$$Z_r = R'_r + j\ X'_r \tag{260}$$

$$Y_r = \frac{1}{Z_r} = \frac{1}{R'_r + j\ X'_r} \tag{261}$$

$$Y_\mu = \frac{1}{R_{HF}} + j\ \frac{1}{X_\mu} \tag{262}$$

$$Y_{\mu r} = Y_\mu + Y_r = \frac{1}{R_{HF}} + j\ \frac{1}{X_\mu} + \frac{1}{R'_r + j\ X'_r} \tag{263}$$

$$Z_{\mu r} = \frac{1}{Y_{\mu r}} = \frac{1}{\frac{1}{R_{HF}} + j\ \frac{1}{X_\mu} + \frac{1}{R'_r + j\ X'_r}} \tag{264}$$

$$Z_{eq} = R_s + j\ X_s + Z_{\mu r} = R_s + j\ X_s + \frac{1}{\frac{1}{R_{HF}} + j\ \frac{1}{X_\mu} + \frac{1}{R'_r + j\ X'_r}} \tag{265}$$

$$Z_{eq} = 1.03 + j\ 1.03 + \frac{1}{\frac{1}{1\,437.8} + j\ \frac{1}{75} + \frac{1}{0.564 + j\ 7.87}} = 9.969\ \angle\ 79.67° \tag{266}$$

$$\boxed{\overline{I_s} = \frac{\overline{V_s}}{Z_{eq}} = \frac{230}{9.969\ \angle\ 79.67°} = 23.07\ \angle\ \underbrace{-79.67°}_{\phi}} \tag{267}$$

$$\boxed{\cos\varphi = \cos(-79.67°) = 0.179} \tag{268}$$

Exercise 20. Asynchronous motor of a fan

On the nameplate of an asynchronous motor of a fan used in an air handling unit, the following characteristics are read:

4.4 kW; 230/400 V; 15.5/9 A; 50 Hz; 4 poles

Using a single-phase equivalent model of the asynchronous motor:

1. Explain the meaning of each element on the nameplate;
2. The motor is used on a 230 V network, explain which winding coupling should be used for the stator;
3. Calculate the synchronous speed of rotation $\dot{\theta}_s$;
4. Given that the (DC) resistance value measured between two stator terminals is $R_a = 0.654\,\Omega$, compute the value of the statoric resistance R_s of the equivalent single-phase model;

5. A calibrated motor is used to rotate the shaft of the unpowered considered motor, upto reaching the synchronous speed, at which the calibrated motor consumes 86 W. Calculate the mechanical losses of the motor and explain why assuming that these mechanical losses remain constant is a good approximation;

6. At the nominal operating point, without mechanical load, the motor draws a current of RMS value $I_{so} = 3.82$ A for an active power $P_{so} = 300$ W. Calculate the resistance modelling ferromagnetic losses R_{H+F} and the statoric inductance L_μ;

7. The rotor shaft of the motor is stalled while a voltage of RMS value $U_{sc} = 57.5$ V is applied for a consumed three-phase active power $P_{sc,3\phi} = 374$ W and three-phase reactive power $Q_{sc,3\phi} = 1.09$ kvar. Calculate the rotoric resistance R'_r and the leak inductance X'_r seen from the stator.

A direct voltage of value V_s and frequency f is applied on each phase of the motor.

8. Using single-phase equivalent model of the asynchronous motor, express the RMS current value I_s in terms of V_s, R_s R'_r, g et X'_r;

9. Calculate the transmitted power from the stator to the rotor;

10. Calculate the electromagnetic torque C_{elm} and give the maximal reachable torque $\Gamma_{\max}$ after showing that C_{elm} is maximal for a slip value g_{max};

11. Plot C with respect to g for an applied voltage V_s equal to V_n, $\frac{V_n}{\sqrt{2}}$ and $\frac{V_n}{2}$;

12. Explain why a control on the rotor voltages is not suitable for speed variation for load having constant resistive torque;

13. To limit the peak current when starting the motor, a star/delta starter is frequently used. Assuming that this transient mode is much more longer compared to period corresponding to the frequency f of the applied voltages, calculate the RMS current values of the line currents compared to those drawn by using a star/delta starter.

Solution

1. Explain the nameplate

Nominal active power : $P_n = 4,4$ kW

RMS voltages : $\underbrace{230}_{\Delta}$ / $\underbrace{400}_{Y}$ V

Line currents : $\underbrace{15,5}_{\Delta}$ / $\underbrace{9}_{Y}$ A

Nominal frequency : $f_n = 50$ Hz

Number of pairs of poles : 4 poles $\longrightarrow$ 2 pairs of poles $\longrightarrow$ $p = 2$

2. Which coupling for a 230 V network ?

230 V network means that the RMS value of thez composed voltages is $U = 230$ V. Then, the armature (stator) of the machine should be connected in Δ.

3. Synchronous speed $\dot{\theta}_s$

$$\dot{\theta}_s = \frac{f_n}{p} = \frac{50}{2} = 25 \text{ Hz} = 1500 \text{ rpm} = 157 \text{ rad/s} \quad (269)$$

4. Stator resistance R_s

R_a corresponds to the total resitance measured between two terminals of the stator. Then, R_s, the resistance of one phase is two times smaller than $R_a = \frac{U_0}{I_0}$.

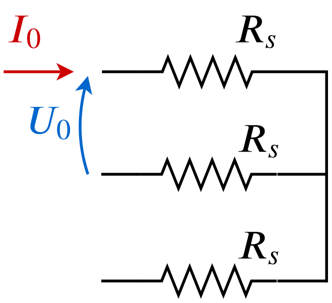


Figure 61: Stator model (star-shaped).

$$R_s = \frac{R_a}{2} = \frac{0,654}{2} = 0,327\ \Omega$$

5. Mechanical losses at synchronous speed

The asynchronous machine is put into potion by an external motor rotating at $\dot{\theta}_s$ and remains unpowered at the stator. Therefore, the mechanical power provided by the external potor exactly compensate the mechanical losses of the asynchronous machine : $p_m = 86$ W.

The mechanical losses depends on the rotation speed. The nominal rotation speed is close to the synchronous rotation speed (low slip), that is why the variation in mechanical losses can be neglected.

6. Determine R_{HF} and L_μ

During the no load test, the active power in the stator P_{so} is

$$P_{so} = \underbrace{p_{js}}_{\text{stator joule losses}} + \underbrace{p_f}_{\text{ferromagnetic losses}} + \underbrace{p_m}_{\text{mechanical losses}} \quad (270)$$

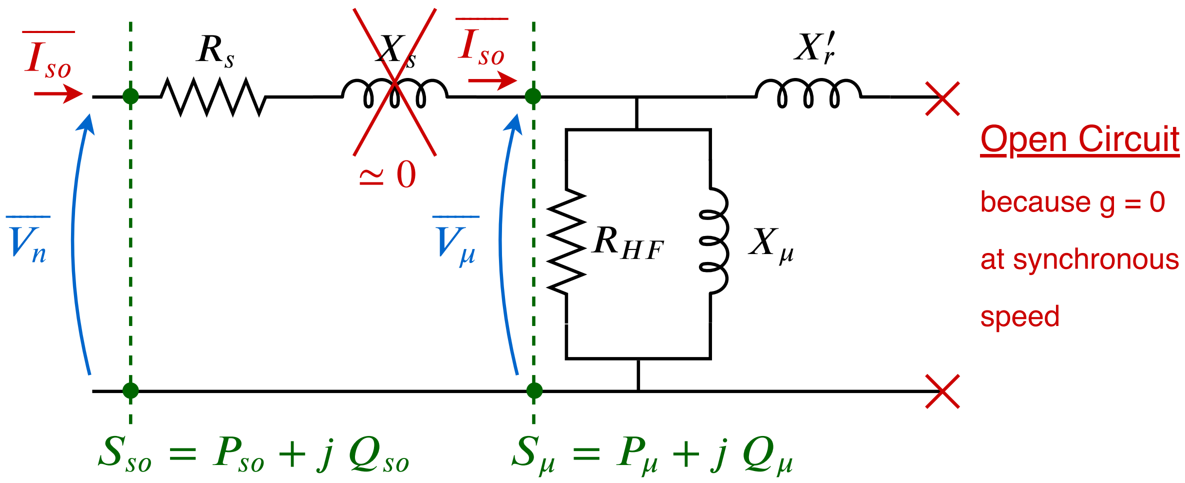


Figure 62: Equivalent circuit of the asynchronous motor running at synchronous speed.

$$I_{so} = 3,82 \text{ A} \quad (271)$$

$$V_n = \frac{U_n}{\sqrt{3}} = \frac{230}{\sqrt{3}} = 132,79 \text{ V} \quad (272)$$

$$P_{so} = \frac{300}{3} = 100 \text{ W} \quad (273)$$

$$S_{so} = V_n\, I_{so} = 507,26 \text{ VA} \quad (274)$$

$$Q_{so} = \sqrt{S_{so}^2 - P_{so}^2} = 497,3\text{var} \quad (275)$$

$$P_\mu = P_{so} - R_s\, I_{so}^2 = 100 - 0,327 \cdot 3,82^2 = 95,23 \text{ W} \quad (276)$$

$$Q_\mu = Q_{so} - \underbrace{X_s}_{\simeq 0}\, I_{so}^2 = Q_{so} = 497,3 \text{ var} \quad (277)$$

$$S_\mu = \sqrt{P_\mu^2 + Q_\mu^2} = 506,34 \text{ VA} \quad (278)$$

$$V_\mu = \frac{S_\mu}{I_{so}} = \frac{506,34}{3,82} = 132,55 \text{ V (relatively close to } V_n) \quad (279)$$

At this point, the mechanical losses are still taken into account and must be subtracted.

