Electromagnetic Energy Conversion
ELEC0431

Exercise session 1: Phasors and power in the sinusoidal steady state

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In this class…

Today's aim: Introduce a **very important** concept: **The phasors**

(pay particular attention to this class!)

- Reminders (one-ports and Kirchhoff laws)
- Harmonic functions
- Root Mean Square value
- The phasors
- Exercise 1
- Product of two harmonic functions
- Active, reactive and apparent power
- Exercises 2 and 3

Reminders

One-ports

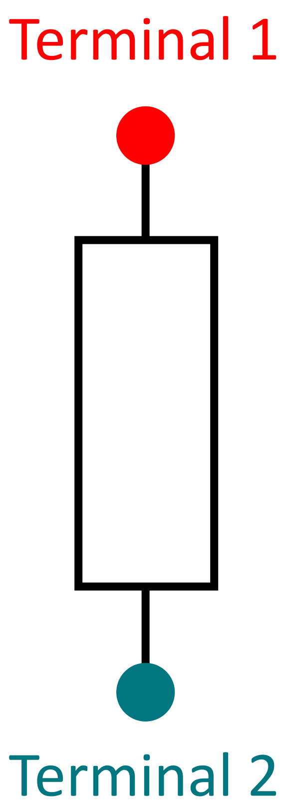
Kirchhoff laws

One-ports (or “dipoles” in French)

A one-port is a two-terminal electric component.

It can be:

- A resistance R
- An inductance L
- A capacitance C
- A power generator
- A diode
- A combination of components
- Etc.



One-ports (or "dipoles" in French)

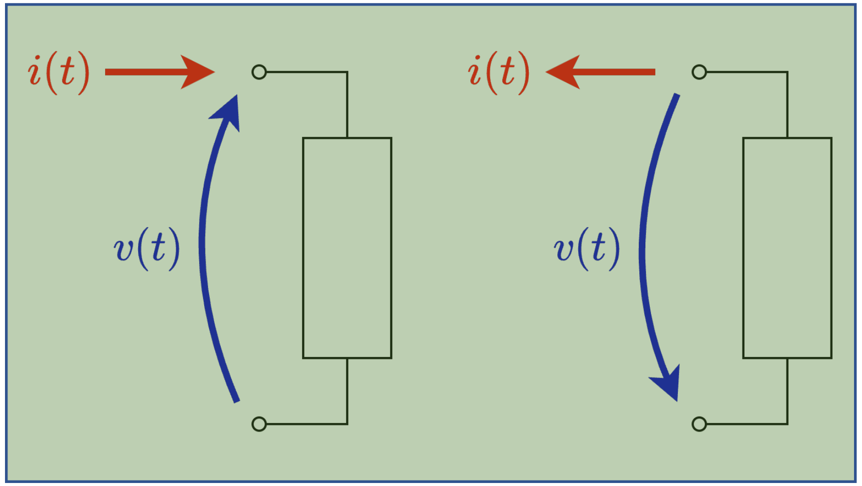
Each one-port is associated with a current $i(t)$ and a voltage $v(t)$.

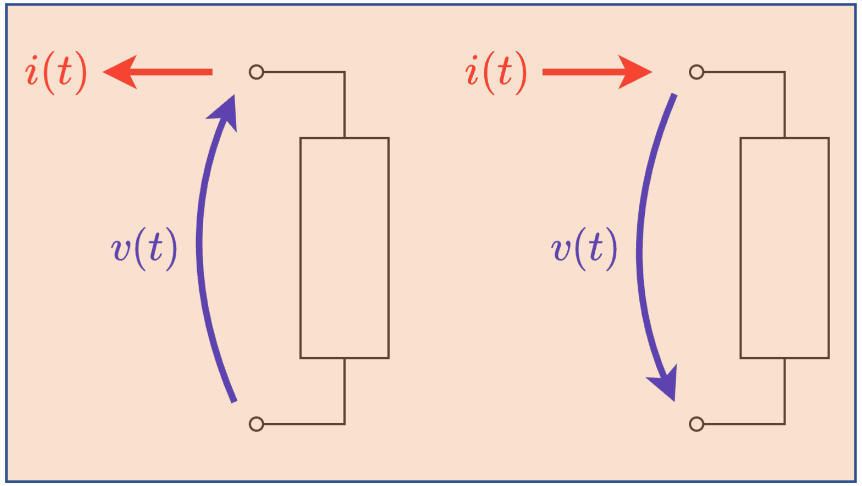
- $i(t)$ is positive if the positive charges flow in the direction indicated by the arrows.
- $v(t)$ is positive if the tension at the head of the arrow is larger than the one at the tip.

Four different configurations can thus be obtained.

if $i(t)$ and $v(t)$ are both positive, two of them consume power and two of them deliver power.

Consume Power (like a **passive** component)





Deliver Power (like an **active** component)

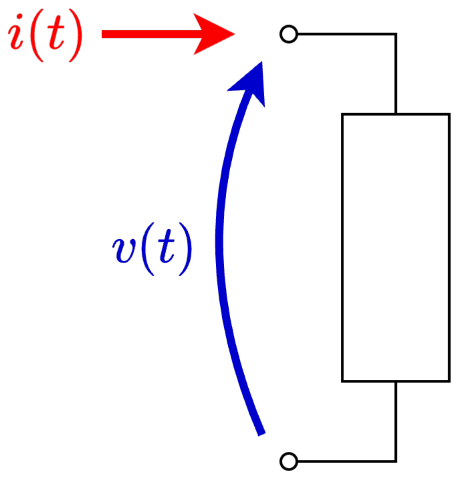
➔ This leads to two different conventions: the passive convention and the active convention.

Passive VS Active conventions

The passive convention

(also called receiver, motor or load convention)

The passive one-port **receives** the power $p(t) = v(t)\, i(t)$

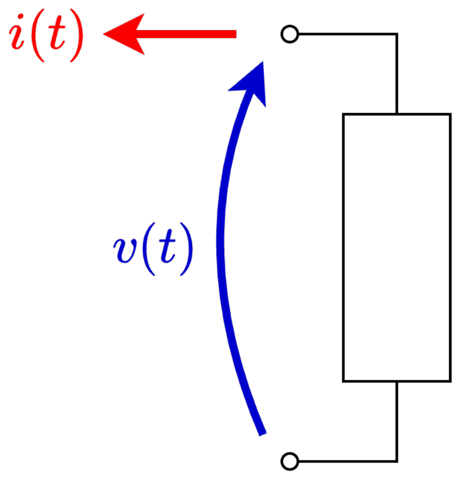


This convention is used for R, L, C, lumped parameters, motors and primary side of transformers.

