

Ordinary and partial differential equations

Numerical treatment of differential equations

Initial value problem: $u'(t) = f(t, u(t))$, $u(0) = u_0$, $t > 0$

Boundary value problem:

$$u''(x) = f(x), \quad x(0) = x_0, x(1) = x_1, \quad x \in [0, 1]$$

General assumption: f has higher derivatives.

IVP stability: solutions corresponding to different u_0 values converge as $t \rightarrow \infty$. Criterium:

$$\frac{\partial}{\partial u} f(t, u) = \begin{cases} > 0 & \text{unstable} \\ = 0 & \text{neutrally stable} \\ < 0 & \text{stable} \end{cases}$$

Simple example: $f(t, u) = -\lambda u$, then $u(t) = u_0 e^{-\lambda t}$;
stable if $\lambda > 0$

Finite difference approximation

We turn the continuous problem into a discrete one, by looking at finite time/space steps.

Assume all functions are sufficiently smooth, and use Taylor series:

$$u(t + \Delta t) = u(t) + u'(t)\Delta t + u''(t)\frac{\Delta t^2}{2!} + u'''(t)\frac{\Delta t^3}{3!} + \dots$$

This gives for u' :

$$u'(t) = \frac{u(t + \Delta t) - u(t)}{\Delta t} + O(\Delta t)$$

So we approximate

$$u'(t) \approx \frac{u(t + \Delta t) - u(t)}{\Delta t}$$

Finite differences 2

How does this help? In $u' = f(t, u)$ substitute

$$u'(t) \rightarrow \frac{u(t + \Delta t) - u(t)}{\Delta t}$$

giving

$$\frac{u(t + \Delta t) - u(t)}{\Delta t} = f(t, u(t))$$

or

$$u(t + \Delta t) = u(t) + \Delta t f(t, u(t))$$

Let $t_0 = 0$, $t_{k+1} = t_k + \Delta t = \dots = (k + 1)\Delta t$, $u(t_k) = u_k$:

$$u_{k+1} = u_k + \Delta t f(t_k, u_k)$$

Discretization

'Explicit Euler' or 'Euler forward'.

Does this compute something close to the true solution?

'Discretization error'

Some error analysis

Local Truncation Error: assume computed solution is exact at step k , how wrong will we be at step $k + 1$?

$$\begin{aligned}u(t_{k+1}) &= u(t_k) + u'(t_k)\Delta t + u''(t_k)\frac{\Delta t^2}{2!} + \dots \\&= u(t_k) + f(t_k, u(t_k))\Delta t + u''(t_k)\frac{\Delta t^2}{2!} + \dots \\u_{k+1} &= u_k + f(t_k, u_k)\Delta t\end{aligned}$$

So

$$\begin{aligned}L_{k+1} &= u_{k+1} - u(t_{k+1}) \\&= u_k - u(t_k) + \Delta t(f(t_k, u_k) - f(t_k, u(t_k))) - u''(t_k)\frac{\Delta t^2}{2!} + \dots \\&= -u''(t_k)\frac{\Delta t^2}{2!} + \dots\end{aligned}$$

Global error: $E_k \approx \sum_k L_k = O(\Delta t)$: First order method

An Euler forward example

Consider $f(t, u) = -\lambda u$, exact solution $u(t) = u_0 e^{-\lambda t}$;
stable if $\lambda > 0$

Explicit Euler scheme

$$u_{k+1} = u_k - \Delta t \lambda u_k = (1 - \lambda \Delta t) u_k = (1 - \lambda \Delta t)^k u_0$$

Then

$$\begin{aligned} u_k &\rightarrow 0 \text{ as } k \rightarrow \infty \\ \Leftrightarrow |1 - \lambda \Delta t| &< 1 \\ \Leftrightarrow -1 < 1 - \lambda \Delta t &< 1 \\ \Leftrightarrow -2 < -\lambda \Delta t &< 0 \\ \Leftrightarrow 0 < \lambda \Delta t &< 2 \\ \Leftrightarrow \Delta t < 2/\lambda \end{aligned}$$

Conditionally stable

Implicit Euler

Or 'Euler backward':

$$u(t - \Delta t) = u(t) - u'(t)\Delta t + u''(t)\frac{\Delta t^2}{2!} + \dots$$

so

$$u'(t) = \frac{u(t) - u(t - \Delta t)}{\Delta t} + u''(t)\Delta t/2 + \dots$$

Compute $u'(t) = f(t, u(t))$ as

$$\begin{aligned}\frac{u(t) - u(t - \Delta t)}{\Delta t} &= f(t, u(t)) \\ \Rightarrow u(t) &= u(t - \Delta t) + \Delta t f(t, u(t)) \\ \Rightarrow u_{k+1} &= u_k + \Delta t f(t_{k+1}, u_{k+1})\end{aligned}$$

Implicit equation for u_{k+1} !

Let $f(t, u) = -u^3$, then $u_{k+1} = u_k - \Delta t u_{k+1}^3$
needs nonlinear solver

Stability of Implicit Euler

Again the $f(t, u) = -\lambda u$ example:

$$\begin{aligned}u_{k+1} &= u_k - \lambda \Delta t u_{k+1} \\(1 + \Delta t)u_{k+1} &= u_k \\u_{k+1} &= \left(\frac{1}{1 + \lambda \Delta t} \right) u_k = \left(\frac{1}{1 + \lambda \Delta t} \right)^k u_0\end{aligned}$$

If $\lambda > 0$ (stable equation), then $u_k \rightarrow 0$ for all values of λ and Δt :
unconditionally stable.

Pro: larger time steps possible, no worries

Con: implicit equation needs to be solved

Higher order methods

Runge-Kutta

Adams-Bashforth

Crank-Nicholson