Then, the ferromagnetic losses correspond to

$$p_f = P_\mu - \frac{p_m}{3} = 95,23 - \frac{86}{3} = 66,56\ \mathrm{W} \tag{280}$$

$$\boxed{R_{HF} = \frac{V_\mu^2}{P_f} = \frac{132,55^2}{66,56} = 264\ \Omega} \tag{281}$$

$$X_\mu = \frac{V_\mu^2}{Q_\mu} = \frac{132,55^2}{497,3} = 35,2\ \Omega \tag{282}$$

$$\boxed{L_\mu = \frac{X_\mu}{100\,\pi} = 0,112\ \mathrm{H}} \tag{283}$$

7. Stalled rotor test

Stalled rotor $\longrightarrow g = 1 \longrightarrow \frac{R_r'}{g} = R_r'$

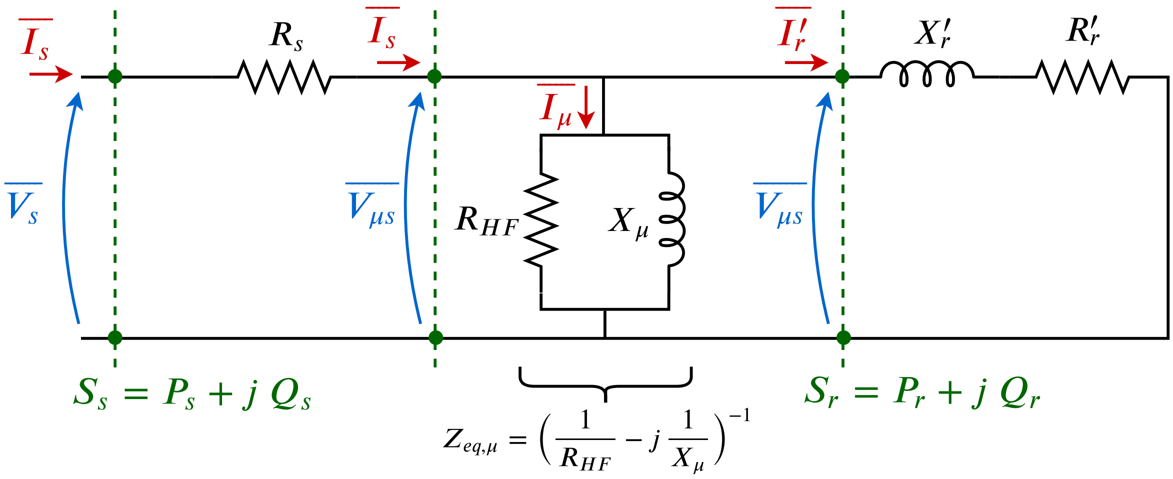


Figure 63: Equivalent circuit of the stalled asynchronous motor.

$$P_s = \frac{374}{3} = 124,67 \text{ W} \tag{284}$$

$$Q_s = \frac{1090}{3} = 363,3 \text{ var} \tag{285}$$

$$S_s = \sqrt{P_s^2 + Q_s^2} = 384,1 \text{ VA} \tag{286}$$

$$V_s = \frac{U_s}{\sqrt{3}} = \frac{57,5}{\sqrt{3}} = 33,2 \text{ V} \tag{287}$$

$$\overline{I_s} = \frac{P_s - j\,Q_s}{V_s} = \frac{124,67 - j\,363,3}{33,2} = 11,57\angle - 71,06^\circ \tag{288}$$

$$I_s = 11,57 \text{ A} \tag{289}$$

$$V_\mu = V_s - R_s\,I_s = 33,2 - 0,327 \cdot 11,57 = 29,42 \text{ V} \tag{290}$$

$$\overline{I_\mu} = \frac{\overline{V_\mu}}{Z_{eq,\mu}} = \frac{\overline{V_\mu}}{\left(\frac{1}{R_{HF}} - j\,\frac{1}{X_\mu}\right)^{-1}} = \frac{29,42}{\left(\frac{1}{264} - j\,\frac{1}{35,2}\right)^{-1}} = 0,843\angle - 82^\circ \tag{291}$$

$$\overline{I'_r} = \overline{I_s} - \overline{I_\mu} = (11,57\angle - 71,06^\circ) \; - (0,843\angle - 82^\circ) = 10,74\angle - 70,21^\circ \tag{292}$$

$$I'_r = 10,74 \text{ A} \tag{293}$$

$$P_r = P_s - R_s\,I_s^2 - \frac{V_\mu^2}{R_{HF}} = 124,67 - 0,327 \cdot 11,57^2 - \frac{29,42^2}{264} = 77,62 \text{ W} \tag{294}$$

$$Q_r = Q_s - \frac{V_\mu^2}{X_\mu} = 363,3 - \frac{29,42^2}{35,2} = 338,71 \text{ var} \tag{295}$$

$$\boxed{R'_r = \frac{P_r}{I_r'^2} = \frac{77,62}{10,74^2} = 0,673\ \Omega} \tag{296}$$

$$\boxed{X'_r = \frac{Q_r}{I_r'^2} = \frac{338,71}{10,74^2} = 2,93\ \Omega} \tag{297}$$

8. <u>Express I_s in terms of V_s, R_s, R'_r, g and X'_r.</u>

Neglecting the magnetizing brnahc and the magnetizing current I_μ,

$$\overline{I_s} = \frac{\overline{V_s}}{R_s + \frac{R'_r}{g} + j\,X'_r} \tag{298}$$

$$\boxed{I_s = \frac{V_s}{\sqrt{\left(R_s + \frac{R'_r}{g}\right)^2 + X_r'^2}}} \tag{299}$$

9. <u>Transmitted power from stator to rotor.</u>

$$P_{st-rot} = 3\,\frac{R'_r}{g}\,I_s^2 = 3\,\frac{R'_r}{g}\,\frac{V_s^2}{\left(R_s + \frac{R'_r}{g}\right)^2 + \left(X'_r\right)^2} \tag{300}$$

10. Show that C_{elm} is maximum for $g_m$and compute C_{elm}.

As $P_{elm} = (1-g)\,P_{st-rot}$ and $\dot{\theta} = (1-g)\,\dot{\theta}_s$, the torque can be expressed as

$$C_{elm} = \frac{P_{elm}}{\dot{\theta}} = \frac{P_{st-rot}}{\dot{\theta}_s} \tag{301}$$

$$C_{elm} = \frac{P_{st-rot}}{\dot{\theta}_s} = 3\,\frac{V_s^2}{\dot{\theta}_s}\,\frac{\frac{R_r'}{g}}{\left(R_s+\frac{R_r'}{g}\right)^2+\left(X_r'\right)^2} \tag{302}$$

$$\frac{d\,C_{elm}}{d\,g} = 3\,\frac{R_r'\,V_s^2}{\dot{\theta}_s}\left(-\frac{\frac{1}{g^2}\left(\left(R_s+\frac{R_r'}{g}\right)^2 + (X_r')^2\right)+\frac{1}{g}\,2\,\left(R_s+\frac{R_r'}{g}\right)\left(-\frac{R_r'}{g^2}\right)}{\left(\left(R_s+\frac{R_r'}{g}\right)^2 + (X_r')^2\right)^2}\right) \tag{303}$$

$$\frac{d\,C_{elm}}{d\,g} = 3\,\frac{R_r'\,V_s^2}{g^2\,\dot{\theta}_s}\left(\frac{R_s^2-\left(\frac{R_r'}{g}\right)^2+X_r'^2}{\left(\left(R_s+\frac{R_r'}{g}\right)^2 + (X_r')^2\right)^2}\right) \tag{304}$$

From that expression, $\frac{d\,C_{elm}}{d\,g}=0$ if $R_s^2-\left(\frac{R_r'}{g}\right)^2+X_r'^2=0$
Meaning, for a slip such that

$$\boxed{g_{max} = +\frac{R_r'}{\sqrt{R_s^2+X_r'^2}}} \quad \text{(positive slip for the motor)} \tag{305}$$

And for such a slip, g_{max}, the maximum electromagnetic torque is

$$\boxed{C_{elm,max} = 3\,\frac{V_s^2}{\dot{\theta}_s}\,\frac{\sqrt{R_s^2+X_r'^2}}{\left(R_s+\sqrt{R_s^2+X_r'^2}\right)^2+X_r'^2} = 3\,\frac{V_s^2}{2\,\dot{\theta}_s}\,\frac{1}{R_s+\sqrt{R_s^2+X_r'^2}}} \tag{306}$$

Neglecting the stator resistance, R_s, the result is

$$C_{elm,max} = 3\,\frac{V_s^2}{\dot{\theta}_s}\,\frac{1}{X_r'} \quad \text{(same as theory)} \tag{307}$$

11. Plot C_{elm} wrt g for $V_s = V_n$, $V_s = \frac{V_n}{\sqrt{2}}$ and $V_s = \frac{V_n}{2}$.

At $V_s = V_n = 132{,}79$ V ; the torque is

$$C_{elm,max} = 3\,\frac{132,79^2}{1\,500 \cdot \frac{2\pi}{60}}\,\frac{\sqrt{0,327^2+2,84^2}}{\left(0,327+\sqrt{0,327^2+2,84^2}\right)^2+2,84^2} = 52,86 \text{ Nm} \tag{308}$$

At $V_s = \frac{V_n}{\sqrt{2}} = 93{,}9$ V ; the torque is

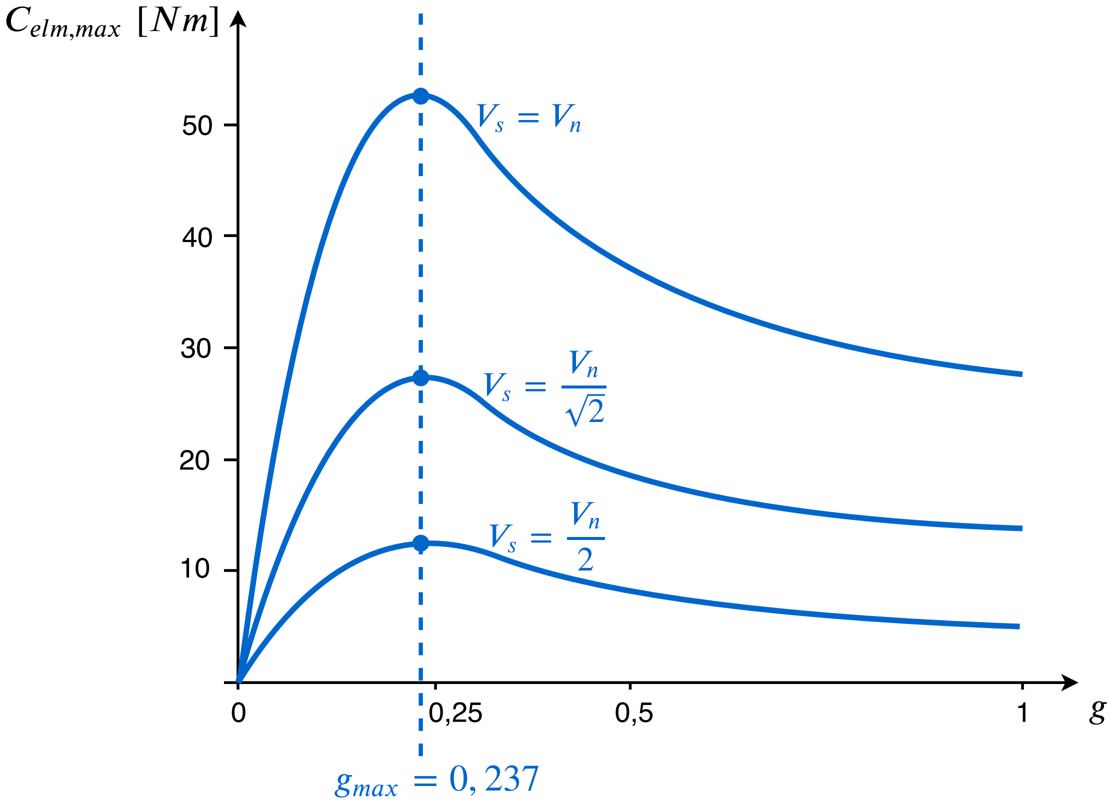
$$C_{elm,max} = 3\,\frac{93,9^2}{1\,500 \cdot \frac{2\pi}{60}}\,\frac{\sqrt{0,327^2+2,84^2}}{\left(0,327+\sqrt{0,327^2+2,84^2}\right)^2+2,84^2} = 26,43 \text{ Nm} \tag{309}$$