The active convention

(also called generator or source convention)

The active one-port **delivers** the power $p(t) = v(t)\, i(t)$

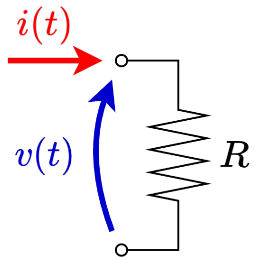


This convention is used for voltage sources, current sources, generator and secondary side of transformers.

$v(t)$ – i(t) relationship of basic components

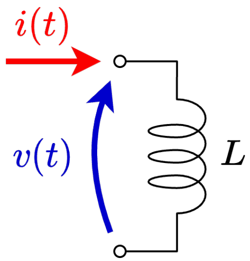
Passive convention

R



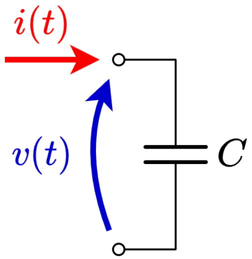
$$v(t) = R\, i(t)$$

L



$$v(t) = L\frac{di(t)}{dt}$$

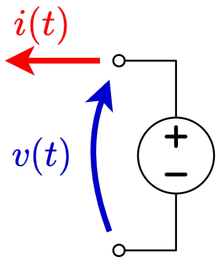
C



$$i(t) = C\frac{dv(t)}{dt}$$

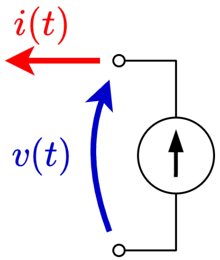
Active convention

Voltage source



$v(t)$ is imposed, $i(t)$ is set by the rest of the circuit

Current source



$i(t)$ is imposed, $v(t)$ is set by the rest of the circuit

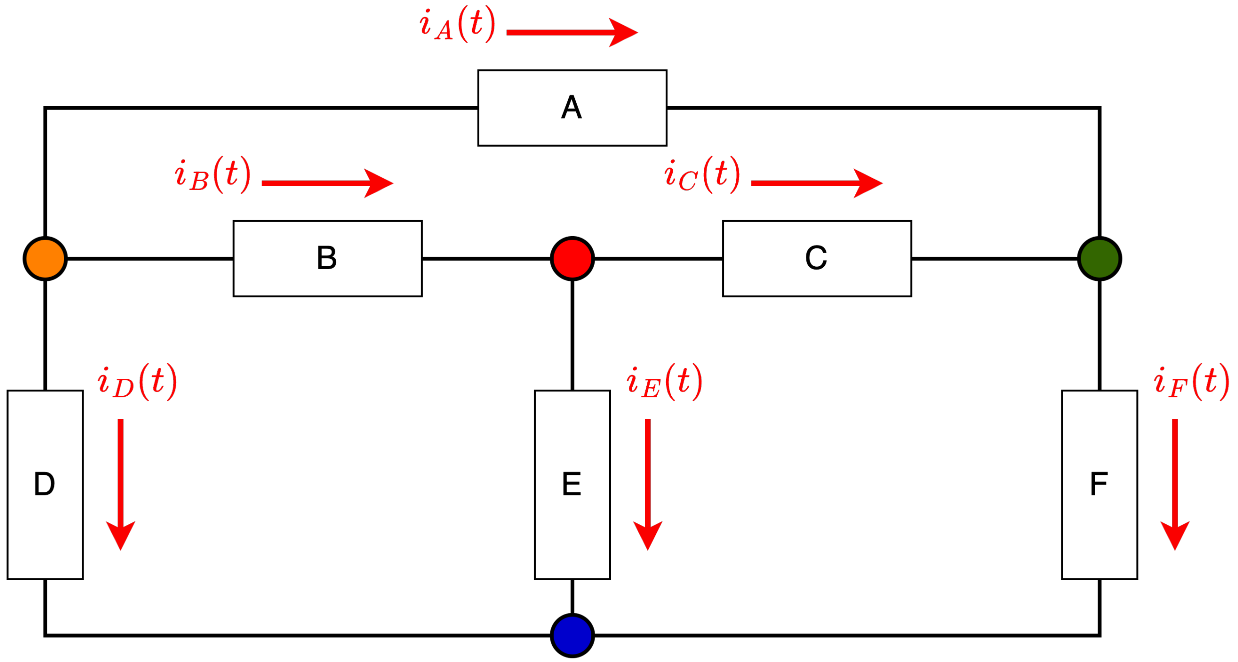
When connected together, use **Kirchhoff laws**

Kirchhoff laws

Kirchhoff's first law

(also called Kirchhoff's junction rule or Kirchhoff's current law)

At any junction in the electrical circuit, the sum of currents flowing into the junction is equal to the sum of currents flowing out of the junction.



