Electromagnetic fields

Maxwell’s equations

\[
\begin{align*}
\text{curl } \mathbf{h} &= \mathbf{j} + \partial_t \mathbf{d} \\
\text{curl } \mathbf{e} &= -\partial_t \mathbf{b} \\
\text{div } \mathbf{b} &= 0 \\
\text{div } \mathbf{d} &= \rho_v
\end{align*}
\]

Physical fields

- \( \mathbf{h} \): magnetic field (A/m)
- \( \mathbf{b} \): magnetic flux density (T)
- \( \mathbf{j} \): current density (A/m\(^2\))
- \( \mathbf{e} \): electric field (V/m)
- \( \mathbf{d} \): electric displacement (C/m\(^2\))
- \( \rho_v \): charge density (C/m\(^3\))
Lorentz force

Interaction of electromagnetic fields with a point charge moving at speed $v$

$$F = q (e + v \times b) \quad \text{(N)}$$

For a conductor (electrically neutral, only negative charges moving)

$$f = j \times b \quad \text{(N/m}^3\text{)}$$

Laplace force
Electromagnetic power

Poynting vector

\[ s = e \times h \]

Power exchanged with a volume (interior normal)

\[ P = \oint_{\partial V} s \cdot n \, ds = - \int_V \text{div} \, s \, dv = \int_V p \, dv \] (W)

Power density

\[ p = -\text{div} \, e \times h = -h \cdot \text{curl} \, e + e \cdot \text{curl} \, h \] (W/m³)

\[ \Rightarrow p = h \cdot \partial_t b + e \cdot j + e \cdot \partial_t d \]
Material constitutive laws

Constitutive laws

\[ b = \mu \ h \]
\[ d = \varepsilon \ e \]
\[ j = \sigma \ e \]

Material characteristics

\( \mu \): magnetic permeability (H/m)
\( \varepsilon \): dielectric permittivity (F/m)
\( \sigma \): electrical conductivity (\( \Omega^{-1}\) m\(^{-1}\))

Magnetic law
Dielectric law
Ohm’s law

Constant (linear materials)
Function of the fields (nonlinear materials)
Tensorial (anisotropic materials)
Function of temperature, mechanical stress, ...
Magnetic constitutive law

\[ b = \mu h \]

\[ \mu = \mu_r \mu_0 \]

- Relative magnetic permeability
- Magnetic permeability of vacuum \((\text{H/m})\)

- **Diamagnetic and paramagnetic materials**
  - Linear materials \(\mu_r \approx 1\) (silver, copper, aluminum)

- **Ferromagnetic materials**
  - Nonlinear materials \(\mu_r \gg 1, \mu_r = \mu_r(h)\) (steel, iron)

**b-h law**

**Hysteresis**

Energy dissipation \((\equiv \text{area of the cycle})\)

\[ p_H = \omega k_h b_{\text{max}}^\nu \quad (\text{W/m}^3) \]

Pulsation \(\omega\), max. flux density \(b_{\text{max}}\), coefficients \(k_h\) and \(\nu\) \((1.5 < \nu < 1.8)\)
Electromagnetic models

- **Electrostatics**
  - Distribution of electric field due to static charges and levels of electric potential

- **Electrokinetics**
  - Distribution of stationary electric current in conductors

- **Electrodynamics (or electroquasistatics, Eqs)**
  - Distribution of electric field and currents in materials (both conductors and insulators)

- **Magnetostatics**
  - Distribution of stationary magnetic field due to magnets and stationary currents

- **Magnetodynamics (or magnetoquasistatics, MQS)**
  - Distribution of magnetic field and eddy currents due to moving magnets and time-dependent currents

- **Wave propagation**
  - Electromagnetic wave propagation

All governed by Maxwell’s equations
MQS: quasi-stationnary approximation

\[ \text{curl } \mathbf{h} = \mathbf{j} + \partial_t \mathbf{d} \]

Small dimensions compared to wavelength

Conduction current density

Displacement current density

\\[>\>\>\\]

\[ \text{curl } \mathbf{h} = \mathbf{j} \]

Ampère’s law

Applications

Electrotechnical devices (motors, generators, power transformers, ...)

Usually, frequencies up to several 100’s of kHz
**Ampère’s law**

\[ \text{curl } \mathbf{h} = \mathbf{j} \]

\[ \oint \mathbf{h} \cdot d\mathbf{l} = I \]

The circulation of the magnetic field along a closed contour is equal to the algebraic sum of the currents crossing any surface bounded by this contour.

\[ \text{div } \mathbf{j} = 0 \]

\[ \oint \mathbf{j} \cdot \mathbf{n} \, ds = 0 \]

The sum of the currents arriving at a given point is zero.
Faraday’s law

\[ \text{curl } \mathbf{e} = - \partial_t \mathbf{b} \]

Faraday’s law

\[ \oint \mathbf{e} \cdot d\mathbf{l} = - \partial_t \Phi \]

e.m.f

Lenz’ law

... which, when this circuit is closed, gives rise to currents generating magnetic flux density opposing these variations

\[ \text{div } \mathbf{b} = 0 \]

Conservation of the magnetic flux

\[ \oint \mathbf{b} \cdot \mathbf{n} \, ds = 0 \]

Magnetic flux lines are closed

Any variation (time, movement or deformation) of the magnetic flux density embraced by a circuit (open or closed) gives rise to an electromotive force (e.m.f.) ...