At $V_s = \frac{V_n}{2} = 66{,}4$ V ; the torque is

$$C_{elm,max} = 3\,\frac{66,4^2}{1\,500 \cdot \frac{2\pi}{60}}\,\frac{\sqrt{0,327^2+2,84^2}}{\left(0,327+\sqrt{0,327^2+2,84^2}\right)^2+2,84^2} = 13,21 \text{ Nm} \tag{310}$$

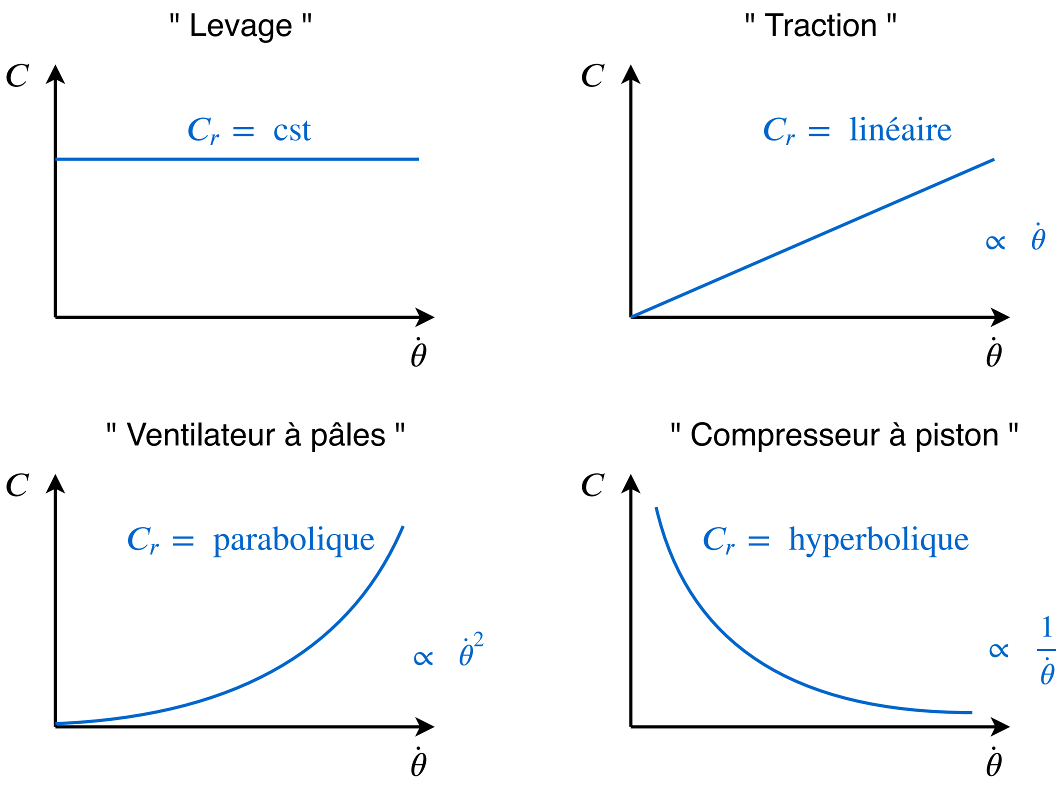
All maximum torques occur at the same slip

$$g_{max} = \frac{0,677}{\sqrt{0,327^2+2,84^2}} = 0,237 \tag{311}$$



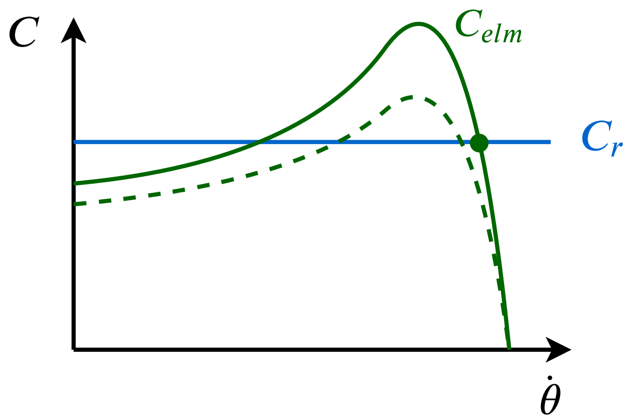
12. Why the control of the voltage is not suited for speed variations at a constant torque.

First of all, the mechanical behavior of different loads (pump, car, fan...) can be separated in 4 main categories.

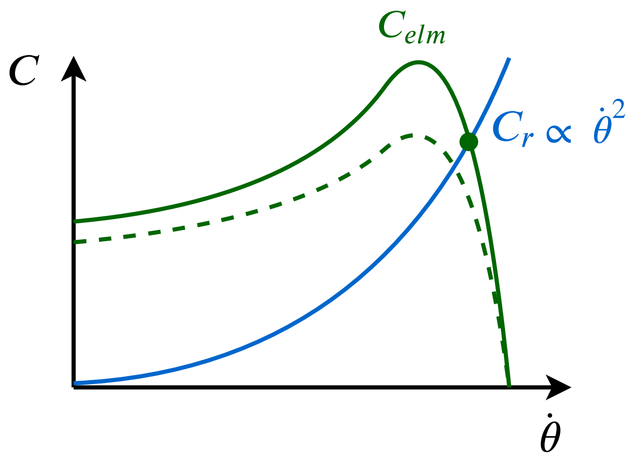


In the first case, decreasing the voltage enables to decrease the speed, but with very small impact. Moreover, the machine must remain in the stable region which is narrow.

→ The voltage control is not suited for speed variation (at a constant resistive torque).

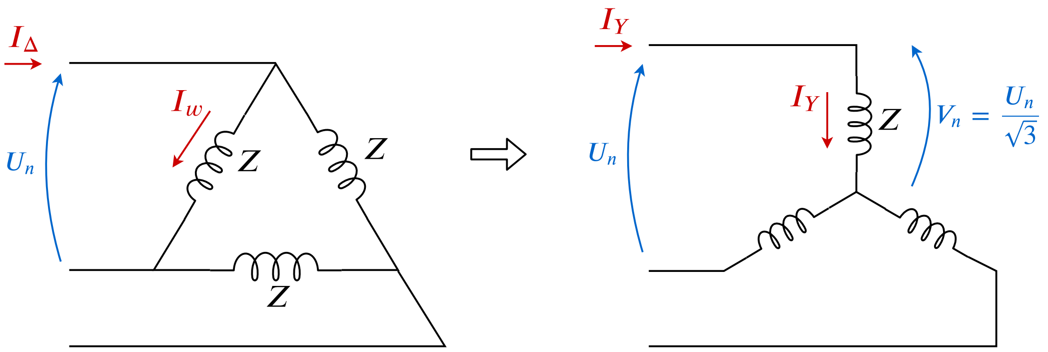


In the other case, if the load is a fan (such as in this exercise), the speed can be easily controlled only by varying the voltage.



13. Comparison without and with star/delta starter.

The use of the star/delta connection allows to reduce the inrush current by a factor of 3.



Winding current : $I_w = \frac{I_\Delta}{\sqrt{3}} = \frac{U_n}{Z}$

Winding current : $I_Y = \frac{V_n}{Z} = \frac{U_n}{\sqrt{3}\,Z} = \frac{I_\Delta}{\sqrt{3}}\frac{1}{\sqrt{3}} = \frac{I_\Delta}{3}$

Exercise 21. Wind turbine ◇

Due to the variations of wind conditions, coupling a synchronous generator, needing a constant speed of rotation, is not feasible. An asynchronous generator (hypersynchronous) is therefore generally used to allow more flexibility in the speed of rotation of the blades. A variable speed wind turbine is studied in this exercise. It is possible to tune the slip by using a wound rotor (not a squirrel cage) typically from 0 % to 10 % by acting on the resistances R of the rotor. During wind gust, the value R is increased, to increase the slip value (in absolute value, since it is negative) allowing thus to increase the speed of rotation and smoothen the power transmitted to the electrical grid.

The studied wind turbine has a 4-poles asynchronous generator of nominal power $P_n = 800\,\mathrm{kW}$ connected to an electrical grid having a composed voltage $U_{sn} = 690\,\mathrm{V}$. The resistance values R of the rotor can be tuned from $0\,\mathrm{m\Omega}$ to $9\,\mathrm{m\Omega}$, in addition to the resistance value of the windings $R_R = 3\,\mathrm{m\Omega}$.

1. Express the input shaft torque Γ_i with respect to the speed of rotation $\dot{\theta}$;
2. Calculate the operating point (speed and slip) of the generator for $R = 0\,\mathrm{m\Omega}$, $P = 2\,\mathrm{MW}$ and $R = 9\,\mathrm{m\Omega}$, $P = 4\,\mathrm{MW}$.

5 DC machines

Exercise 22. Brushed DC motor

The motor of a hammer drill has the following characteristics:

- independant excitation DC machine,
- 2 poles (1 pair),
- Nominal power $P_n = 800\,\text{W}$,
- Nominal speed of rotation $\dot{\theta}_n = 1500\,\text{RPM}$,
- Nominal power voltage $U_n = 220\,\text{V}$,
- Nominal rotor current intensity $I_n = 4.6\,\text{A}$,
- Nominal stator current intensity $I_{en} = 0.35\,\text{A}$.

Using two different excitation currents I_e, the electromotive force has been determined for different rotation speeds (Tables 6 et 7).

Table 6: Electromotive force with respect to the rotation speed for an excitation current $I_{e1} = 0.35\,\text{A}$.

n [RPM]	E [V]
1670	240
1510	220
1380	200
1040	150
820	120
510	75
110	20
0	0

Table 7: Electromotive force with respect to the rotation speed for an excitation current $I_{e2} = 0.20\,\text{A}$.

n [RPM]	E [V]
1800	186
1450	150
1150	120
850	90
560	60
260	30
0	0

1. Plot E with respect to $\dot{\theta}$ for I_{e1} and I_{e2} and justify the shape of the curves;
2. Show that the flux Φ is not proportional to the excitation current intensity I_e.

Maintaining the nominal speed of rotation, the electromotive force is measured for different excitation currents I_e (Table 8).

3. Plot E with respect to I_e and justify the shape of the curves.

Some measurements have allowed to quantify the stator resistance value, which is $R_e = 512.1\,\Omega$ and the rotor resistance value is $R = 4.6\,\Omega$.

4. Draw the equivalent model of the motor.

A test at constant nominal speed has been performed to measure the voltage across the rotor U and the current drawn in the rotor I for different excitation currents value I_e (Table 9).

Table 8: Electromotive force with respect to the excitation current at nominal constant rotation speed.

I_e [A]	E [V]
0.39	219
0.35	210
0.33	204
0.31	198
0.3	194
0.28	188
0.26	179
0.24	168
0.22	158
0.2	147
0.18	137
0.17	130
0.14	107
0.13	100
0.11	87
0.1	78
0.08	65
0.07	56

Table 9: Voltage across the rotor U and current drawn in the rotor I for different excitation currents value I_e.