$$0 = i_A(t) + i_B(t) + i_D(t)$$

$$i_B(t) = i_C(t) + i_E(t)$$

$$i_A(t) + i_C(t) = i_F(t)$$

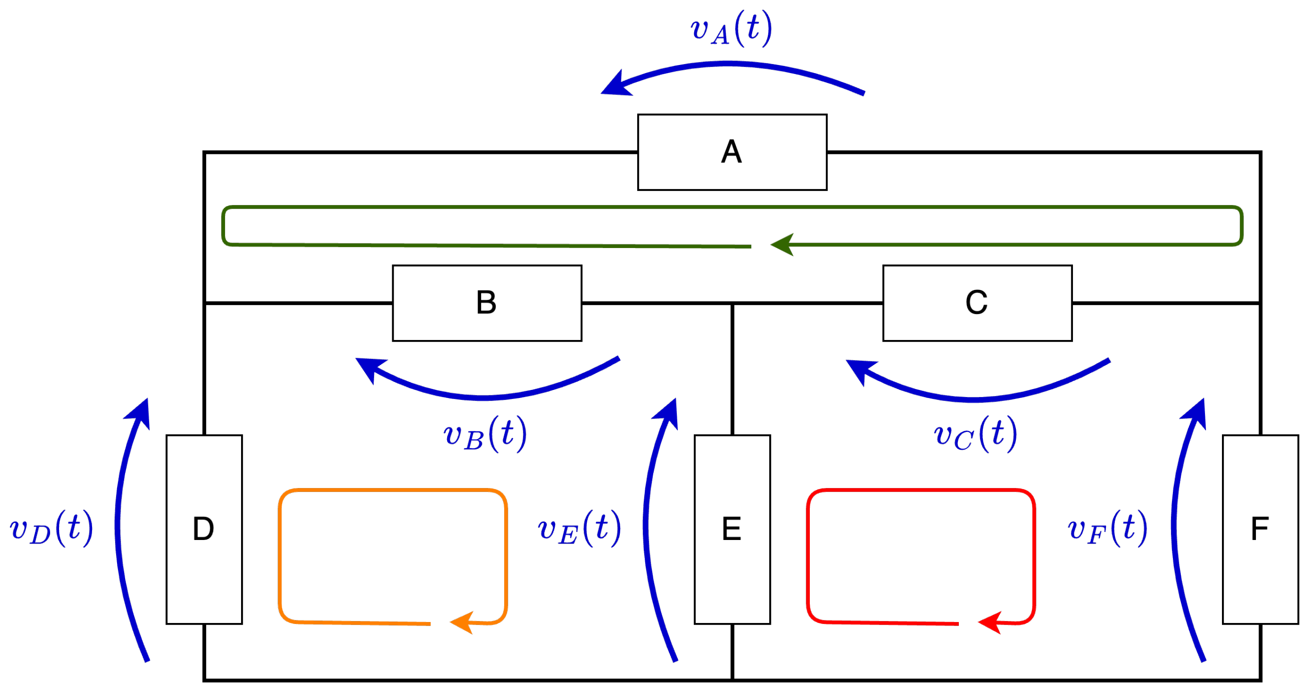
$$i_D(t) + i_E(t) + i_F(t) = 0$$

Kirchhoff laws

Kirchhoff's second law

(also called Kirchhoff's loop rule or Kirchhoff's voltage law)

Around any closed loop in a circuit, the directed sum of potential differences across components is zero.



$$v_D(t) - v_B(t) - v_E(t) = 0$$

$$v_E(t) - v_C(t) - v_F(t) = 0$$

$$v_B(t) - v_A(t) + v_C(t) = 0$$

Introduction to phasors

Harmonic functions
Root Mean square value
The phasors
Harmonic function ⇔ Phasor equivalence
Phasors applied to circuits
Application: Exercise 1

Harmonic functions

In this class, we deal with harmonic functions of the type

$$f(t) = F_{peak} \cos(\omega t + \theta)$$

where:

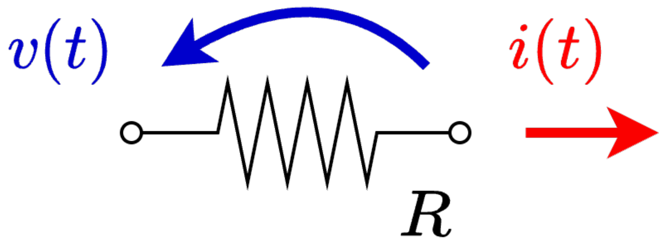
- F_{peak} is the amplitude,
- ω is the angular frequency,
- θ is the phase angle.

The angular frequency is linked to the frequency f and period T as

$$\omega = 2\pi f = \frac{2\pi}{T}.$$

Root Mean square (RMS) value

Let's consider a resistance R:



The instantaneous power $p(t)$ dissipated in the resistance is

$$p(t) = v(t)\, i(t) = R\, i(t)^2 = \frac{v(t)^2}{R}$$

And the mean dissipated power is

$$P = \frac{1}{T}\int_0^T v(t)\, i(t)\, dt = R\, \underbrace{\frac{1}{T}\int_0^T i(t)^2\, dt}_{I_{RMS}^2} = \frac{1}{R}\, \underbrace{\frac{1}{T}\int_0^T v(t)^2\, dt}_{V_{RMS}^2}$$

Root Mean square (RMS) value

The Root Mean Square (RMS) value (or effective value) of $f(t)$ is defined as:

$$F_{RMS} = \sqrt{\frac{1}{T}\int_0^T f(t)^2\,dt}.$$

With $f(t) = F_{peak}\cos(\omega t + \theta)$, we obtain:

$$F_{RMS} = \sqrt{\frac{1}{T}\int_0^T F_{peak}^2 cos^2(\omega t + \theta)\,dt} = F_{peak}\sqrt{\frac{1}{T}\int_0^T cos^2\,(\omega t)dt}$$

$$= F_{peak}\sqrt{\frac{1}{T}\int_0^T \frac{1+\cos(2\omega t)}{2}dt} = F_{peak}\sqrt{\frac{1}{T}\int_0^T \frac{1}{2}dt} = \frac{F_{peak}}{\sqrt{2}}$$

Root Mean square (RMS) value

Remark:

The following conventions will be used in the rest of the course for RMS and peak/max values:

$$F_{RMS} \quad \Rightarrow \quad F$$

$$F_{peak} \quad \Rightarrow \quad F_m$$

In this class, we deal only with harmonic functions $f(t) = F_m \cos(\omega t + \theta)$ for which we have

