Mathématiques appliquées (MATH0504-1)

B. Dewals, C. Geuzaine

Objective of the course

Introduce mathematical topics and methods useful for a wide range of engineering applications:

- Partial Differential Equations (PDEs), including simple numerical methods

- Linear algebra complements: subspace methods and Singular Value Decomposition (SVD)

Organization

Schedule

- Theory: Friday 1:45 PM - 3:45 PM
  B4 A604, B31 De Méan, B7a A500 – Check your calendars!
- Exercises: Friday 4 PM – 5:45 PM (B5b) – Except Today and October 18th

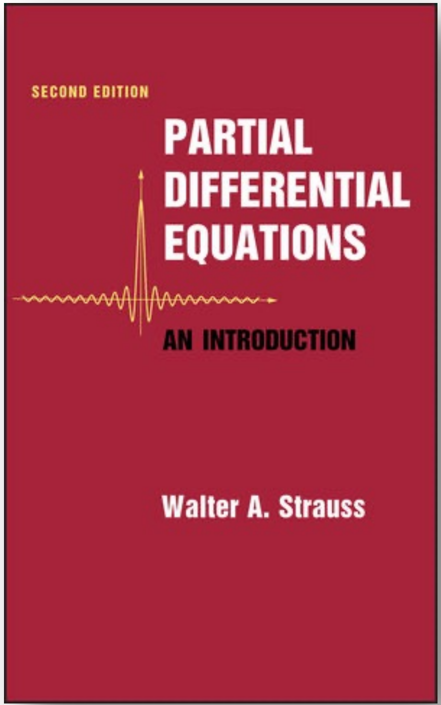
Course website with the latest information, slides and exercise booklet in PDF format:
http://people.montefiore.ulg.ac.be/geuzaine/MATH0504

Videos (Theory 2020) are available on eCampus

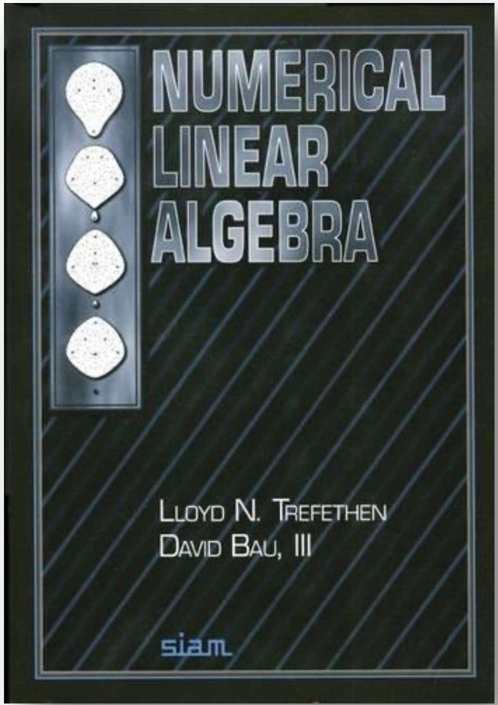
Written exam in January (theory + exercises)

Textbooks

1. Strauss, W. (2008). *Partial Differential Equations: An Introduction*, Wiley.



2. Trefethen, L.N. & Bau, D. (1997). *Numerical Linear Algebra*, SIAM.



Part I

Introduction to Partial Differential Equations

Lecture 1 Introduction

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Partial Differential Equations (*PDE*s)

The understanding of the fundamental processes of the natural world is based to a large extent on partial differential equations:

- deformation and vibration of solids
- fluids flow
- dispersion of chemicals
- heat diffusion
- radiation of electromagnetic waves
- structure of molecules
- interactions of photons and electrons …

Partial Differential Equations (*PDEs*)

Partial differential equations also play a central role in modern mathematics, especially in geometry and analysis.

The availability of powerful computers is gradually shifting the emphasis in PDEs away from the analytical computation of solutions and toward both their numerical analysis and the qualitative theory.

Partial Differential Equations (*PDE*s)

In this course we will

- motivate with physics but then do mathematics,
- focus on three classical equations,
  All key ideas can be understood from them!
- consider one spatial dimension before extending to two and three dimensions with their more complicated geometries,
- address problems without boundaries before bringing in boundary conditions.

Learning objectives of this class

Learn elementary properties of PDEs

Solve simple first-order linear PDEs

Understand the link between some classical PDEs and the modelling of physical phenomena

Outline

What is a partial differential equation?

First-order linear equations

Flows, vibrations and diffusions

1 – What is a Partial Differential Equation?

Definition of a PDE

Linear vs. nonlinear, homogeneous vs. inhomogeneous

Examples

What is a PDE?

Let $x, y, \ldots$ denote independent variables and let u denote a dependent variable that is an unknown function of these variables, i.e. $u(x, y, \ldots)$.

A PDE is an identity that relates the independent variables, the dependent variable u, and the **partial derivatives** of u.

We will often denote the derivatives by subscripts, thus e.g. $\partial u / \partial x = u_x$.

A *solution* of a PDE is a function $u(x, y, \ldots)$ that satisfies the equation identically, at least in some region of the $x, y, \ldots$ variables.