I_e [A]	U [V]	I [A]
0.4	222	0.43
0.35	213	0.44
0.3	198	0.45
0.25	176	0.48
0.2	151	0.56
0.15	120	0.66
0.1	85	0.92

5. Plot the collective (i.e. ferromagnetic plus mechanical) losses p_c with respect to I_e;
6. For the linear part of the curve, determine the mechanical losses p_m at the nominal speed of rotation.

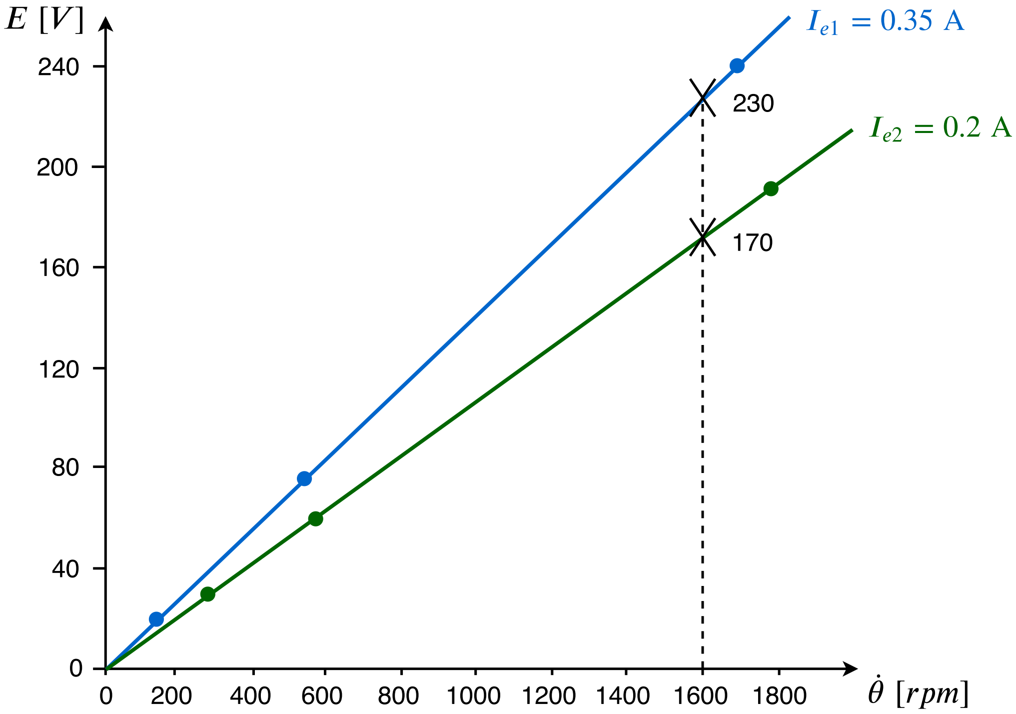
A hole is drilled using the drill. The nominal speed of rotation remains constant while the rotor draws a current of $I_0 = 3$ A when a voltage $U_0 = 212$ V is measured on the rotor terminals.

7. Calculate the electromotive force and deduce the value of the excitation current I_{e0};
8. Compute the shaft output power P_u;
9. Deduce the resistive torque C_r induced by the drilling process.

Solution

1. Plot E wrt $\dot{\theta}$ for different excitation current (I_{e1} and I_{e2}).

   In both cases, the emf E is proportionnal to the rotation speed. The excitation current I_{e1} is 75 % higher than I_{e2} ($\frac{I_{e1}}{I_{e2}} = 1.75$ (*)) whereas the ratio of the emf is lower than 1.75 due to the saturation.



2. Show that the flux Φ is not proportionnal to the excitation current I_e.

   First, the electromotive force E is proportionnal to the flux Φ : $E = k\,\dot{\theta}\,\Phi$.
   Then, at 1600 rpm, $\frac{E_1}{E_2} = \frac{230}{170} = 1.35$ such that $\frac{\Phi_1}{\Phi_2} = 1.35$ (**).

   Since (*) $\neq$ (**), the flux Φ is not proportionnal to the excitation current I_e. This is due to the saturation phenomenon.

3. Plot the voltage E wrt the excitation current I_e and justify.

   The curve $E(I_e)$ behaves as the $\mathcal{B}(\mathcal{H})$. The behaviour is linear for low excitation current I_e and the saturation occurs at some point, when I_e increases more.

4. Draw the equivalent circuit of the motor.

   The total power consumed by the motor corresponds to

$$P = U\,I + U_e\,I_e \tag{312}$$

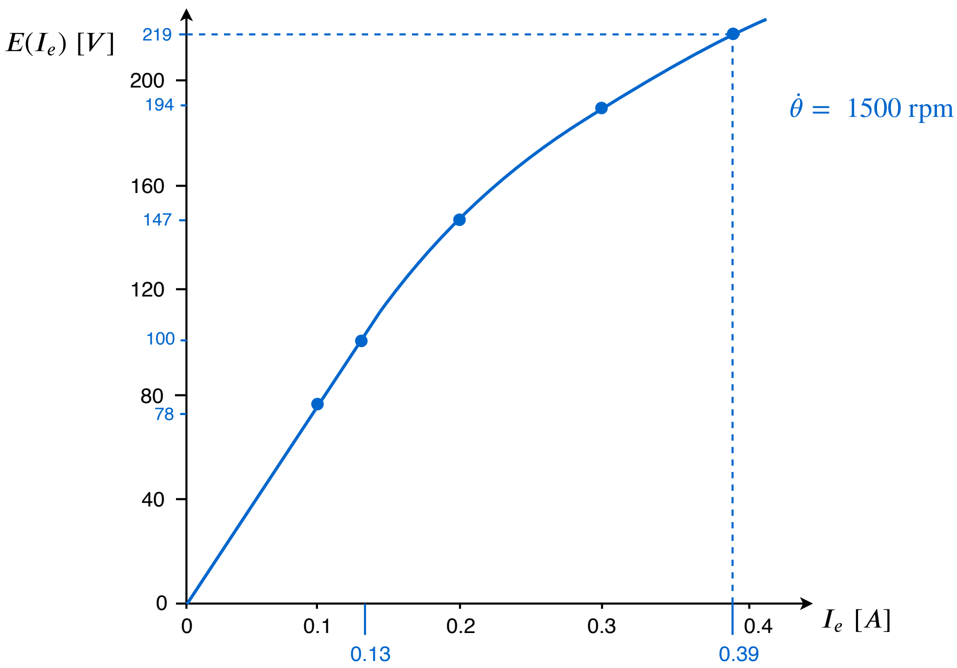


Figure 64: DC motor emf as a function of the excitation current.

The joule losses in the stator are

$$p_{je} = R_e\, I_e^2 = U_e\, I_e \tag{313}$$

and the joule losses in the rotor are

$$p_{ja} = R\, I^2 \tag{314}$$

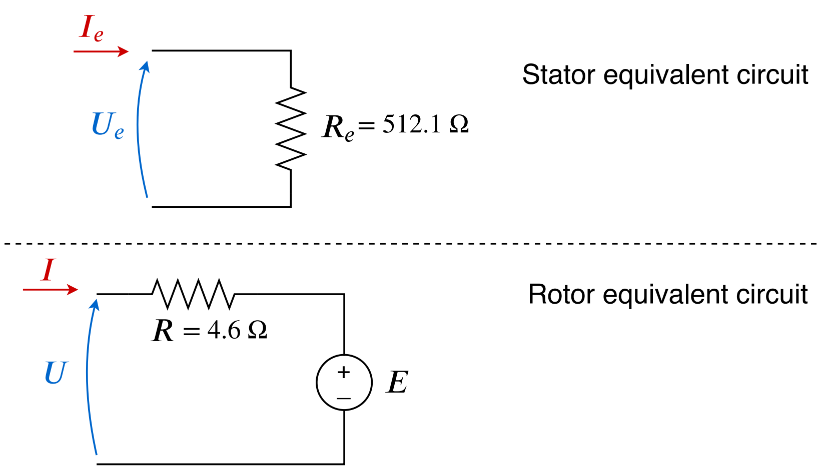


Figure 65: Equivalent circuit of the DC motor.

5. Plot the collective losses p_c wrt the excitation current I_e.

$$\underbrace{p_c}_{\text{collective losses}} = \underbrace{p_m}_{\text{mechanical losses}} + \underbrace{p_f}_{\text{ferromagnetic losses}} \tag{315}$$

For a no load test, the useful mechanical power (available et the shaft) is 0. Meaning that all the electrical power P, injected into the machine, is consumed by the losses.

$$P = p_{ja} + p_{je} + p_c \tag{316}$$
$$p_c = P - p_{ja} - p_{je} \tag{317}$$

Remark that,

$$P = U\,I + U_e\,I_e \tag{318}$$

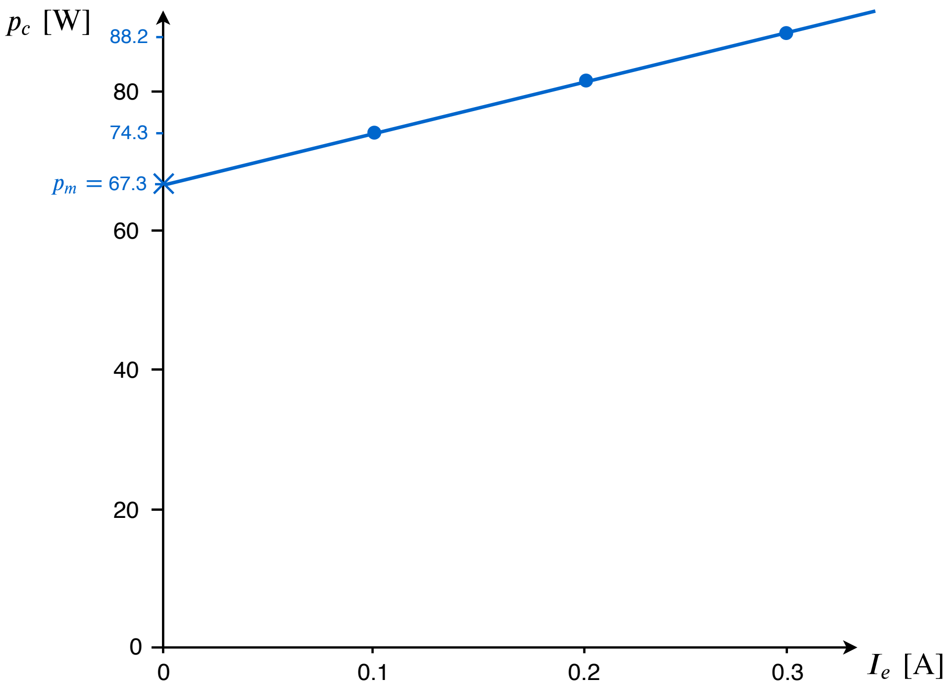
which leads to

$$\boxed{p_c = U\,I - R\,I^2} \tag{319}$$

Then, the collective losses can be computed for different excitation currents.

I_e [A]	U [V]	I [A]	p_c [W]
0.1	85	0.92	74.3
0.2	151	0.56	83.1
0.3	198	0.45	88.2

Table 10: Collective losses p_c as a function of I_e.