$$\boxed{F_m = \sqrt{2}\, F}$$

The phasors

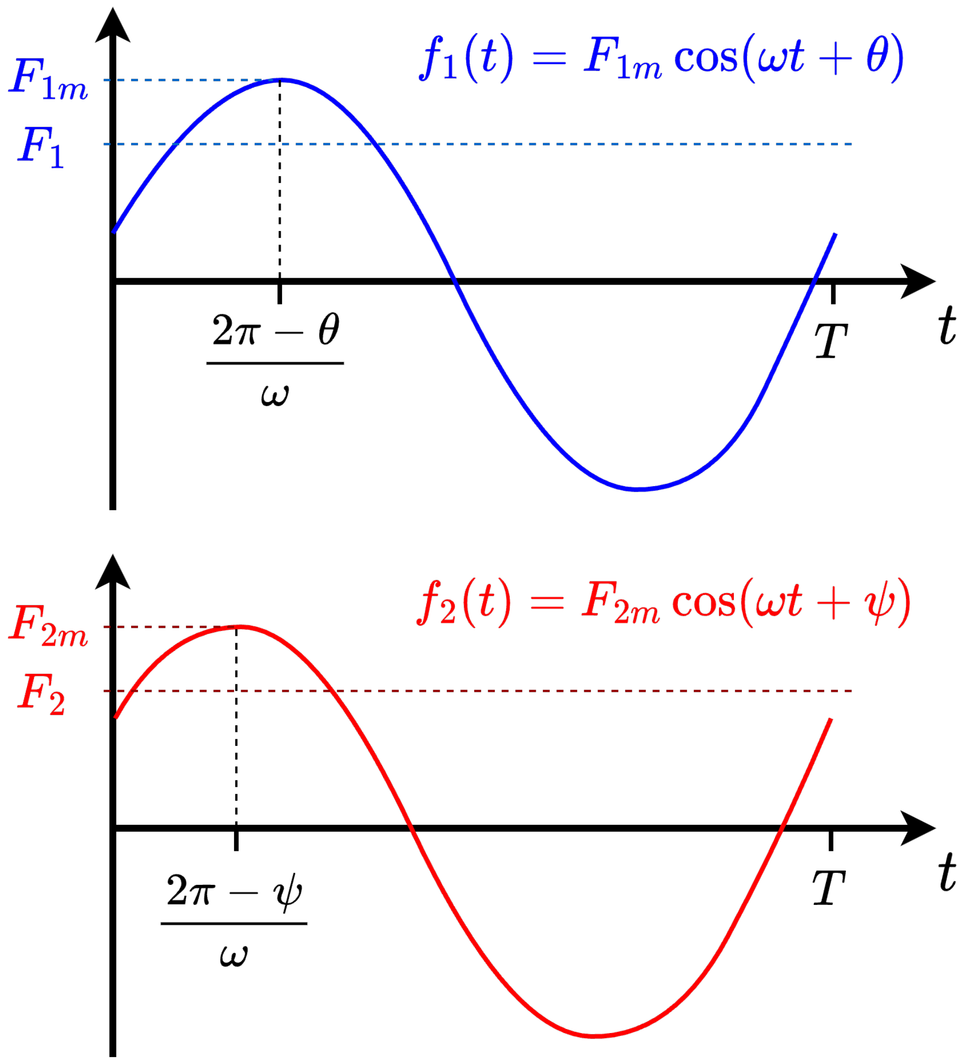
The definition of the phasors is obtained by manipulating the expression of $f(t)$:

$$\begin{aligned} f(t) &= F_m \cos(\omega t + \theta) \\ &= \sqrt{2}\, F \cos(\omega t + \theta) \\ &= \boldsymbol{R}\left(\sqrt{2}\, F\, e^{j(\omega t + \theta)}\right) \\ &= \boldsymbol{R}\Big(\sqrt{2}\, \underbrace{F\, e^{j\theta}}_{\bar{F}}\, e^{j\omega t}\Big) \end{aligned}$$

$$\bar{F} = F\, e^{j\theta}$$

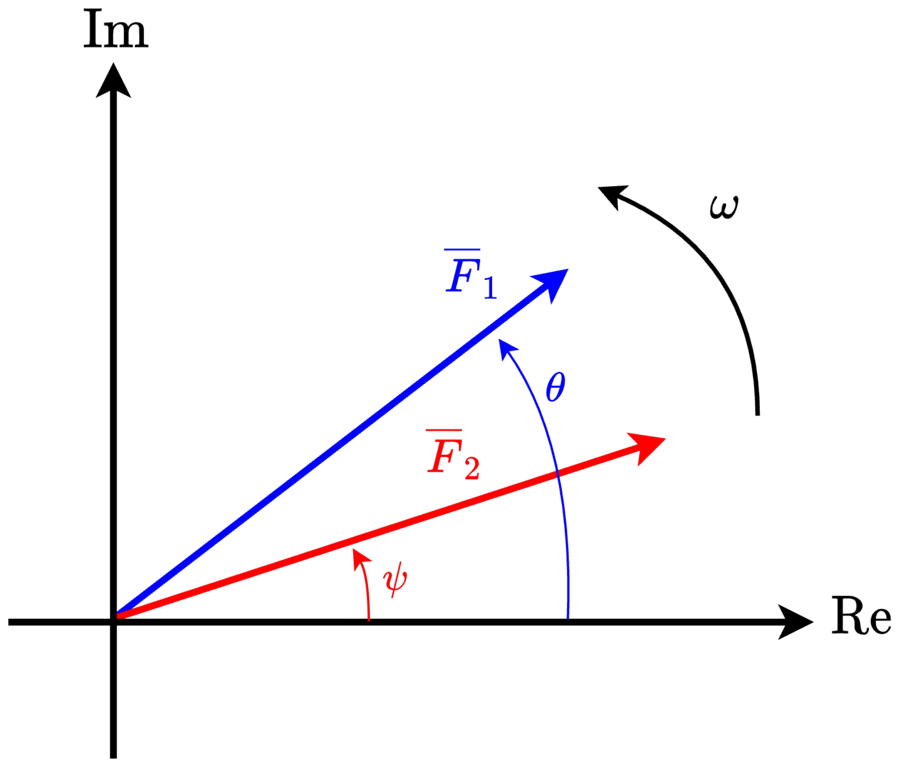
is the phasor of $f(t)$. It is a complex number with information on the signal amplitude (F) and the signal phase (θ). For a system operating at a fixed frequency, F and θ holds all the relevant information to describe the system.

The phasors



For signals evolving at a same angular frequency ω, phasors are like a snapshot representation containing all the necessary information (phase angle and amplitude) and allowing easy computations.

$\Longleftrightarrow$



Harmonic function ⇔ Phasor equivalence

Linearity:

$$
\begin{aligned}
f_1(t) + f_2(t) &= F_{1m}\cos(\omega t + \theta) + F_{2m}\cos(\omega t + \psi) \\
&= \sqrt{2}\,F_1\cos(\omega t + \theta) + \sqrt{2}\,F_2\cos(\omega t + \psi) \\
&= \boldsymbol{R}\left(\sqrt{2}\,F_1\,\mathrm{e}^{j(\omega t + \theta)}\right) + \boldsymbol{R}\left(\sqrt{2}\,F_2\,e^{j(\omega t + \psi)}\right) \\
&= \boldsymbol{R}\left(\sqrt{2}\,e^{j\omega t}\,\underbrace{\left(F_1 e^{j\theta} + F_2 e^{j\psi}\right)}_{\bar{F}_1 + \bar{F}_2}\right)
\end{aligned}
$$

$$
a\,f_1(t) + b\,f_2(t) \quad\Leftrightarrow\quad a\,\bar{F}_1 + b\,\bar{F}_2
$$