Examples with two independent variables

1. $u_x + u_y = 0$ (transport)
2. $u_x + yu_y = 0$ (transport)
3. $u_x + uu_y = 0$ (shock wave)
4. $u_{xx} + u_{yy} = 0$ (Laplace's equation)
5. $u_{tt} - u_{xx} + u^3 = 0$ (wave with interaction)
6. $u_t + uu_x + u_{xxx} = 0$ (dispersive wave)
7. $u_{tt} + u_{xxxx} = 0$ (vibrating bar)
8. $u_t - iu_{xx} = 0$ $(i = \sqrt{-1})$ (quantum mechanics)

Order of a PDE

The *order* of a PDE is the highest derivative that appears in the equation.

Most general *first order* PDE in two independent variables:

$$F(x, y, u, u_x, u_y) = 0$$

Most general *second order* PDE in two independent variables:

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0.$$

Linear vs. nonlinear PDEs

Write the equation in the form $\mathscr{L}u = g$, where $\mathscr{L}$ is an operator and g is a function of the independent variables (or zero).

A PDE is linear if

$$\mathscr{L}(u+v) = \mathscr{L}u + \mathscr{L}v \qquad \mathscr{L}(cu) = c\mathscr{L}u$$

for any functions u, v and any constant c.

If one or both conditions do not hold, the PDE is nonlinear.

We will mostly study linear PDEs (often with constant coefficients).

Homogeneous vs. inhomogeneous PDEs

If $\mathscr{L}$ is a linear operator, the equation

$$\mathscr{L}u = 0$$

is called a *homogeneous* linear equation.

The equation

$$\mathscr{L}u = g$$

where $g \neq 0$ is a given function of the independent variables, is called an *inhomogeneous* linear equation.

Wooclap

Go to https://wooclap.com and enter event code *LIMUZD*

Which of the following *operators* are linear?

$$\mathscr{L}(u+v) = \mathscr{L}u + \mathscr{L}v \qquad \mathscr{L}(cu) = c\mathscr{L}u$$

(a) $\mathscr{L}u = u_x + xu_y$

(b) $\mathscr{L}u = u_x + uu_y$

(c) $\mathscr{L}u = u_x + u_y^2$

(d) $\mathscr{L}u = u_x + u_y + 1$

(e) $\mathscr{L}u = \sqrt{1+x^2}\,(\cos y)u_x + u_{yxy} - [\arctan(x/y)]u$

Wooclap

Nonlinear, linear inhomogeneous or linear homogeneous PDE?

(a) $u_t - u_{xx} + 1 = 0$

(b) $u_t - u_{xx} + xu = 0$

(c) $u_t - u_{xxt} + uu_x = 0$

(d) $u_{tt} - u_{xx} + x^2 = 0$

(e) $iu_t - u_{xx} + u/x = 0$

(f) $u_x(1 + u_x^2)^{-1/2} + u_y(1 + u_y^2)^{-1/2} = 0$

(g) $u_x + e^y u_y = 0$

(h) $u_t + u_{xxxx} + \sqrt{1 + u} = 0$

2 – First-Order Linear Equations

Constant coefficient equation

Variable coefficient equation

Constant coefficient equation

Let us solve

$$a\,u_x + b\,u_y = 0$$

where a and b are not both zero.

The quantity $a\,u_x + b\,u_y$ is the directional derivative of u in the direction of the vector $\mathbf{V} = (a,b)$.

The PDE thus means that $u(x,y)$ is constant along the direction of $\mathbf{V}$.

Note: equation of a line parallel to vector $\mathbf{V}$

General formulation:

$$y = m\,x + c$$

where m is the slope, and c the intercept.

If the line is parallel to vector $\mathbf{V} = (a,b)$, then the slope m is given by $m = b\,/\,a$.

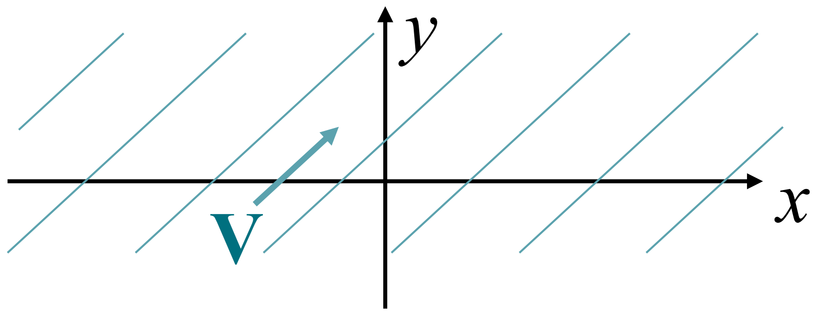
Hence the equation of the line becomes:

$$y = \frac{b}{a}\,x + c \quad \Rightarrow \quad b\,x - a\,y = -a\,c = \text{constant}$$

Constant coefficient equation

The lines parallel to $\mathbf{V} = (a,\, b)$ are described by the equation:

$$b\,x - a\,y = \text{constant}$$



These lines are called *characteristic lines*.

The solution $u(x,y)$ is constant on each characteristic line.

Constant coefficient equation

On the line $b\,x - a\,y = \mathrm{cst}$, the solution u takes a constant value, which depends on the particular line, i.e., on the value of the constant.

Call the constant c,

and the value of the solution $f(\,c\,)$:

$$u\left(x,y\right)=f\left(c\right)=f\left(bx-ay\right)$$

Constant coefficient equation

We have: $u(x,y) = f(c) = f(bx - ay)$

Since c is arbitrary, this holds for all values of x and y. It follows that

$$u(x,y) = f(bx - ay)$$

is the *general solution* of the PDE, with f any function of one variable.