6. Determine the mechanical losses p_m.

For $I_e = 0$, there are no ferromagnetic losses. Then, only the mechanical losses p_m occur. Considering that the collective losses depend linearly on the excitation current, one can deduce p_m such that

$$p_m = 88.2 - \frac{88.2 - 74.3}{0.3 - 0.1} \cdot 0.3 = 67.3 \text{ W} \tag{320}$$

7. Calculate the emf E_o and deduce $I_{e,o}$, the corresponding excitation current.

As the hole is drilled, one can measure $I_o = 3$ A and $U_o = 212$ V. Then,

$$E_o = U_o - R\,I_o = 212 - 4.6 \cdot 3 = 198 \text{ V} \tag{321}$$

From the $E(I_e)$ characteristics, one can deduce $I_{e,o} = 0.31$ A .

8. Compute the shaft output power, P_u.

The power balance and other power equations of the machine can be established as :

$$P = P_u + p_{ja} + p_{je} + p_c \tag{322}$$
$$P = U_o\,I_o + U_e\,I_e \tag{323}$$
$$p_{ja} = R\,I_o^2 \tag{324}$$
$$p_{je} = U_e\,I_e \tag{325}$$

Then,

$$P_u = P - p_{ja} - p_{je} - p_c \tag{326}$$
$$= U_o\,I_o + U_e\,I_e - R\,I_o^2 - U_e\,I_e - p_c \tag{327}$$
$$= U_o\,I_o - R\,I_o^2 - p_c \tag{328}$$
$$= 212 \cdot 3 - 4.6 \cdot 3^2 - 89 = 506 \text{ W} \tag{329}$$

9. Deduce the resistive torque C_r.

During the nominal regime, the two torques balance each other such that

$$\underbrace{C_u}_{\text{useful torque}} = \underbrace{C_r}_{\text{resistive torque}} \tag{330}$$

$$C_u = \frac{P_u}{\dot{\theta}} = \frac{506}{1\,500 \cdot \frac{2\pi}{60}} = 3.22 \text{ Nm} \tag{331}$$

Exercise 23. Brushed DC motor with series excitation

Consider a brushed DC motor with its excitation in series. This motor is fed by a constant voltage source $U = 220$ V. In order to simplify the study, the resistances of the armature and the inductor as well as the collective losses will be neglected.

1. Show that the electromagnetic torque is proportionnal to the square of the consumed current.
2. Show that the electromagnetic torque is inversely proportionnal to the speed of rotation of the motor.
3. Deduce that there is a runaway of the motor at no load.
4. According to the second sub-question, one can write that :

$$\boxed{C_u = \frac{a}{\dot{\theta}^2}} \tag{332}$$

   where C_u corresponds to the useful torque of the motor in Nm, $\dot{\theta}$ is the speed of rotation in rpm and a is the constant to be determined. The nameplate of the machine indicates a nominal voltage of 220 V, a nominal speed of rotation of 1 200 rpm and a nominal current of 7.8 A. Deduce the value of the constant a.

In the following, we will take the value of $a = 20 \cdot 10^6 \quad [Nm \cdot rpm^2]$

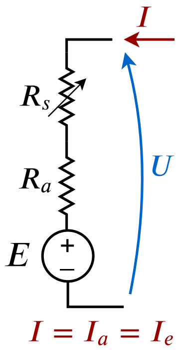
5. Draw the mechanical characteristics of C_u.
6. The motor drives a hoist whose resistive torque is constant : $C_r = 10$ Nm. Deduce the rotation speed of the whole set.

Solution

1. Electromagnetic torque proportionnal to the square of the current :

   The general expression of the electromotive force of the DC machine is

$$E = k_e\ \dot{\theta}\ \left(\Phi(I_e) - \Delta\Phi(I_a)\right) \tag{333}$$



   In the series excitation motor, the armature current I_a and the excitation current I_e are common. One refers to them as $I = I_e = I_a$, which leads to

$$E = k_e\ \dot{\theta}\ \left(\Phi(I) - \Delta\Phi(I)\right) \tag{334}$$

   Moreover, we will make the assumption that there is no armature reaction, such that $\Delta\Phi(I) = 0$ and we will consider that the magnetic flux Φ is directly proportionnal to the current I. This can be synthetized as

$$\Phi = L\ I \tag{335}$$

   where L is the self-inductance of the field winding (also called the inductor). Therefore, the electromotive force E is

$$E = k_E\ \dot{\theta}\ L\ I \tag{336}$$

   The electromotive power P_{elm} is

$$P_{elm} = E\ I = k_E\ \dot{\theta}\ L\ I^2 \tag{337}$$

   Finally, the electromotive torque C_{elm} is

$$\boxed{C_{elm} = \frac{P_{elm}}{\dot{\theta}} = k_E\ L\ I^2} \tag{338}$$

   The flux Φ is proportionnal to the excitation current, which is also the inductor current in the case of a series excitation machine. Therefore, the torque is proportionnal to the square of the current it consumes.

2. Electromagnetic torque inversely proportionnal to the square of the rotation speed :

   Based on the equivalent circuit of the series motor, one can write that

$$E = U - (R_a + R_s)\ I \tag{339}$$

   By neglecting the series resistances of the armature and the excitation (which are low by construction), one can get

$$U \simeq E = k_E\ \dot{\theta}\ L\ I \tag{340}$$

   Therefore, the current and the input voltage are linked by

$$I = \frac{U}{k_E\ \dot{\theta}\ L} \tag{341}$$

   Finally, by replacing the current in the previous result, we obtain that

$$\boxed{C_{elm} = \frac{U^2}{k_E\ L\ \dot{\theta}^2}} \tag{342}$$

The current is proportionnal to $\frac{U}{\dot{\theta}}$, the torque is proportionnal to $\left(\frac{U}{\dot{\theta}}\right)^2$ and with the voltage U being constant, the torque is inversely proportionnal to the square of the speed of rotation.

3. Deduce that there is a runaway of the motor at no load :

At no load, the resisitve torque is small and the motor torque is also small. From expression (342), one can deduce that the speed of rotation is very high and that is why the motor is in runaway.

4. Deduce the value of the constant a :

Since the resitances are negligeable, the input power ($P = U\,I$) is equal to the electromagnetic power ($P_{elm} = E\,I$), with

$$P_{elm} = U\ I = 220 \cdot 7.8 = 1\,716\ [\text{W}] \tag{343}$$

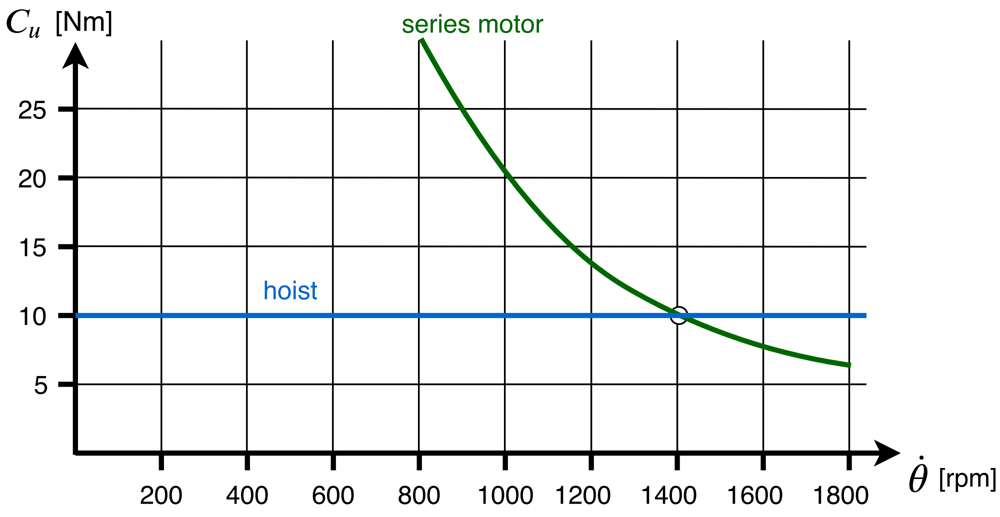
$$C_{elm} = \frac{P_{elm}}{\dot{\theta}} = \frac{1\,716}{1\,200 \cdot \frac{2\pi}{60}} = 13.6555\ [\text{Nm}] \tag{344}$$

Also, because the collective losses are neglected, the useful torque C_u is equal to the electromagnetic torque C_{elm}, as there are no mechanical losses.

Then, the constant a can be computed based on the nominal voltage, current and speed of rotation such that

$$\boxed{a = C_u \cdot \dot{\theta}^2 = 13.6555 \cdot 1\,200^2 = 19.66 \cdot 10^6 [\text{Nm rpm}^2]} \tag{345}$$

5. Series motor characteristics :



6. Rotation speed for the hoist :

The rotation speed of the complete set (DC motor and hoist attached) can be determined by

$$\boxed{\dot{\theta} = \sqrt{\frac{a}{C_r}} = \sqrt{\frac{20 \cdot 10^6}{10}} = 1\,414\ [\text{rpm}]} \tag{346}$$

Exercise 24. Regenerative braking

Hybrid electric vehicles are generally provided with regenerative braking, allowing to load onboard battery when the vehicle is braking or when the vehicle acts as a driving load. In this exercise, the DC motor, having an electromotive force E and internal resistance $R = 0.5\,\Omega$ is connected (when the regenerative breaking is active) to a battery delivering a current I under the voltage $V = 100\,\text{V}$ using a chopper DC-DC converter (Fig. 66).

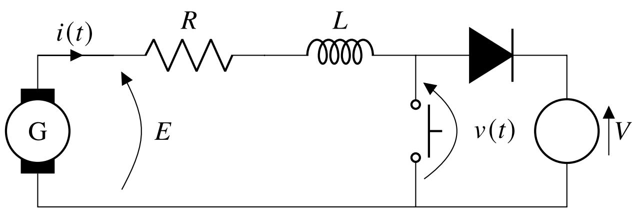


Figure 66: DC-DC converter for regenerative braking.

1. Find the mean value of $v(t)$: V_m;
2. Find the link between the mean input current I_m and the mean output current I;
3. Express the voltage V with respect to I_m, E, R and D, the duty cycle;
4. Compute the duty cycle allowing to obtain $V_m = 60\,\text{V}$;
5. Compute the mean braking current I_m when the motor delivers an electromotive force $E = 70\,\text{V}$ for $V_m = 60\,\text{V}$;
6. Calculate the braking power EI_m and the braking torque C_m if the motor speed of rotation is $\dot{\theta} = 955\,\text{RPM}$.

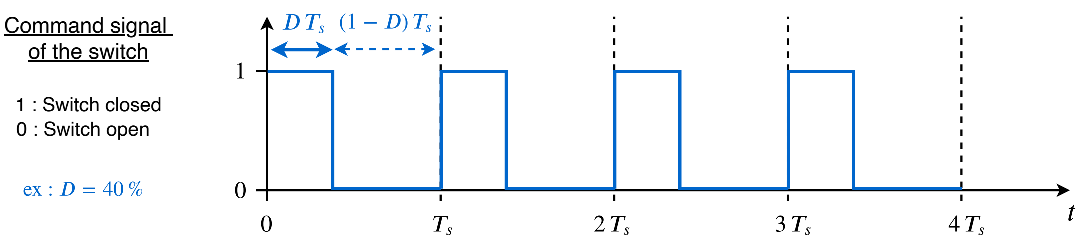
Basic principle of the switched DC-DC converter

The switches are supposed ideal. Meaning that they act as perfect short-circuits when closed and perfect open-circuits when open. In practice, transistors are used and can not be considered as ideal (*i.e.* ON resistance, OFF resistances,...).