Harmonic function ⇔ Phasor equivalence

Differentiation:

$$\frac{df(t)}{dt} = \frac{d}{dt}(F_m \cos(\omega t + \theta))$$

$$= \frac{d}{dt}\, \boldsymbol{R}\left(\sqrt{2}\, F\, e^{j(\omega t + \theta)}\right)$$

$$= \boldsymbol{R}\left(\sqrt{2}\, F\, e^{j(\omega t + \theta)} j\omega\right)$$

$$= \boldsymbol{R}\left(\underbrace{\sqrt{2}\, j\omega\, F\, e^{j\theta}}_{j\omega \bar{F}}\, e^{j\omega t}\right)$$

$$\frac{df(t)}{dt} \Leftrightarrow j\omega\, \bar{F}$$

Harmonic function ⇔ Phasor equivalence

Integration:

$$\int f(t)\,dt = \int F_m \cos(\omega t + \theta)\,dt$$

$$= \int \boldsymbol{R}\left(\sqrt{2}\,F\,e^{j(\omega t+\theta)}\right) dt$$

$$= \boldsymbol{R}\left(\sqrt{2}\,F\,e^{j(\omega t+\theta)}\, {}^{1}\!/_{j\omega}\right)$$

$$= \boldsymbol{R}\left(\sqrt{2}\ \underbrace{{}^{1}\!/_{j\omega}\,F\,e^{j\theta}}_{\bar{F}/_{j\omega}}\ e^{j\omega t}\right)$$

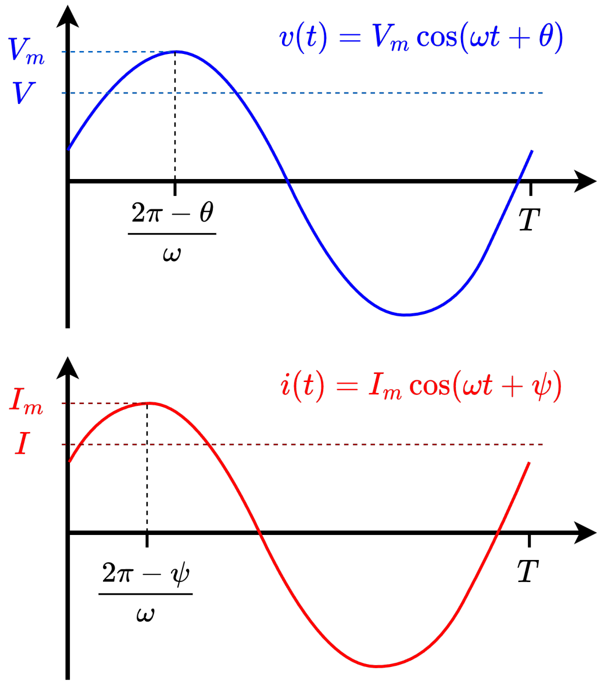
$$\int f(t)\,dt \quad \Leftrightarrow \quad \frac{\bar{F}}{j\omega}$$

Phasors applied to circuits

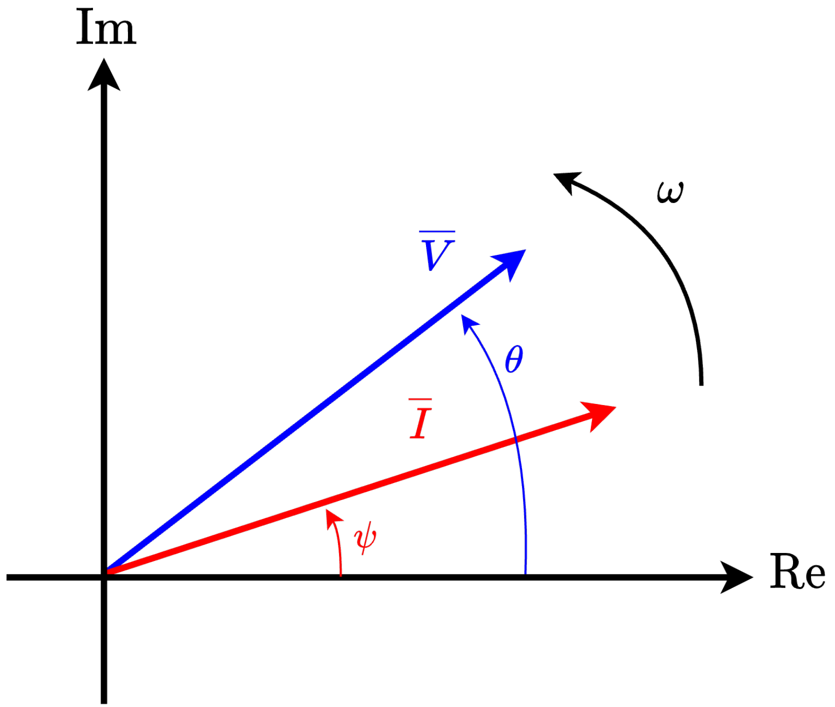
The phasor formalism can be exported to voltages and currents:

$$v(t) = \sqrt{2}\, V \cos(\omega t + \theta) \quad \Longleftrightarrow \quad \bar{V} = V\, e^{j\theta}$$

$$i(t) = \sqrt{2}\, I \cos(\omega t + \psi) \quad \Longleftrightarrow \quad \bar{I} = I\, e^{j\psi}$$