Moral: a PDE *has arbitrary functions in its solution*.

These functions need to be fixed thanks to *auxiliary* ("boundary" or "initial") conditions.

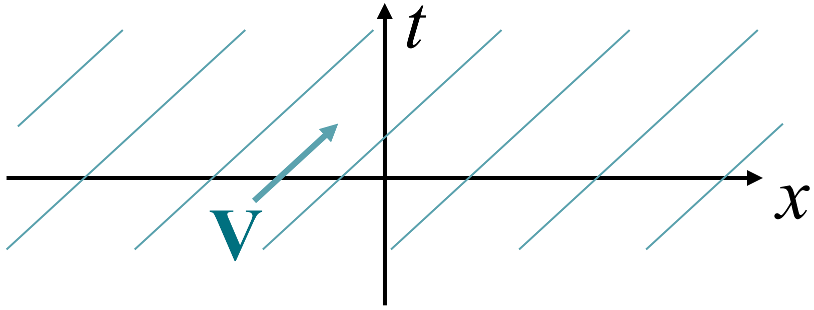
Example: solve $2\, u_t + 3\, u_x = 0$ with the auxiliary condition $u = \sin x$ when $t = 0$

The quantity $3\, u_x + 2\, u_t$ is the directional derivative of u in the direction of the vector $\mathbf{V} = (3, 2)$.

This means that $u(x, t)$ must be constant in the direction of $\mathbf{V}$.

The lines parallel to $\mathbf{V}$ have the equations

$$2\, x - 3\, t = \text{constant}.$$



Example: solve $2\,u_t + 3\,u_x = 0$ with the auxiliary condition $u = \sin x$ when $t = 0$

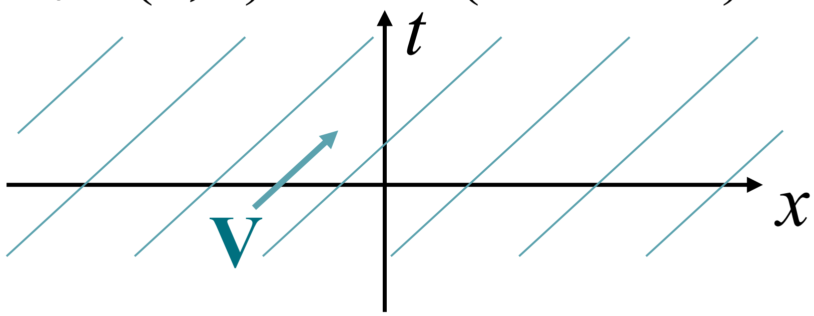
Thus the *general solution* of the PDE is

$$u(x, t) = f\,(2\,x - 3\,t),$$

where f is any function of one variable.

Auxiliary condition (initial condition)

- setting $t = 0$ yields the equation $f\,(2\,x) = \sin x$
- Letting $s = 2\,x$ yields $f(s) = \sin\,(s/2)$.
- Therefore, $u(x, t) = \sin\,(x - 3/2\;t)$.



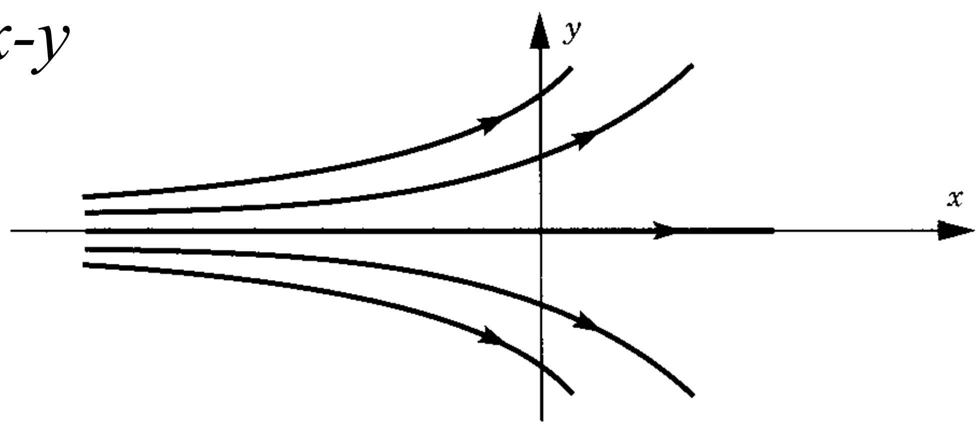
Variable coefficient equation

Let us solve

$$u_x + y\,u_y = 0$$

This is a linear PDE, but with a variable coefficient. The PDE asserts that the directional derivative of u in the direction of the vector $\mathbf{V} = (\,1\,,\,y\,)$ is zero.

The curves in the x-y plane with $(1,y)$ as tangent vectors have slopes y.