Diodes are also supposed ideal even if, in practice, every diode will present a voltage drop when the current flows.

For a switched-mode converter, one can consider that each period of time T_s (T_s : the switching period) is divided into 2 sub-intervals : During sub-interval 1, the switch (transistor) is closed and during sub-interval 2, the switch is open.

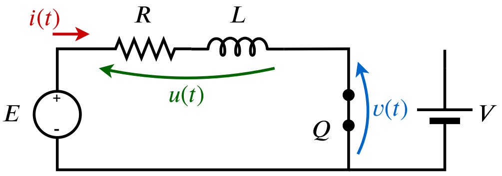
The duty cycle (D) corresponds to the proportion of T_s during which the switch remains closed.



Therefore, 2 sub-circuits exist (one for each sub-interval) and the converter's behavior can be summarized as following.

Sub-interval 1	Sub-interval 2
$D\,\%$ of T_s (The switch is conducting)	$(1-D)\,\%$ of T_s (The diode is conducting)

Sub-interval 1



$$v_D = 0$$

$$u_D = E \qquad (>0)$$

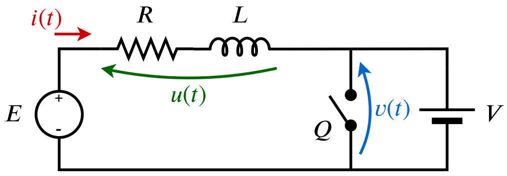
$$i_D = \left(i_0 - \frac{E}{R}\right) e^{-\frac{R}{L}t} + \frac{E}{R}$$

Assumption :

R is small compared to L .
The exponential can be replaced by a first order function considering that $\frac{R}{L}t$ is close to 0 .

$$i_D \simeq i_0 + \frac{E}{L}\,t$$

Sub-interval 2



$$v_{(1-D)} = V$$

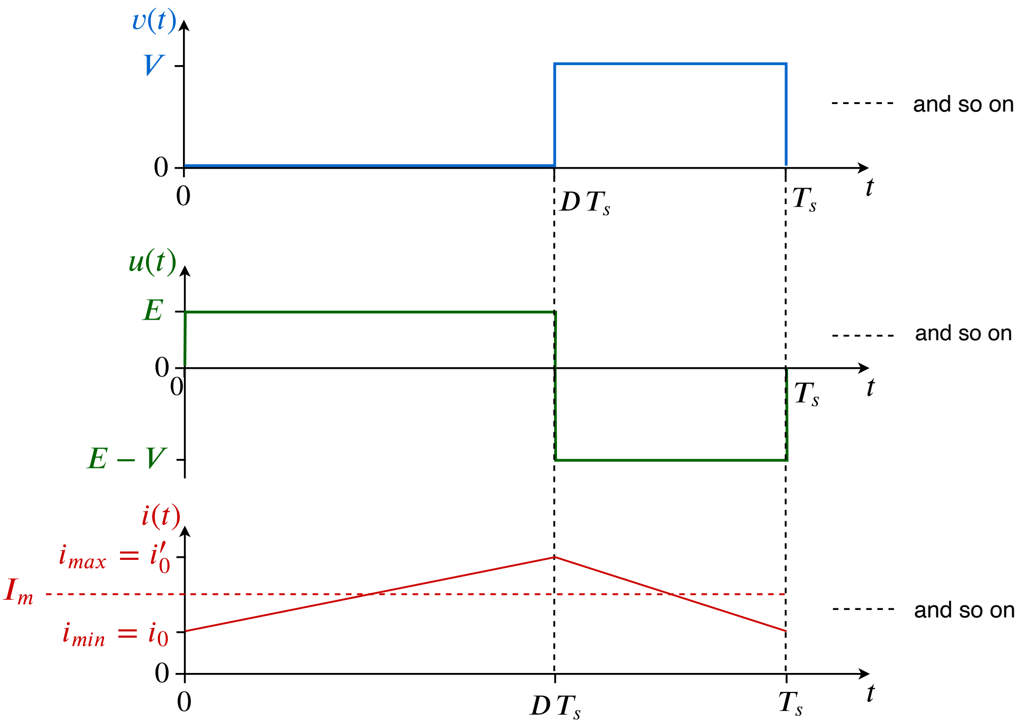
$$u_{(1-D)} = E - V \qquad (<0)$$

$$i_{(1-D)} = \left(i'_0 - \frac{E-V}{R}\right) e^{-\frac{R}{L}t} + \frac{E-V}{R}$$

Assumption :

R is small compared to L .
The exponential can be replaced by a first order function considering that $\frac{R}{L}t$ is close to 0 .

$$i_{(1-D)} \simeq i'_0 + \frac{(E-V)}{L}\,t'$$

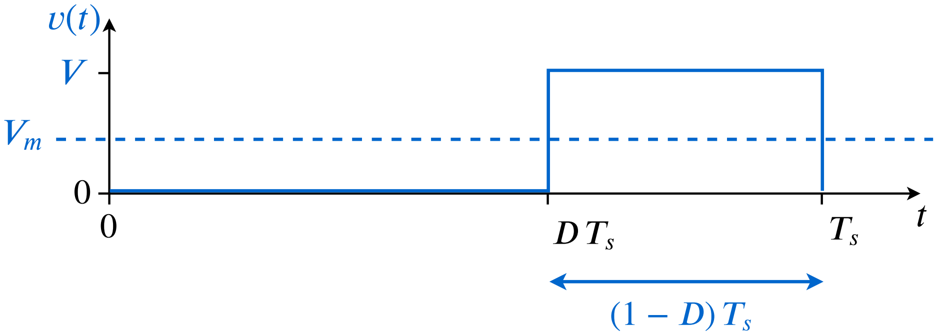


As the inductance L is designed with a high value, the inductor current $i(t)$ does not vary much over a switching period (indeed, the goal of the inductance is to smoothen the current). The inductor current can be correctly approximated by its mean value I_m.

On average over a switching period the inductance current remains constant such that : $i(t) \simeq I_m$

1. Find the mean value of $v(t)$, here denoted V_m.

   Over the switching period T_s, the mean value of $v(t)$ can be defined as

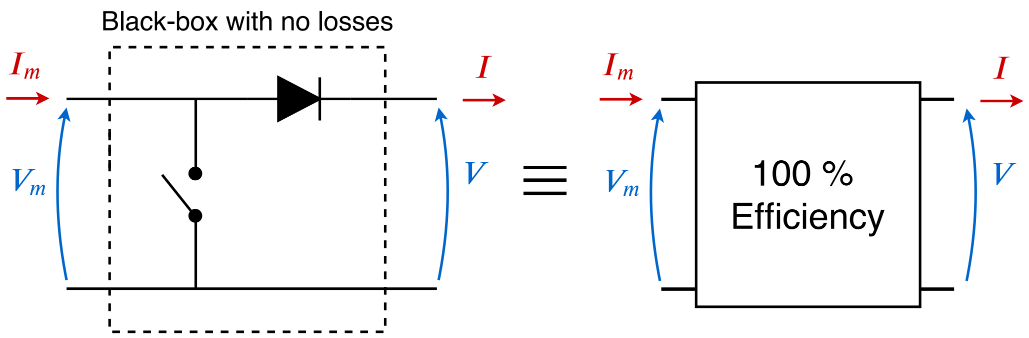


$$V_m = \frac{1}{T_s}\int_0^{T_s} v(t)\,dt = \frac{1}{T_s}\int_{DT_s}^{T_s} V\,dt = \frac{1}{T_s}(1-D)\,V\,T_s \tag{347}$$

$$\boxed{V_m = (1-D)\,V} \tag{348}$$

2. Find the link between the output current I and the input current I_m.

The average input current I_m is the current through the inductor. This current can be linked to the output current by writing the power balance around the semiconductors (diode and switch).



As the semiconductors are considered lossless, the power balance can directly be written as

$$V_m\,I_m = V\,I \tag{349}$$

Using the result of the previous question, one can find the link between the input and output currents :

$$\boxed{I = I_m\,(1-D)} \tag{350}$$

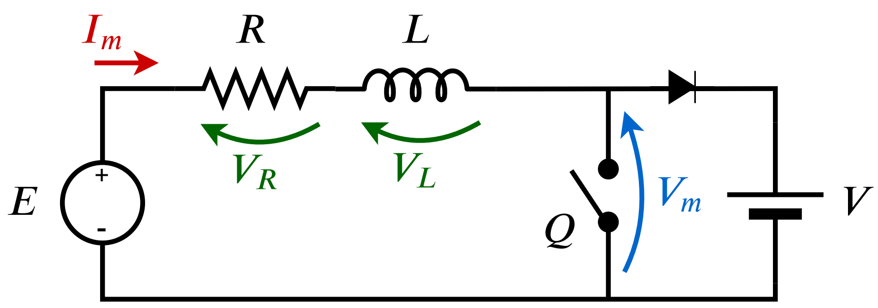
3. Express the voltage V with respect to I_m, E, R and D.

Considering the time-averaged equivalent circuit,
one can apply the Kirchhoff Voltage Law to relate the switch average voltage V_m to the input voltage E :

$$V_m = E - V_R - V_L \tag{351}$$

Remark that the average voltage of the inductance is 0 : $V_L = 0$ because the inductor average current is constant over time. This leads to

$$V_m = E - V_R \tag{352}$$
$$V_m = E - R\,I_m \tag{353}$$



The relationship between the switch average voltage V_m and the output voltage V is already known as $V_m = (1 - D)\, V$, which finally leads to

$$\boxed{V = \frac{E - R\, I_m}{(1 - D)}} \tag{354}$$

4. Compute the duty cycle D allowing to obtain $V_m = 60$ V.

The input voltage is $V = 100$ V and $V_m = (1 - D)\, V$. Then, the duty cycle can be expressed as :

$$D = 1 - \frac{V_m}{V} \tag{355}$$

$$\boxed{D = 1 - \frac{60}{100} = 0.4} \tag{356}$$

5. Compute the average braking current I_m for $E = 70$ V, $V = 100$ V and $V_m = 60$ V.

The duty cycle has already been computed for $V = 100$ V and $V_m = 60$ V. The value remains $D = 0.4$. Then, from question 3, one can use the expression

$$V = \frac{E - R\, I_m}{(1 - D)} \tag{357}$$

to isolate the average inductor current, such that :

$$\boxed{I_m = \frac{E - (1 - D)\, V}{R} = \frac{70 - 0.6 \cdot 100}{0.5} = 20 \text{ A}} \tag{358}$$

6. Braking power $E\, I_m$ and the braking torque for $\dot{\theta} = 955$ rpm.

$$\boxed{E\, I_m = 70 \cdot 20 = 1400 \text{ W}} \tag{359}$$

$$\boxed{C_m = \frac{E\, I_m}{\dot{\theta}} = \frac{1400}{955 \cdot \frac{2\pi}{60}} = 14 \text{ Nm}} \tag{360}$$

Exercise 25. DC generator-motor mechanical coupling ◇

Two identical DC machines, rated $P_{u,n} = 200\,\text{kW}$, $V_n = 520\,\text{V}$, $I_n = 420\,\text{A}$, $\dot{\theta}_n = 1000\,\text{RPM}$, are tested by coupling their shafts together, so that one will act as a generator to supply power to the other, which will act as a motor and drive the generator. This is a common test procedure to avoid the use of high-power electrical supplies and heavy loading rigs. The two rotors are connected in parallel (Fig. 67) while both machines are separately excited by manually adjusting the excitation current. To startup, the excitation current is set to its nominal value while the applied voltage V is progressively increased to V_n to reach the unloaded speed $\dot{\theta}_u = 1040\,\text{RPM}$. The couple generator-motor draws 37 A from the DC source, which is mainly due the torque required to overcome windage, friction, and iron losses. Knowing the the armature resistance value of each machine is $R = 0.05\,\Omega$ and ignoring commutation losses (brush volt-drop):

1. Calculate the electromotive force at no-load;
2. Compute the power loss at no-load;
3. Compute the mechanical power losses;
4. Calculate the torque at no-load.