$\Longleftrightarrow$

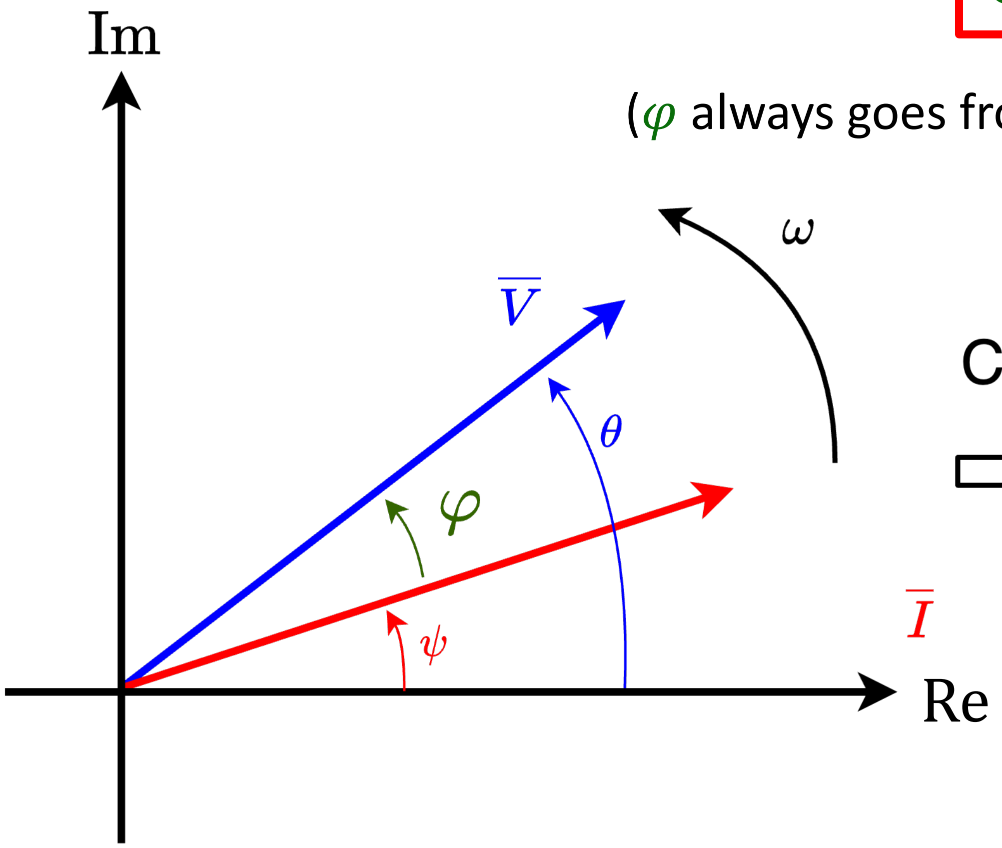


Phasors applied to circuits

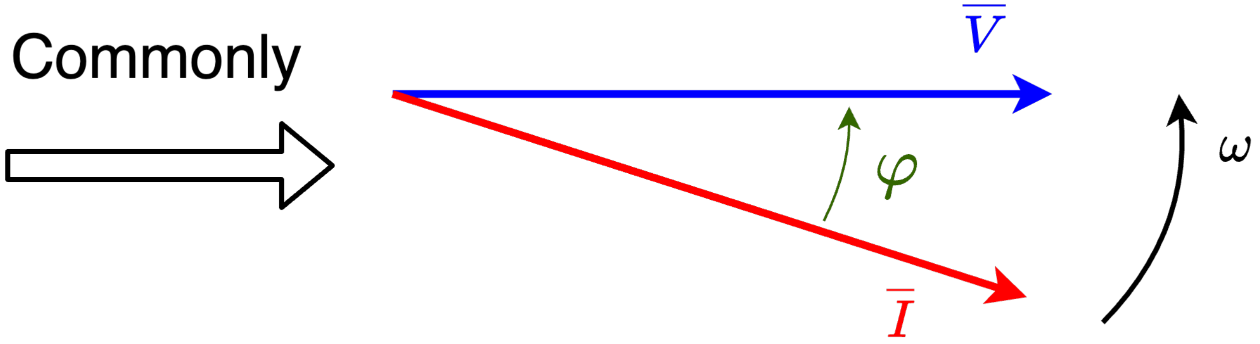
Remark: The phase angle difference (or phase lag) is noted φ.
It is defined as

$$\varphi = \theta - \psi$$

(φ always goes from the current to the voltage !)



Commonly



It is common to take $\bar{V}$ as the reference with $\theta = 0$ and $\varphi = -\psi$.

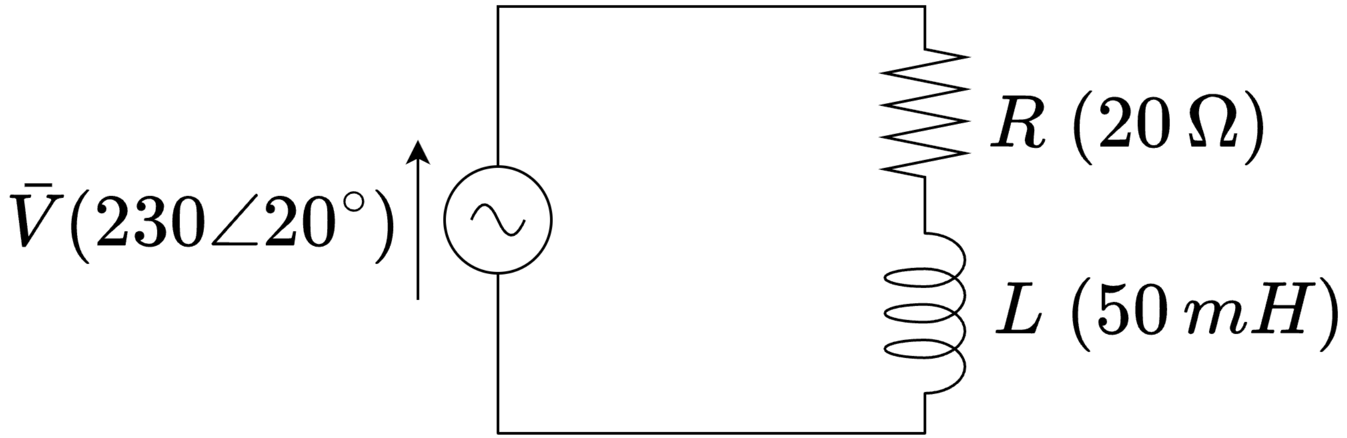
Phasors applied to circuits

	Time domain	$\Leftrightarrow$	Frequency domain
	$f(t) = \sqrt{2}\, F \cos(\omega t + \theta)$	$\Leftrightarrow$	$\bar{F} = F\, e^{j\theta}$
	$a\, f_1(t) + b\, f_2(t)$	$\Leftrightarrow$	$a\, \bar{F}_1 + b\, \bar{F}_2$
	$df(t)/dt$	$\Leftrightarrow$	$j\omega\, \bar{F}$
	$\int f(t)\, dt$	$\Leftrightarrow$	$\bar{F}/j\omega$
Resistance:	$v(t) = R\, i(t)$	$\Leftrightarrow$	$\bar{V} = R\, \bar{I}$
Inductance:	$v(t) = L\ di(t)/dt$	$\Leftrightarrow$	$\bar{V} = j\omega L\, \bar{I}$
Capacitance:	$v(t) = (1/C) \int i(t)\, dt$	$\Leftrightarrow$	$\bar{V} = \bar{I}/j\omega C$

Thanks to the linearity equivalence, the Kirchhoff laws can be applied to phasors. In addition, resistances, inductances and capacitances can be replaced by complex impedances $Z = \frac{\bar{V}}{\bar{I}}$ of value respectively equal to R, $j\omega L$ and $\frac{1}{j\omega C}$.