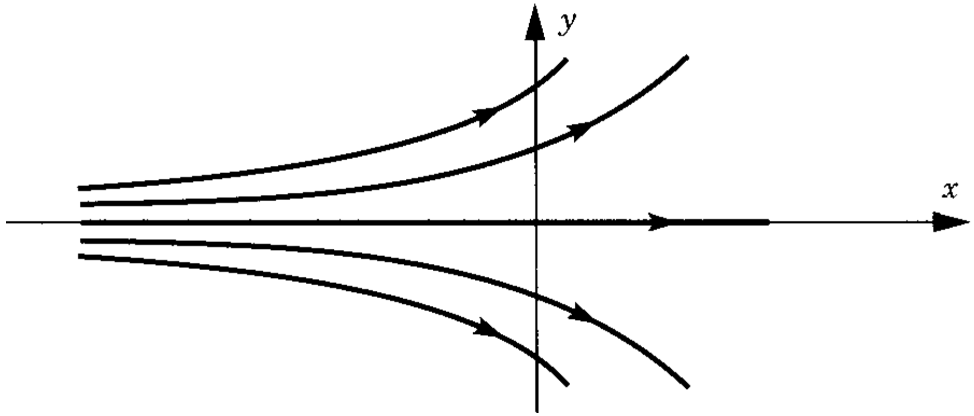


Variable coefficient equation

The equations of these *characteristic curves* are

$$\frac{dy}{dx} = \frac{y}{1} \qquad \Rightarrow \qquad y = C\, e^{x}$$

As the value of C is varied, these curves fill the x-y plane without intersecting.



Variable coefficient equation

On the paths defined by the characteristic curves the PDE reduces to an ordinary differential equation (ODE):

$$\frac{d}{dx}u\left(x, Ce^{x}\right) = \frac{\partial u}{\partial x} + Ce^{x}\frac{\partial u}{\partial y} = u_{x} + yu_{y} = 0$$

Hence, $u = f(C)$, with f an arbitrary function of the characteristic coordinate $C = e^{-x}y$.

The *general solution* of the PDE is thus

$$u\left(x, y\right) = f\left(e^{-x}y\right)$$

Variable coefficient equation

This geometric method works nicely for any PDE of the form $a(x, y)\, u_x + b(x, y)\, u_y = 0$.

It reduces the solution of the PDE to the solution of the ODE

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$$

If the ODE can be solved, so can the PDE. Every solution of the PDE is constant on the solution curves of the ODE.

Alternative formulation

It is also possible to transform the following PDE

$$a(x, y)\, u_x + b(x, y)\, u_y = 0$$

from the definition of the total derivative

$$\frac{d}{dx} u\left[x, y(x)\right] = \frac{\partial u}{\partial x} + \frac{dy}{dx}\frac{\partial u}{\partial y} = 0,$$

into the following 2 ODEs, by identification

$$\frac{du}{dx} = 0 \qquad \& \qquad \frac{dy}{dx} = \frac{b(x, y)}{a(x, y)}$$

Example: solve $\sqrt{1-x^2}u_x + u_y = 0$ with $u(0,y) = y$

On the characteristic curves defined by

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

the PDE reduces to an ODE. Integrating, we get

$$y = \arcsin x + C$$

The general solution of the PDE is thus

$$u(x,y) = f(y - \arcsin x)$$

Since we are told that $u(0,y) = y$, we need $f(y) = y$ and thus:

$$u(x,y) = y - \arcsin x$$

Alternative : solve $\sqrt{1-x^2}u_x + u_y = 0$ with $u(0,y) = y$

The 2 ODEs are

$$\frac{du}{dx} = 0 \qquad \& \qquad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Integrating, we get

$$u = C_1 \qquad \& \qquad y = \arcsin x + C_2$$

If we define $y_0 \triangleq y(x=0)$, we have

$$u = y_0 \qquad \& \qquad y = \arcsin x + y_0$$

From the characteristic line equation: $y_0 = -\arcsin x + y$

Finally, replacing the above expression in the solution $u(x,y) = y - \arcsin x$

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3 – Flows, Vibrations and Diffusions

PDE Examples from Physics

In physical problems the independent variables are often those of space x, y, z, and time t.

Let us look at some classical problems:

- transport of a pollutant in a fluid flow
- vibrating string
- vibrating drumhead
- diffusion of a dye in a motionless fluid
- stationary diffusion

Simple Transport

Consider a fluid (e.g., water) flowing at a constant rate c along a horizontal pipe of fixed cross section in the positive x direction.

A substance (e.g., a pollutant) is suspended in the water. Let $u(x, t)$ be its *linear* concentration (e.g., in g/cm) at time t.

The amount of pollutant in the interval [0, b] at the time t is (e.g., in g):

$$M = \int_0^b u(x,t)\,dx$$

Simple Transport

At the later time $t + h$, the same molecules of pollutant have moved to the right by a distance $c\,h$ (e.g., in centimeters), i.e.,

$$M = \int_0^b u(x,t)\,dx = \int_{ch}^{b+ch} u(x,t+h)\,dx$$

Differentiating with respect to b, we get

$$u(b,t) = u(b+ch,t+h)$$

Differentiating this last result with respect to h, and setting h = 0, we get

$$c\,u_x(b+ch,t+h) + u_t(b+ch,t+h) = 0$$

$$c\,u_x(b,t) + u_t(b,t) = 0$$

Leibniz integral rule

To evaluate the derivative of these integrals

$$\int_0^b u(x,t)\,dx = \int_{ch}^{b+ch} u(x,t+h)\,dx$$

with respect to b, the Leibniz integral rule has been used:

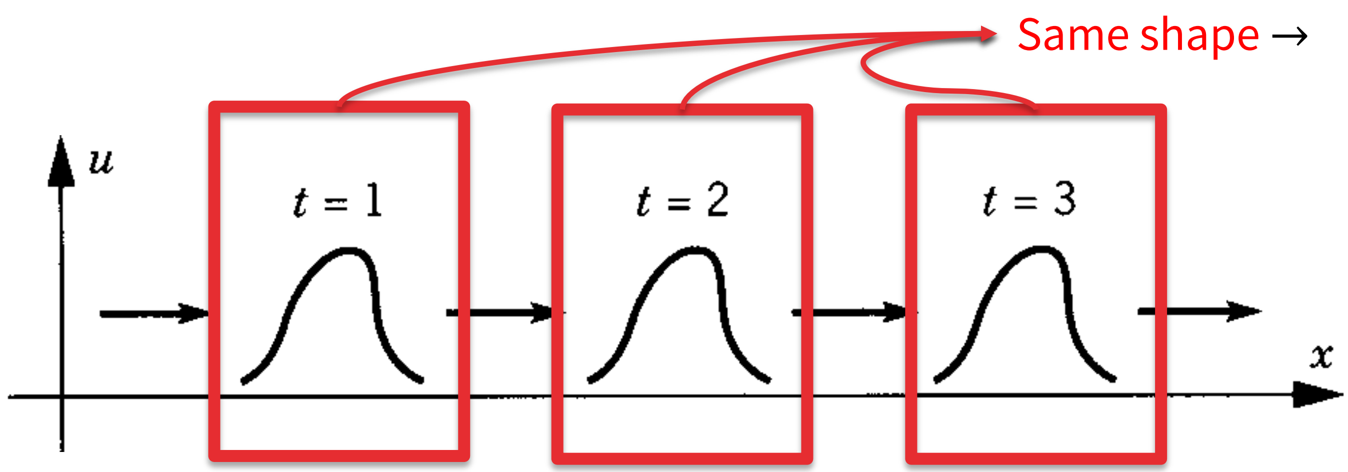
$$\frac{d}{dz}\left[\int_{a(z)}^{b(z)} u(x,z)\right] dx = \int_{a(z)}^{b(z)} \frac{\partial}{\partial z} u(x,z)\,dx$$

$$+u(b,z)\frac{d}{dz}b(z) - u(a,z)\frac{d}{dz}a(z)$$

Simple Transport

The simple transport PDE is thus $u_t + cu_x = 0$.

The general solution is a function of $(x - ct)$ only: the substance is transported to the right at a fixed speed c. Each individual particle moves to the right at speed c; in the x-t plane, each particle moves precisely along a characteristic line.



$$\frac{du}{dt} = 0$$

Simple Transport

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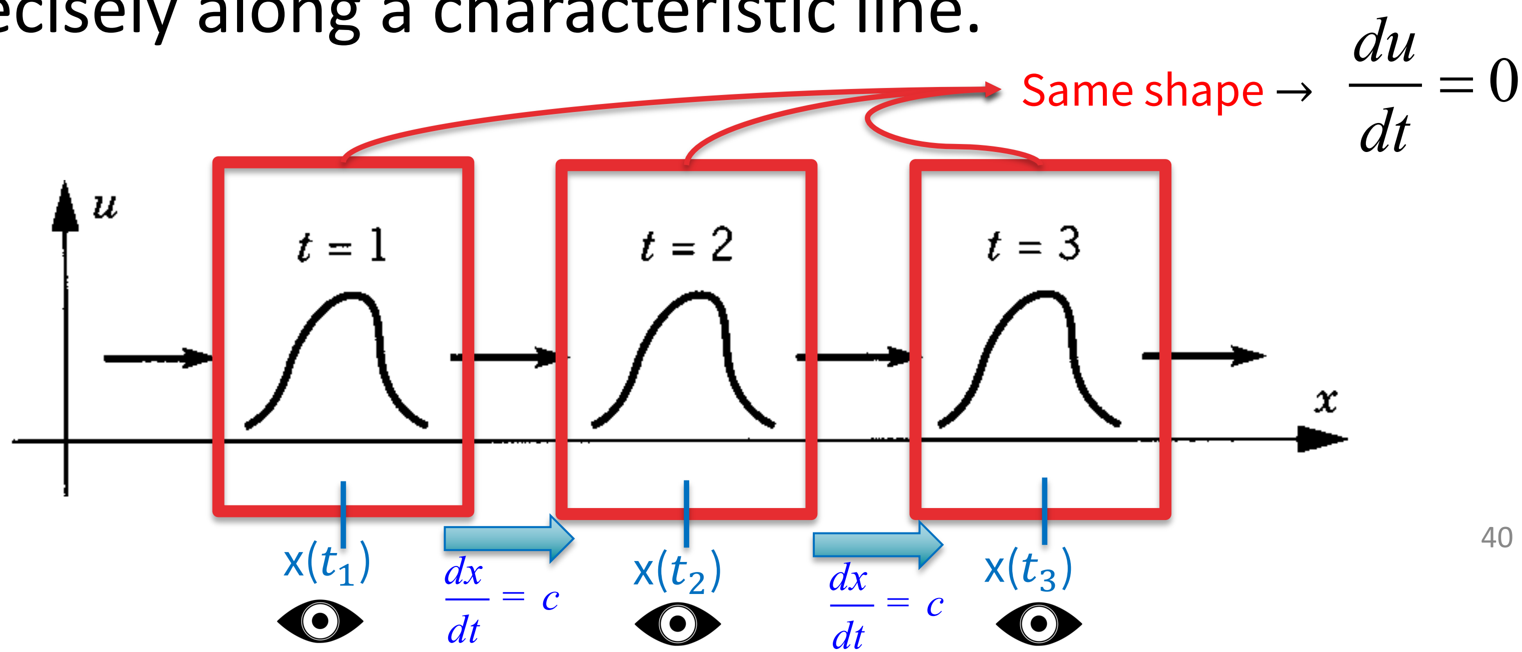
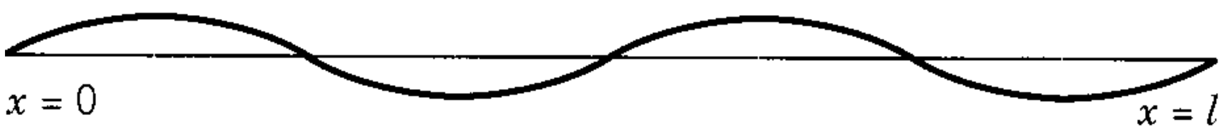