Movement, velocity \( \mathbf{v} \)

\[ \mathbf{e} = \mathbf{v} \times \mathbf{b} \]

Principles of Electromagnetism
Faraday’s law – Eddy currents

\[ \text{curl } \mathbf{e} = -\partial_t \mathbf{b} \]

Faraday’s law

\[ \oint e \cdot dl = -\partial_t \Phi \]

In a massive conductor subject to time-varying magnetic field, e.m.f.s appear that give rise to currents

Eddy (or induced) currents

Heating by Joule effect
(degrades efficiency)

Reduction of the global magnetic flux (Lenz’s law) (degrades material efficiency)

Laminated magnetic materials

Stacks of thin magnetic sheets, parallel to the magnetic flux density and electrically isolated

For thin sheets, eddy current losses:

\[ p_F = \frac{\omega^2 e_t^2 \sigma b_{max}^2}{16} \quad (\text{W} / \text{m}^3) \]

pulsation \( \omega \), sheet thickness \( e_t \),
electrical conductivity \( \sigma \),
max. magnetic flux density \( b_{max} \)

Principles of Electromagnetism
Skin effect

\[ \text{curl } \mathbf{e} = - \partial_t \mathbf{b} \]

Faraday’s law

Skin effect

The skin depth \( \delta \) characterizes the depth in the material at which the current (and the magnetic field) tend to concentrate. Increasing the frequency leads to smaller \( \delta \), which leads to currents concentrated closer to the surface of the conductor.

\[ \delta = \sqrt{\frac{2}{\omega \sigma \mu}} \text{ (m)} \]

- \( \omega \): pulsation (rad/m)
- \( \sigma \): electrical conductivity (\( \Omega^{-1} \text{ m}^{-1} \))
- \( \mu \): magnetic permeability (H/m)
Magnetic circuits

Produced by electric currents (e.g. in windings) or magnets

through which the transfer of conversion of energy is carried out (e.g. between windings for electrical energy)

Interest in high magnetic coupling (good magnetic link)

Magnetic circuits with magnetic materials to channel the magnetic flux density

with airgaps (e.g. separating moving parts)
Ideal magnetic circuit

\[ \Phi_{t1} = n_1 \Phi = \frac{n_1^2}{R} I_1 + \frac{n_1 n_2}{R} I_2 = \lambda_1 I_1 + M_{12} I_2 \]

\[ \Phi_{t2} = n_2 \Phi = \frac{n_1 n_2}{R} I_1 + \frac{n_2^2}{R} I_2 = M_{21} I_1 + \lambda_2 I_2 \]

Inductances

\[ \lambda_1 = \frac{n_1^2}{R} , \quad \lambda_2 = \frac{n_2^2}{R} , \quad M = M_{12} = M_{21} = \frac{n_1 n_2}{R} \]

\[ \oint h \cdot dl = h \ell = n_1 I_1 + n_2 I_2 \]

Magnetomotive forces (m.m.f.)

\[ \Phi = b S = \mu h S = (n_1 I_1 + n_2 I_2) \frac{\mu S}{\ell} = \frac{n_1 I_1 + n_2 I_2}{R} \]

Reluctance of the circuit

\[ R = \frac{\ell}{\mu S} \]

Perfect magnetic coupling

\[ (\lambda_1 \lambda_2 = M^2) \]

Principles of Electromagnetism
Real magnetic circuit

\[ \Phi = \frac{n_1 I_1 + n_2 I_2}{R} \]

Useful flux

Leakage flux

\[ \Phi_{f1} = \frac{n_1 I_1}{R_{f1}} \quad \text{et} \quad \Phi_{f2} = \frac{n_2 I_2}{R_{f2}} \]

Leakage reluctances

\[ \Phi_{t1} = n_1 (\Phi + \Phi_{f1}) = \left( \frac{n_1^2}{R} + \frac{n_1^2}{R_{f1}} \right) I_1 + \frac{n_1 n_2}{R} I_2 = \lambda_1 I_1 + M_{12} I_2 \]

\[ \Phi_{t2} = n_2 (\Phi + \Phi_{f2}) = \frac{n_1 n_2}{R} I_1 + \left( \frac{n_2^2}{R} + \frac{n_2^2}{R_{f2}} \right) I_2 = M_{21} I_1 + \lambda_2 I_2 \]

Inductances

\[ \lambda_1 = \frac{n_1^2}{R} + \frac{n_1^2}{R_{f1}}, \quad \lambda_2 = \frac{n_2^2}{R} + \frac{n_2^2}{R_{f2}}, \quad M = M_{12} = M_{21} = \frac{n_1 n_2}{R} \]

Non-ideal magnetic coupling

\( (\lambda_1 \lambda_2 \geq M^2) \)