Application: Exercise 1

The circuit hereunder presents a resistive-inductive load powered with an AC generator of sinusoidal voltage ($230\ V$, $50\ Hz$). Find the voltages across R and L (magnitude and phase angle) and represent all the voltages on a phasor diagram.



Active, reactive, complex and apparent power

Product of two harmonic functions

Active, reactive, complex and apparent power

Application: Exercise 2

Homework: Exercise 3

Product of two harmonic functions

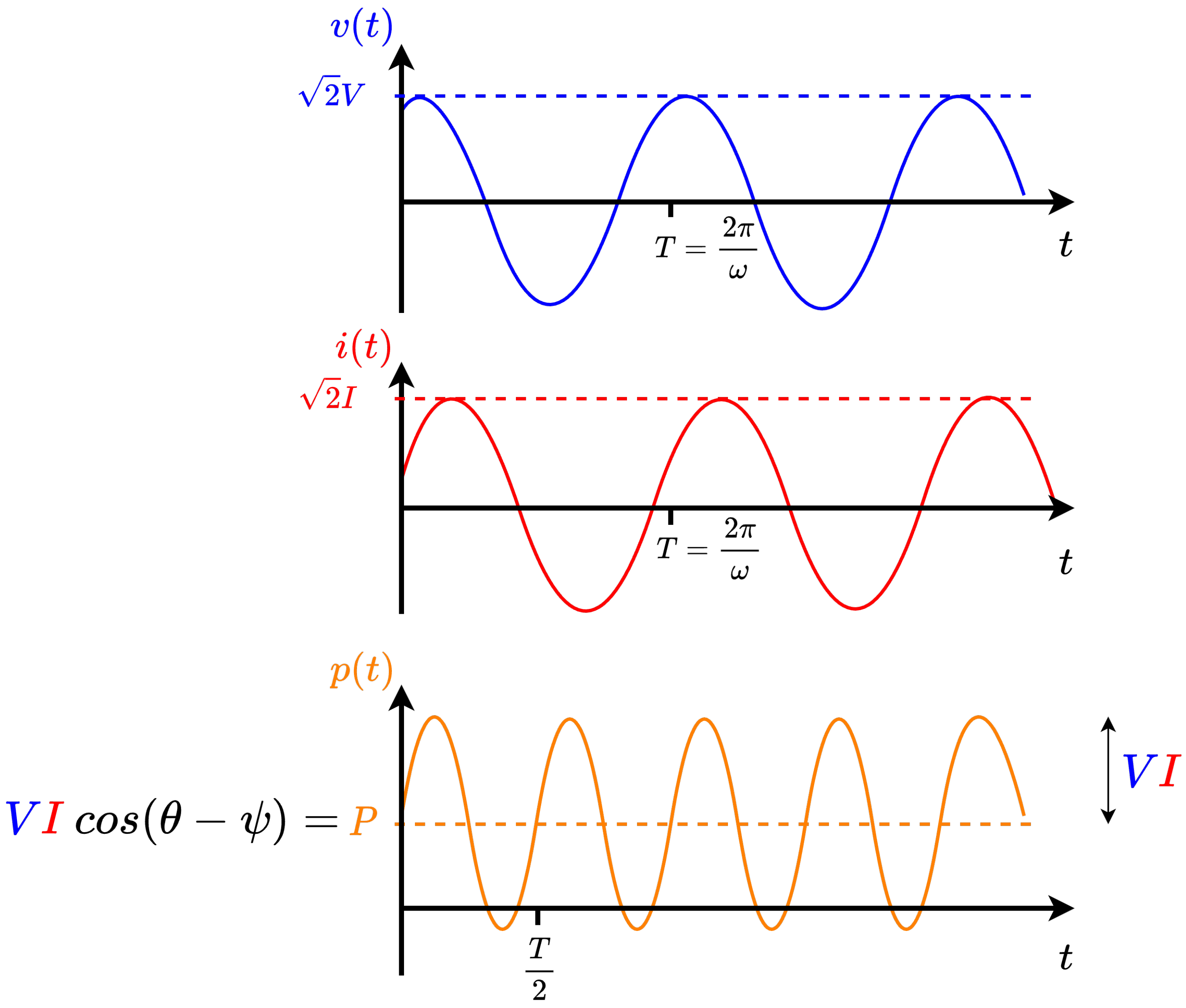
The instantaneous power $p(t)$ consumed by passive components (or delivered by active components) is obtained as by the product of the instantaneous voltage $v(t)$ with the instantaneous current $i(t)$:

$$\begin{aligned} p(t) &= v(t)\, i(t) \\ &= \sqrt{2}\, V \cos(\omega t + \theta)\ \sqrt{2}\, I \cos(\omega t + \psi) \\ &= 2\, V\, I \cos(\omega t + \theta)\ \cos(\omega t + \psi) \\ &= V\, I\, [\underbrace{\cos(\theta - \psi)}_{\text{DC component}} + \underbrace{\cos(2\omega t + \theta + \psi)}_{\text{AC component}}] \end{aligned}$$

DC component

AC component (at twice the original frequency)

Product of two harmonic functions



The instantaneous power can be split in two parts:

- $\mathrm{VI}\cos(\theta - \psi)$: A DC component called the Active power P.
- $V\ I\cos(2\omega t + \theta + \psi)$: An AC component oscillating at twice the original frequency.