Vibrating String

Consider a flexible, elastic homogenous string of length l, undergoing small transverse vibrations.

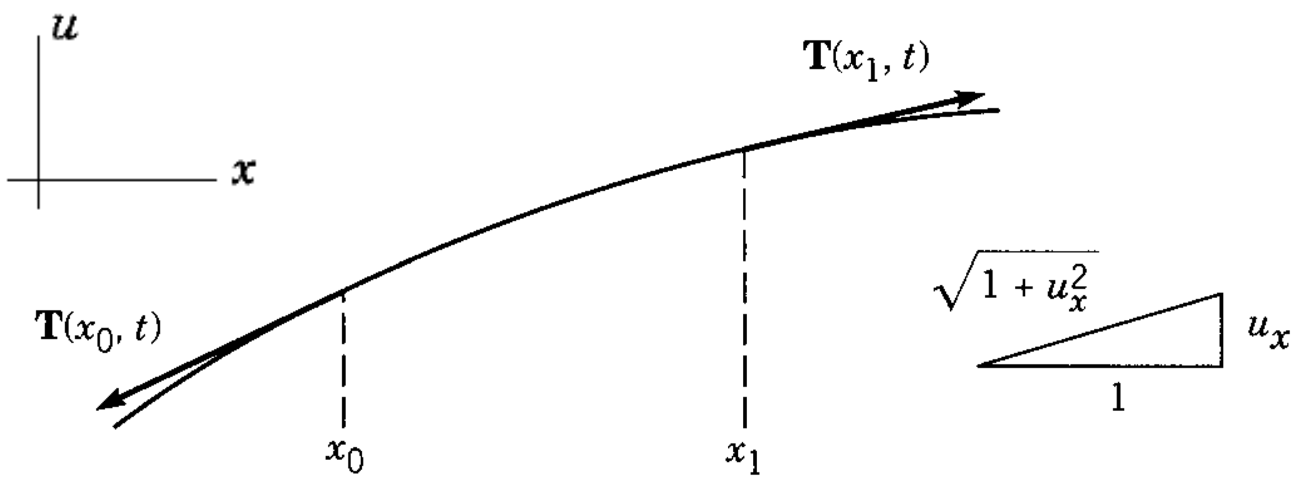
Let $u(x, t)$ be its displacement from equilibrium position at time t and position x:



Let ρ be the (constant) *linear* density (mass per unit length) of the string.

Vibrating String

Because the string is perfectly flexible, the tension (force) is directed tangentially along the string:



Let us denote by $T(x, t)$ the magnitude of this tension vector and let us write Newton's law for the part of the string between x_0 and x_1.

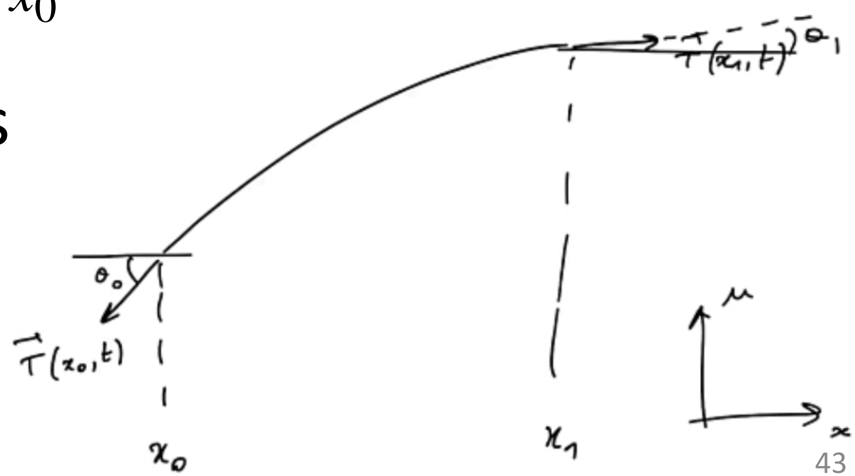
Vibrating String

We have $\sum \mathbf{F} = m\mathbf{a}$, i.e., in the x-u coordinate system (with the x-component of $\mathbf{a}$ = 0 since we assume purely transversal vibrations):

$$-T\cos\theta_0 + T\cos\theta_1 = 0 \qquad \text{along } x$$

$$-T\sin\theta_0 + T\sin\theta_1 = \int_{x_0}^{x_1} \rho u_{tt}\,dx \qquad \text{along } u$$

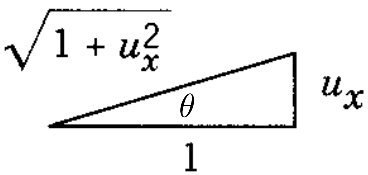
with θ_0 and θ_1 the angles between the tension vector and the x-axis at x_0 and x_1, respectively.