The excitation current of the generator is now reduced to reach the nominal value of the output current I_n.

5. Calculate the speed of rotation of each machine;
6. Calculate the current drawn by the motor;
7. Calculate the shaft output torque;
8. Compare the power supplied by the DC source to the nominal power of each machine and conclude about the relevance of such a test and about the load level reached during this test.

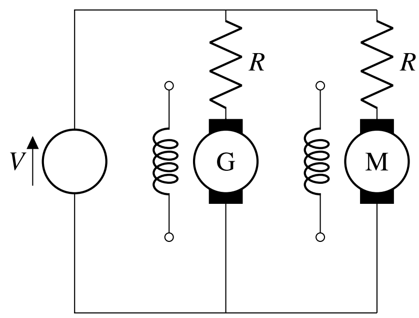


Figure 67: DC generator-motor mechanical coupling.

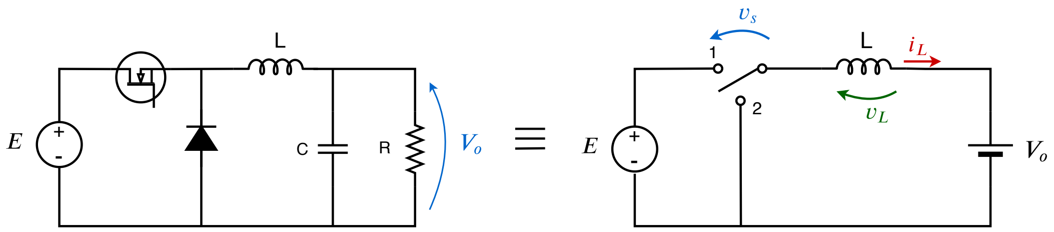
Answers

1. $emf = 519.075$ V
2. $p = 17.11$ W
3. $P = 9603$ W
4. $C = 88.17$ Nm
5. $E_m = 502$ V ; $\dot{\theta} = 1076$ rpm
6. $I = 360$ A
7. $C = 1604$ Nm
8. Ratio = 14.2 %

6 Electronic control system

Exercise 26. DC-DC buck converter

DC-DC converters are used to adapt two different voltage levels. For instance, in particular model of an electric car, the battery voltage is set to $E = 302$ V, whereas the auxiliaries (lights, cigar lighter, window and wiper motors, ...) are working with $V_o = 12$ V. A DC-DC buck converter is used to reduce the battery high voltage to the lower value (12 V) ensuring high efficiency. The DC-DC buck converter can be modelled by the following circuit.



1. Find the waveforms of the voltage across the ideal switch (v_s) and the voltage across the inductance (v_L).
2. Deduce the inductance current waveform from it.

Now, suppose that the average voltage across the inductance is 0 during a switching period (corresponding to a steady-state condition).

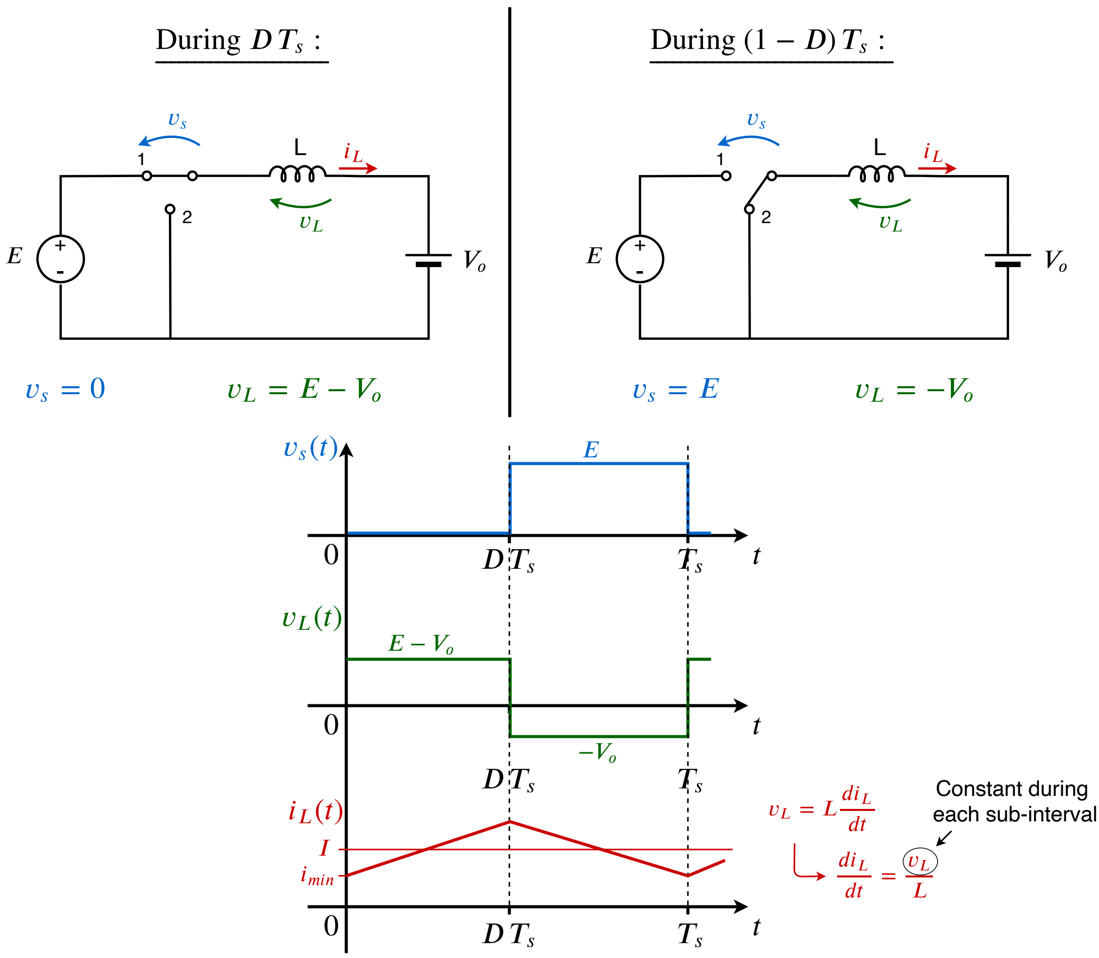
3. Express the ratio $\frac{V_o}{E}$ in terms of the duty cycle D.
4. Give the value of D in this situation.

The current ripple Δi is defined as the absolute difference between the maximum of current (during a switching period) and the average current I (over the same switching period).

5. Find the expression of the inductor current ripple Δi_L in terms of V_o, E, D, T_s and L.
6. Estimate the inductor current ripple Δi_L for a switching frequency $f_s = 1000$ Hz and an inductance of 50 mH. Compare the value of the current ripple to the value of the output current if the auxiliaries draw 12 W.

Solution

1. Find the waveforms of the voltages v_s and v_L.



2. Deduce the waveform of the inductance current i_L.

See the figure below. The current in the inductance behaves as

$$v_L(t) = L\,\frac{di_L(t)}{dt} \tag{361}$$

$$i_L(t_2) = i_L(t_1) + \int_{t_1}^{t_2} \frac{v_L(t)}{L}\,dt \tag{362}$$

The voltage across the inductance remains constant during each sub-interval. The current first increases linearly during $D\,T_s$ and then decreases linearly during $(1-D)\,T_s$.

Remark that the steady-state condition imposes that the current must return to its initial value i_{min} at the end of the switching period. With this condition, the successive switching periods have the same waveforms and the steady-state condition can be considered.

$$\underbrace{i_L(T_s)}_{=i_{min}} = \underbrace{i_L(0)}_{=i_{min}} + \underbrace{\int_0^{T_s} \frac{v_L(t)}{L}\,dt}_{=0} \tag{363}$$

The steady-state condition can be fulfilled if

$$\int_0^{T_s} v_L(t)\, dt = 0 \tag{364}$$

3. Express the ratio $\frac{V_o}{E}$ in terms of the duty cycle D.

On average, the inductance voltage is 0

$$\int_0^{T_s} v_L(t)\, dt = 0 \tag{365}$$

$$\int_0^{T_s} v_L(t)\, dt = \int_0^{DT_s} (E - V_o)\, dt + \int_{DT_s}^{T_s} (-V_o)\, dt \tag{366}$$

$$= (E - V_o)\, D T_s + (-V_o)\,(1 - D)\, T_s = 0 \tag{367}$$

Simplifying the switching period T_s leads to

$$(E - V_o)\, D + (-V_o)\,(1 - D) = 0 \tag{368}$$

$$D E - V_o = 0 \tag{369}$$

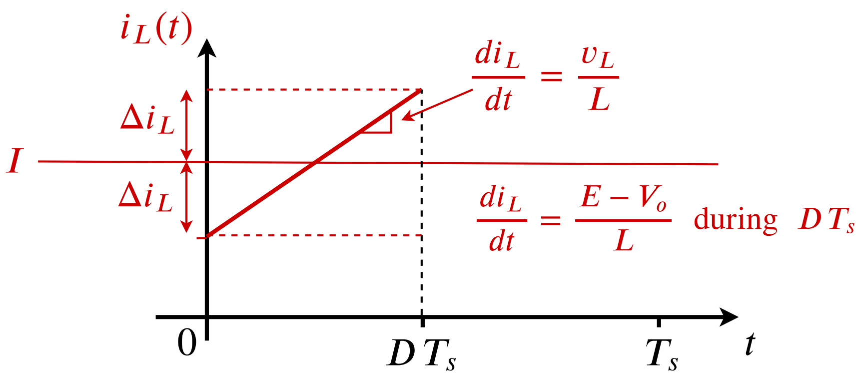
$$\boxed{\frac{V_o}{E} = D} \tag{370}$$

4. Give the value of D in this situation.

In this case,

$$\boxed{D = \frac{V_o}{E} = \frac{12}{302} = 3.973\%} \tag{371}$$

5. Find the expression of the inductor current ripple Δi_L in terms of V_o, E, D, T_s and L.



The ripple can be computed during the first sub-interval by observing that the slope $\frac{di_L}{dt}$ is equal to $\frac{E-V_o}{L}$:

$$2\,\Delta i_L = \frac{E - V_o}{L}\, D\, T_s \tag{372}$$

$$\boxed{\Delta i_L = \frac{E - V_o}{2\,L}\, D\, T_s} \tag{373}$$

6. Estimate the inductor current ripple Δi_L and compare it to the output current.

$$\Delta i_L = \frac{E - V_o}{2\,L}\, D\, T_s = \frac{302 - 12}{2 \cdot 0.05}\,\frac{12}{302} \cdot 0.001 = 0.11523 \text{ A} \tag{374}$$