Active, reactive, complex and apparent power

$$
\begin{aligned}
p(t) &= v(t)\, i(t) \\
&= V\, I\, [\cos(\theta - \psi) + \cos(2\omega t + \theta + \psi)] \\
&= V\, I\, \left[\cos(\theta - \psi) + \cos\big(2(\omega t + \theta) - (\theta - \psi)\big)\right] \\
&= V\, I\, \left[\cos(\theta - \psi) + \cos\big(2(\omega t + \theta)\big)\cos(\theta - \psi) + \sin\big(2(\omega t + \theta)\big)\sin(\theta - \psi)\right] \\
&= \underbrace{V\, I \cos(\theta - \psi)}_{P} \underbrace{\left[1 + \cos\big(2(\omega t + \theta)\big)\right]}_{\geq 0} + \underbrace{V\, I \sin(\theta - \psi)}_{Q} \underbrace{\sin\big(2(\omega t + \theta)\big)}_{\text{Oscillating +-}}
\end{aligned}
$$

$p(t)$ can be decomposed into:

- A flow of energy going always in the same direction with an average value $P = VI\, cos(\theta - \psi)$. The value P is called the active power.
- A fluctuating flow of energy, exchanged back and forth twice a period, with an average value $Q = VI \sin(\theta - \psi)$. The value Q is called the reactive power.

Active, reactive, complex and apparent power

Although $p(t) = P\left[1 + \cos\left(2(\omega t + \theta)\right)\right] + Q\ \sin\left(2(\omega t + \theta)\right)$ **is a real number**, one defines the complex power S as:

$$S = P + jQ$$

S **IS NOT A PHASOR**, it is just a convenient complex number allowing to describe easily and completely the instantaneous power (real number).

One also shows that is is really easy to determine S using phasors:

$$S = P + jQ = VI\cos(\theta - \psi) + j\,VI\sin(\theta - \psi) = VI\left[\cos(\theta - \psi) + j\sin(\theta - \psi)\right]$$
$$= VI\ e^{j(\theta - \psi)} = V\ e^{j\theta}\ I\ e^{-j\psi} = \bar{V}\ \bar{I}^{*}$$

$$S = \bar{V}\ \bar{I}^{*}$$

Active, reactive, complex and apparent power

One also defines the apparent power:

$$
\begin{aligned}
|S| &= \sqrt{P^2 + Q^2} \\
&= \sqrt{V^2 I^2 cos^2(\theta - \psi) + V^2 I^2 \sin^2(\theta - \psi)} \\
&= VI
\end{aligned}
$$

And the power factor:

$$PF = \cos(\theta - \psi) = \cos(\varphi) = \frac{P}{|S|}$$

Let's now suppose that we want to deliver an active power P to a load. What current I should we inject for a fixed voltage V ?

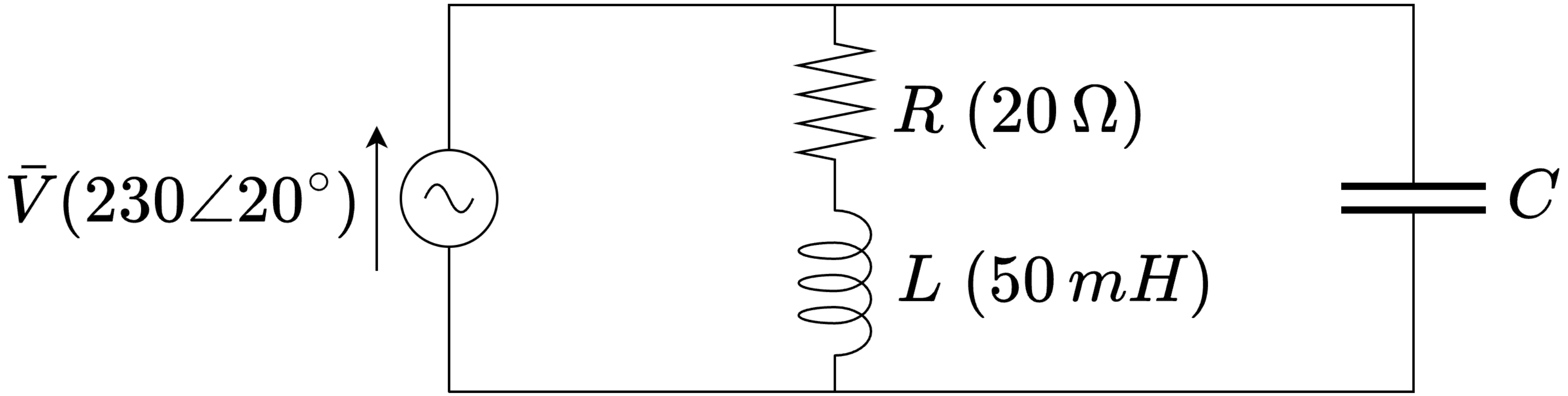
$$I = \frac{|S|}{V} = \frac{P}{V\cos(\varphi)}$$

I is larger when $\cos(\varphi)$ is small ➔ If $\cos(\varphi)$ is small, there are more transmission losses

➔ We want $\cos(\varphi)$ to be 1 ➔ We want the reactive power Q to be as small as possible.

Application: Exercise 2

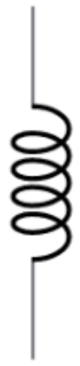
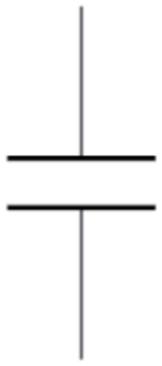
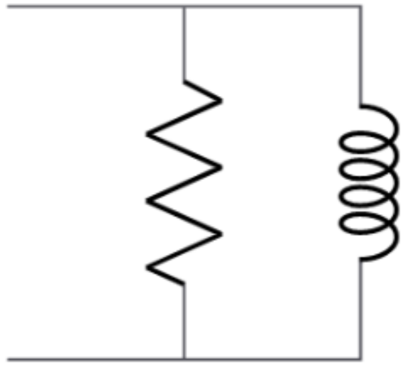
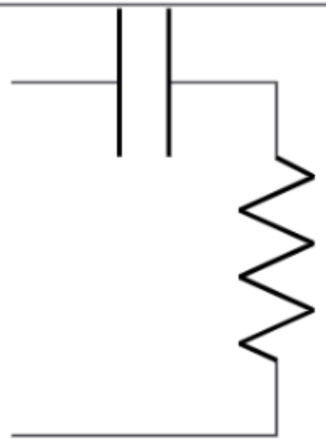
Your colleague suggests to add a $50\ \mu F$ capacitor in parallel of the RL load. It is supposed to compensate the reactive power consumed by the inductive load. Is it a good idea ? If so, what would be the exact value of C needed ?



Homework: Exercise 3

Fill the cells of the table below with the most appropriate answer among

$$= 0 \qquad < 0 \qquad > 0 \qquad = 1 \qquad < 1 \qquad +\infty \qquad -\infty$$

one-port:					
active power consumed					
reactive power produced					
$\cos\phi$					
$\tan\phi$					