Vibrating String

Since $\tan\theta$ is equal to the slope $\frac{\partial u}{\partial x} = u_x$, we have

$$\cos\theta = \frac{1}{\sqrt{1+u_x^2}} \qquad \sin\theta = \frac{u_x}{\sqrt{1+u_x^2}}$$



and thus

$$-T\cos\theta_0 + T\cos\theta_1 = 0 \quad \Rightarrow \quad \left. \frac{T}{\sqrt{1+u_x^2}} \right|_{x_0}^{x_1} = 0$$

$$-T\sin\theta_0 + T\sin\theta_1 = \int_{x_0}^{x_1} \rho\, u_{tt}\, dx \quad \Rightarrow \quad \left. \frac{T\, u_x}{\sqrt{1+u_x^2}} \right|_{x_0}^{x_1} = \int_{x_0}^{x_1} \rho\, u_{tt}\, dx$$

Vibrating String

Assuming that the motion is small, i.e.

$$\sqrt{1+u_x^2} = 1 + \frac{1}{2}u_x^2 + \ldots \approx 1$$

the first equation leads to:

$$\left.\frac{T}{\sqrt{1+u_x^2}}\right|_{x_0}^{x_1} = 0 \qquad \Rightarrow \qquad T\big|_{x=x_0} \approx T\big|_{x=x1}$$

i.e., the tension T is constant along the string.

Vibrating String

Assuming that the motion is small, i.e.

$$\sqrt{1+u_x^2} = 1 + \frac{1}{2}u_x^2 + \ldots \approx 1$$

the second equation leads to:

$$\left. \frac{T\, u_x}{\sqrt{1+u_x^2}} \right|_{x_0}^{x_1} = \int_{x_0}^{x_1} \rho\, u_{tt}\, dx$$

$$\Rightarrow \quad T\, u_x(x_1,t) - T\, u_x(x_0,t) = \int_{x_0}^{x_1} \frac{\partial}{\partial x}(T\, u_x)\, dx \approx \int_{x_0}^{x_1} \rho\, u_{tt}\, dx$$

This leads us to a second order PDE: $(T\, u_x)_x = \rho\, u_{tt}$

Vibrating String

Since T is independent of x, we can rewrite the equation as:

$$u_{tt} = c^2\, u_{xx} \qquad \text{with} \qquad c = \sqrt{\frac{T}{\rho}}$$

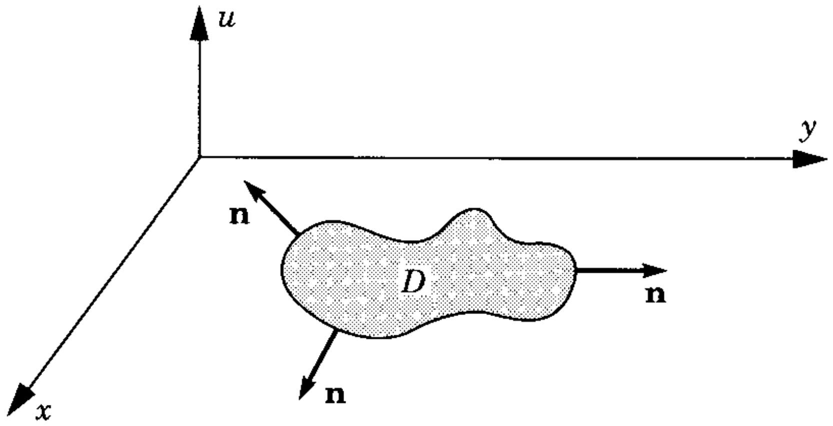
This is the *wave equation.*

We will see in Lecture 2 that c is the wave speed.

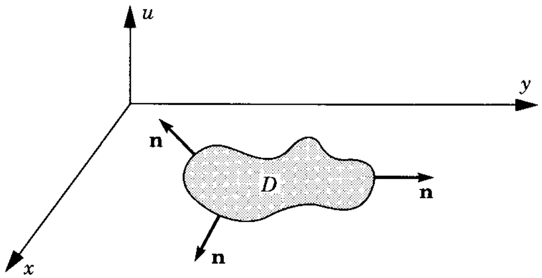
Vibrating Drumhead

The two-dimensional version of a string is an elastic, flexible membrane stretched over a frame (a homogeneous drumhead).

Say the frame lies in the x-y plane, with $u(x, y, t)$ the vertical displacement (there is no horizontal motion).