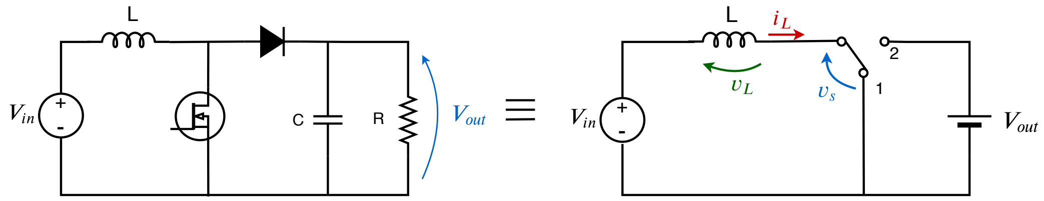
For a 12 W output power, the output current is

$$I_o = \frac{P_o}{V_o} = \frac{12}{12} = 1 \text{ A} \tag{375}$$

Then the ripple is 12 % of the output current which is small enough.

Exercise 27. DC-DC boost converter

DC-DC converters are used to adapt two different voltage levels. In some electronic calculator, the battery voltage is set as V_{in} = 3 V, whereas the electronic parts work under V_{out} = 9 V. A DC-DC boost converter is used to increase the battery low voltage to the higher value (9 V) ensuring high efficiency. The DC-DC boost converter can be modelled by the following circuit.



1. Find the waveforms of the voltage across the ideal switch (v_s) and the voltage across the inductance (v_L).
2. Deduce the inductance current waveform from it.

Now, suppose that the average voltage across the inductance is 0 during a switching period (corresponding to a steady-state condition).

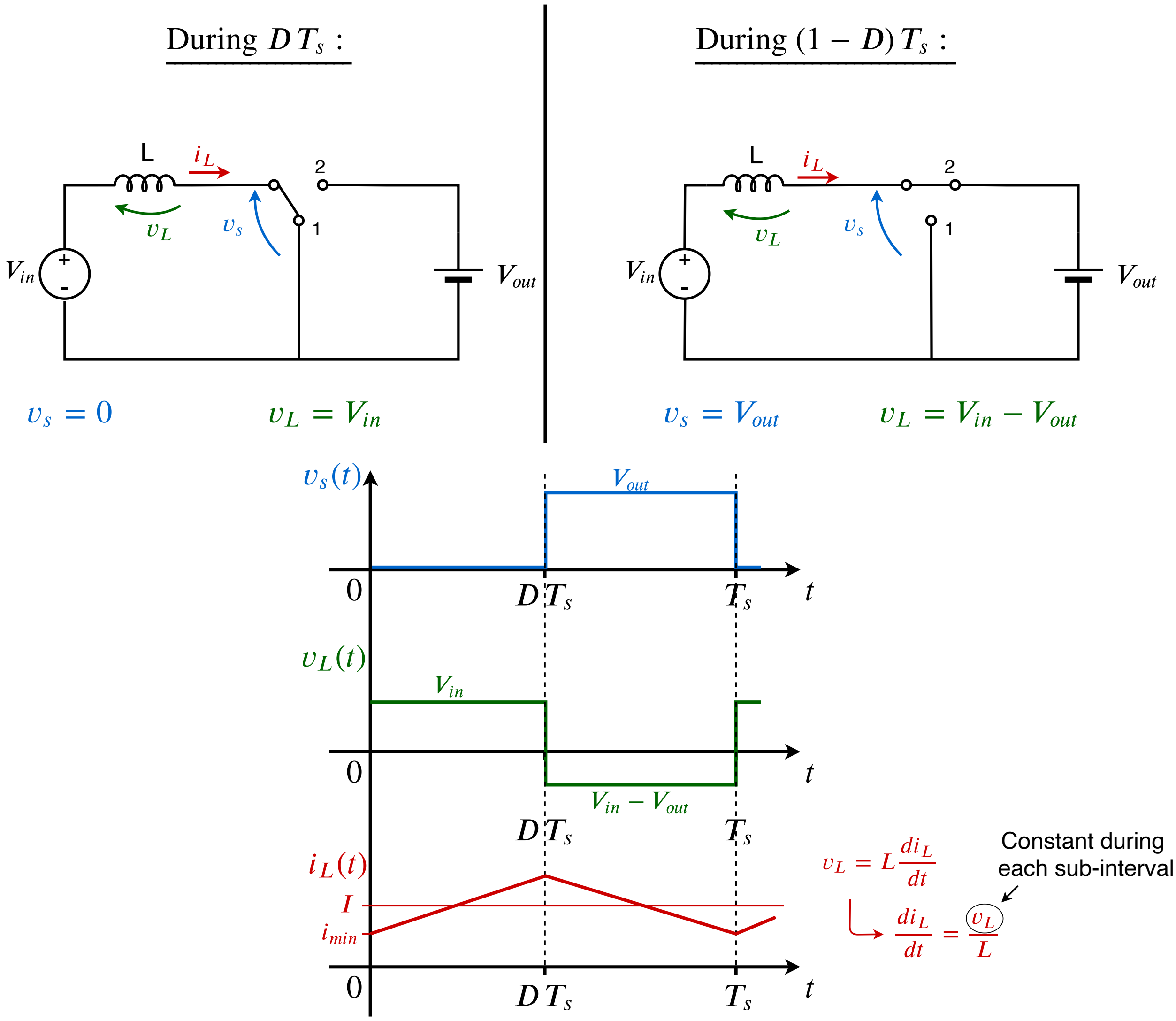
3. Express the ratio $\frac{V_{out}}{V_{in}}$ in terms of the duty cycle D.
4. Give the value of D in this situation.

The current ripple Δi is defined as the absolute difference between the maximum of current (during a switching period) and the average current I (over the same switching period).

5. Find the expression of the inductor current ripple Δi_L in terms of V_{out}, V_{in}, D, T_s and L.
6. Estimate the inductor current ripple Δi_L for a switching frequency f_s = 30 kHz and an inductance of 75 mH. Compare the value of the current ripple to the value of the output current if the system draws 15 mW.

Solution

1. Find the waveforms of the voltages v_s and v_L.



2. Deduce the waveform of the inductance current i_L.

See the figure below. The current in the inductance behaves as

$$v_L(t) = L\,\frac{di_L(t)}{dt} \tag{376}$$

$$i_L(t_2) = i_L(t_1) + \int_{t_1}^{t_2} \frac{v_L(t)}{L}\,dt \tag{377}$$

The voltage across the inductance remains constant during each sub-interval. The current first increases linearly during $D\,T_s$ and then decreases linearly during $(1-D)\,T_s$.

Remark that the steady-state condition imposes that the current must return to its initial value i_{min} at the end of the switching period. With this condition, the successive switching periods have the same waveforms and the steady-state condition can be considered.

$$\underbrace{i_L(T_s)}_{=i_{min}} = \underbrace{i_L(0)}_{=i_{min}} + \underbrace{\int_0^{T_s} \frac{v_L(t)}{L}\, dt}_{=0} \tag{378}$$

The steady-state condition can be fulfilled if

$$\int_0^{T_s} v_L(t)\, dt = 0 \tag{379}$$

3. Express the ratio $\frac{V_o}{E}$ in terms of the duty cycle D.

   On average, the inductance voltage is 0

$$\int_0^{T_s} v_L(t)\, dt = 0 \tag{380}$$

$$\int_0^{T_s} v_L(t)\, dt = \int_0^{D\,T_s} (V_{in})\, dt + \int_{D\,T_s}^{T_s} (V_{in} - V_{out})\, dt \tag{381}$$

$$= (V_{in})\, D\, T_s + (V_{in} - V_{out})\,(1 - D)\, T_s = 0 \tag{382}$$

   Simplifying the switching period T_s leads to

$$(V_{in})\, D + (V_{in} - V_{out})\,(1 - D) = 0 \tag{383}$$

$$V_{in} - (1 - D)\, V_{out} = 0 \tag{384}$$

$$\boxed{\frac{V_{out}}{V_{in}} = \frac{1}{1 - D}} \tag{385}$$

4. Give the value of D in this situation.

   In this case,

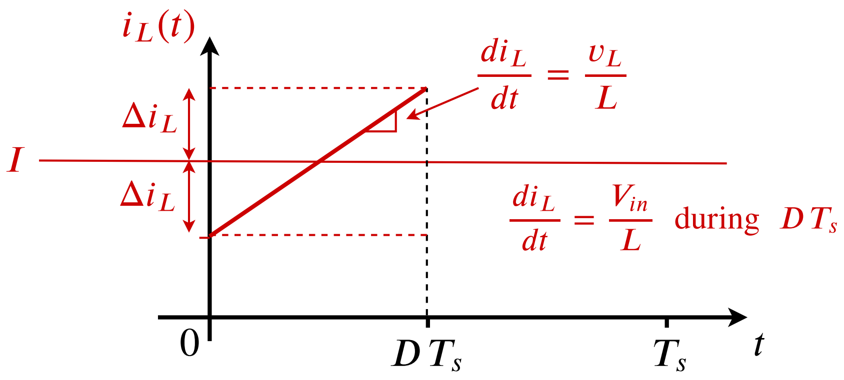
$$\boxed{D = 1 - \frac{V_{in}}{V_{out}} = \frac{2}{3} = 66.6\%} \tag{386}$$

5. Find the expression of the inductor current ripple Δi_L in terms of V_o, E, D, T_s and L.

   The ripple can be computed during the first sub-interval by observing that the slope $\frac{di_L}{dt}$ is equal to $\frac{V_{in}}{L}$:

$$2\,\Delta i_L = \frac{V_{in}}{L}\, D\, T_s \tag{387}$$

$$\boxed{\Delta i_L = \frac{V_{in}}{2\,L}\, D\, T_s} \tag{388}$$



6. Estimate the inductor current ripple Δi_L and compare it to the output current.

$$\Delta i_L = \frac{V_{in}}{2L} D T_s = \frac{3}{2 \cdot 0.075} \cdot \frac{2}{3} \cdot \frac{1}{30\,000} = 0.44 \text{ mA} \tag{389}$$

For a 15 mW output power, the output current is

$$I_o = \frac{P_o}{V_o} = \frac{0.015}{9} = 1.67 \text{ mA} \tag{390}$$

Then the ripple is 27 % of the output current.