Vibrating Drumhead



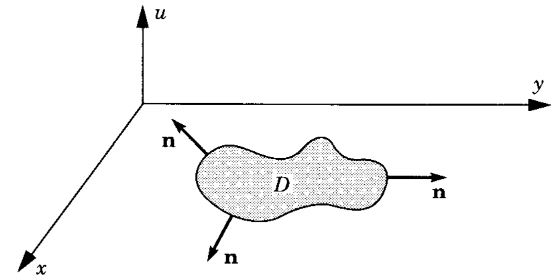
The horizontal components of Newton's law again give constant tension T.

Let D be any domain in the x-y plane (e.g., a circle or a rectangle). Let bdy D be its boundary curve. The vertical component gives (like 1D string case):

$$F = \int_{\text{bdy } D} T \frac{\partial u}{\partial n}\, ds = \iint_D \rho u_{tt}\, dx\, dy = ma,$$

where $\partial u/\partial n = \mathbf{n} \cdot \nabla u$ is the directional derivative in the outward normal direction, $\mathbf{n}$ being the unit outward normal vector on bdy D.

Vibrating Drumhead



Using the Gauss theorem, we get

$$\iint_D \nabla \cdot (T\nabla u)\,dx\,dy = \iint_D \rho u_{tt}\,dx\,dy.$$

Since D is arbitrary, we have $\rho u_{tt} = \nabla \cdot (T\nabla u)$.

Since T is constant, we eventually get:

$$u_{tt} = c^2\nabla \cdot (\nabla u) \equiv c^2(u_{xx} + u_{yy})$$

where $c = \sqrt{T/\rho}$ as before: this is the *2D wave equation*.

The operator $\nabla \cdot (\nabla u) = \text{div grad}\, u = u_{xx} + u_{yy}$ is known as the 2D *Laplacian.*

Wave Equation in Three Dimensions

The pattern is now clear. Simple three-dimensional vibrations obey the equation:

$$u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz}).$$

The operator $\mathscr{L} = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial/\partial z^2$ is called the *three-dimensional Laplacian* operator, usually denoted by Δ or ∇^2.

Physical examples described by the 3D wave equation or a variation of it include the vibrations of an elastic solid, sound waves in air, electromagnetic waves, linearized supersonic airflow, free mesons in nuclear physics, seismic waves propagating through the earth…

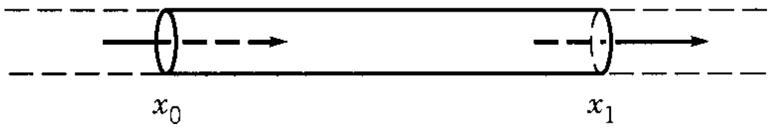
Diffusion

Imagine a motionless liquid filling a straight pipe and a chemical dye, which is diffusing through the liquid.

The dye moves from regions of higher concentration to regions of lower concentration. The rate of motion is proportional to the concentration gradient. (This is known as Fick's law of diffusion.)

Let $u(x, t)$ be the concentration (mass per unit length) of the dye at position x of the pipe at time t.

Diffusion



In the section of pipe from x_0 to x_1, the mass of dye is

$$M(t) = \int_{x_0}^{x_1} u(x,t)\,dx, \quad \text{so} \quad \frac{dM}{dt} = \int_{x_0}^{x_1} u_t(x,t)\,dx.$$

The mass in this section of the pipe cannot change except by flowing in or out at its ends.

By Fick's law:

$$\frac{dM}{dt} = \text{flow in} - \text{flow out} = ku_x(x_1,t) - ku_x(x_0,t),$$

where k is a proportionality constant.

Diffusion

We thus have

$$\int_{x_0}^{x_1} u_t(x,t)dx = ku_x(x_1,t) - ku_x(x_0,t) = \int_{x_0}^{x_1} \frac{\partial}{\partial x}(ku_x)dx$$

leading to

$$u_t = k\, u_{xx}$$

which is the diffusion equation.

Diffusion in Three Dimensions

In three dimensions we have

$$\iiint_D u_t \, dx\, dy\, dz = \iint_{\text{bdy } D} k(\mathbf{n} \cdot \nabla u)\, dS,$$

where D is any solid domain and bdy D is its bounding surface.

By the divergence theorem, we get the *three-dimensional diffusion equation*

$$u_t = k\left(u_{xx} + u_{yy} + u_{zz}\right) = k\,\Delta u$$

Diffusion in Three Dimensions

If there is an external source (or a "sink") of the dye, and if the rate k of diffusion is a variable, we get the more general inhomogeneous equation

$$u_t = \nabla \cdot (k \nabla u) + f(x, t).$$

The same equation describes the conduction of heat, brownian motion, diffusion models of population dynamics, and many other phenomena.

Laplace Equation

Consider the wave and the diffusion equations in a situation where the physical state does not change with time.

Then $u_t = u_{tt} = 0$ and *both* the wave and the diffusion equations reduce to

$$\Delta u = u_{xx} + u_{yy} + u_{zz} = 0.$$

This is called the Laplace equation. Its solutions are called *harmonic functions.*

Take-home messages

- General solutions of PDEs involve arbitrary functions ; well-posed problems require the prescription of initial and/or boundary conditions.
- Simple first-order linear PDEs can be recasted as ODEs; their solution is constant along characteristic curves (for homogeneous PDE).
- The wave, diffusion and Laplace equations are representative of the three main families of second-order PDEs: hyperbolic, parabolic and elliptic.

Next lecture

Initial and boundary conditions

Well-posed problems

Classification of PDEs