Lecture 2 Well-posed problems and classification of PDEs

Mathématiques appliquées (MATH0504-1)

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+ Linear algebra (Classes 11, 12, 13)

Learning objectives of this lecture

① Understand the notion of "well-posed" problem, together with the concepts of boundary and/or initial conditions

② Recognize the main families of

- 2^{nd}-order PDEs
- **systems of 1^{st}-order PDEs**

Outline

Initial and boundary conditions

Well-posed problems

Types of second-order PDEs

Types of systems of first-order PDEs

Reminder

What is a PDE? What is the order of a PDE?

A PDE is an identity that relates

- independent variables (e.g. x, y, t …)
- to a dependent variable u, and its partial derivatives.

We will often denote the derivatives by subscripts, thus e.g. $u_x = \partial u / \partial x$.

The *order* of a PDE is the order of the highest derivative which appears in the equation:

- E.g. 1st order: $F(x, y, u, u_x, u_y) = 0$
- E.g. 2nd order: $F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0.$

What is a linear PDE? When is it homogeneous?

Let us write the PDE in the form $\mathscr{L}(u) = g$, where

- $\mathscr{L}$ is an operator
- and g is a function of the independent variables (or zero).

We will somehow generalize this later in this class

A PDE is linear if

$$\mathscr{L}(u+v) = \mathscr{L}(u) + \mathscr{L}(v) \text{ and } \mathscr{L}(c\,u) = c\,\mathscr{L}(u)$$

for any functions u and v, and any constant c.

Besides, it is homogeneous if $g = 0$, i.e. $\mathscr{L}(u) = 0$.

1^{st} order linear PDEs can be reformulated as ODEs; their solution is constant along characteristic curves

Consider the 1^{st} order PDE

$$a(x, y)\, u_x + b(x, y)\, u_y = 0$$

where $a(x, y)$ and $b(x, y)$ are not both equal to zero.

It expresses actually a directional derivative of u.

Hence, solving the PDE reduces to solving the ODE:

$$dy \,/\, dx = b(x, y) \,/\, a(x, y)$$

and the solution of the PDE is constant along the solution curves of this ODE, referred to as characteristic curves.

For a 1st order linear PDE with constant coefficients, the characteristic curves are straight lines

Consider the 1st order PDE

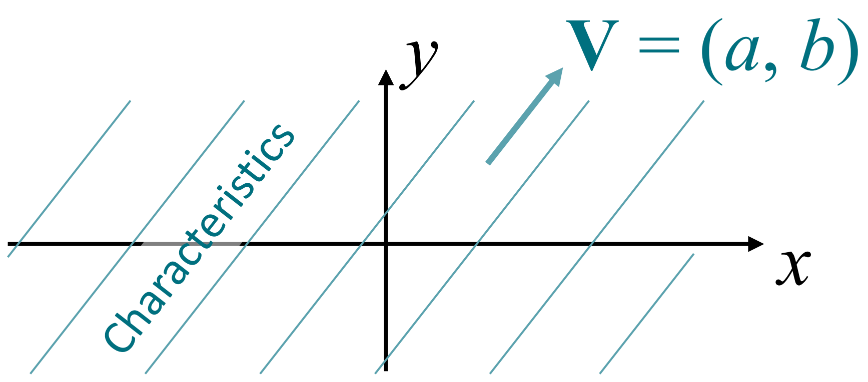
$$a\,u_x + b\,u_y = 0$$

where a and b are not both equal to zero.

The general solution of this PDE writes:

$$u(x, y) = f(\,b\,x - a\,y\,)$$

with f <u>any function</u> of one variable.



Need for initial / boundary conditions!

Paradigmatic PDEs

Simple transport

$$u_t + c\, u_x = 0$$

Wave equation

$$u_{tt} = c^2 \left(u_{xx} + u_{yy} + u_{zz} \right) = c^2 \, \Delta u$$

Diffusion equation

$$u_t = k \left(u_{xx} + u_{yy} + u_{zz} \right) = k \, \Delta u$$

Laplace equation

$$u_{xx} + u_{yy} + u_{zz} = \Delta u = 0$$

1 – Initial and boundary conditions, and the concept of well-posed problems

Initial and boundary conditions

General PDE solutions involve arbitrary functions: to single out one solution we need auxiliary conditions.

For PDEs describing physical phenomena these conditions are motivated by physics and take the form of initial or boundary conditions:

- an *initial condition* specifies the physical state at a particular time t_0.
- a *boundary condition* specifies the specifies the physical state on the boundary of the domain D in which the PDE is valid.

Initial conditions

For the diffusion equation $u_t = k\,\Delta u$,
the initial condition is

$$u(\mathbf{x}, t_0) = \phi(\mathbf{x}),$$

E.g. initial temperature, initial concentration ...

where $\phi(\mathbf{x}) = \phi(x, y, z)$ is a given function.

For the wave equation $u_{tt} = c^2\,\Delta u$,
a pair of initial conditions is needed:

This will be proven later in the course.

$$u(\mathbf{x},\ t_0) = \phi(\mathbf{x}) \quad \text{and} \quad \frac{\partial u}{\partial t}(\mathbf{x},\ t_0) = \psi(\mathbf{x}),$$

where $\phi(\mathbf{x})$ is the initial position and $\psi(\mathbf{x})$ is the initial velocity.

Boundary conditions

The three most common types of boundary conditions are:

E.g. violin string

- (D) u is specified ("*Dirichlet* condition")

E.g. string free to move transversally

- (N) the normal derivative $\partial u / \partial n$ is specified ("*Neumann* condition")

E.g. string attached to a spring

- (R) $\partial u / \partial n + a\, u$ is specified ("*Robin* condition")

where a is a given function of x, y, z, and t.

Each is to hold for all t
and for some $\mathbf{x} = (x, y, z)$ belonging to bdy D.

Boundary conditions (cont'd)

Usually we write (D), (N), and (R) as equations.

For instance, (N) is written as the equation

$$\partial u / \partial n = g(x, t)$$

where g is a given function that could be called the boundary data.

Any of these boundary conditions is called *homogeneous* if the specified function equals zero. Otherwise it is called *inhomogeneous*.

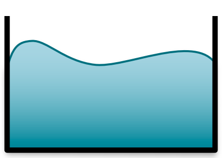
Initial and boundary conditions

We will come back later on which initial and/or boundary conditions must be specified to set up a problem that has a unique solution.

This depends on the PDE being considered, and can be analyzed mathematically.

Some PDEs are posed in an *unbounded* domain D. In that case conditions "at infinity" are needed.

"Jump" conditions apply when the domain is made of two parts, such as two media for instance.



E.g. waves at the air-water interface

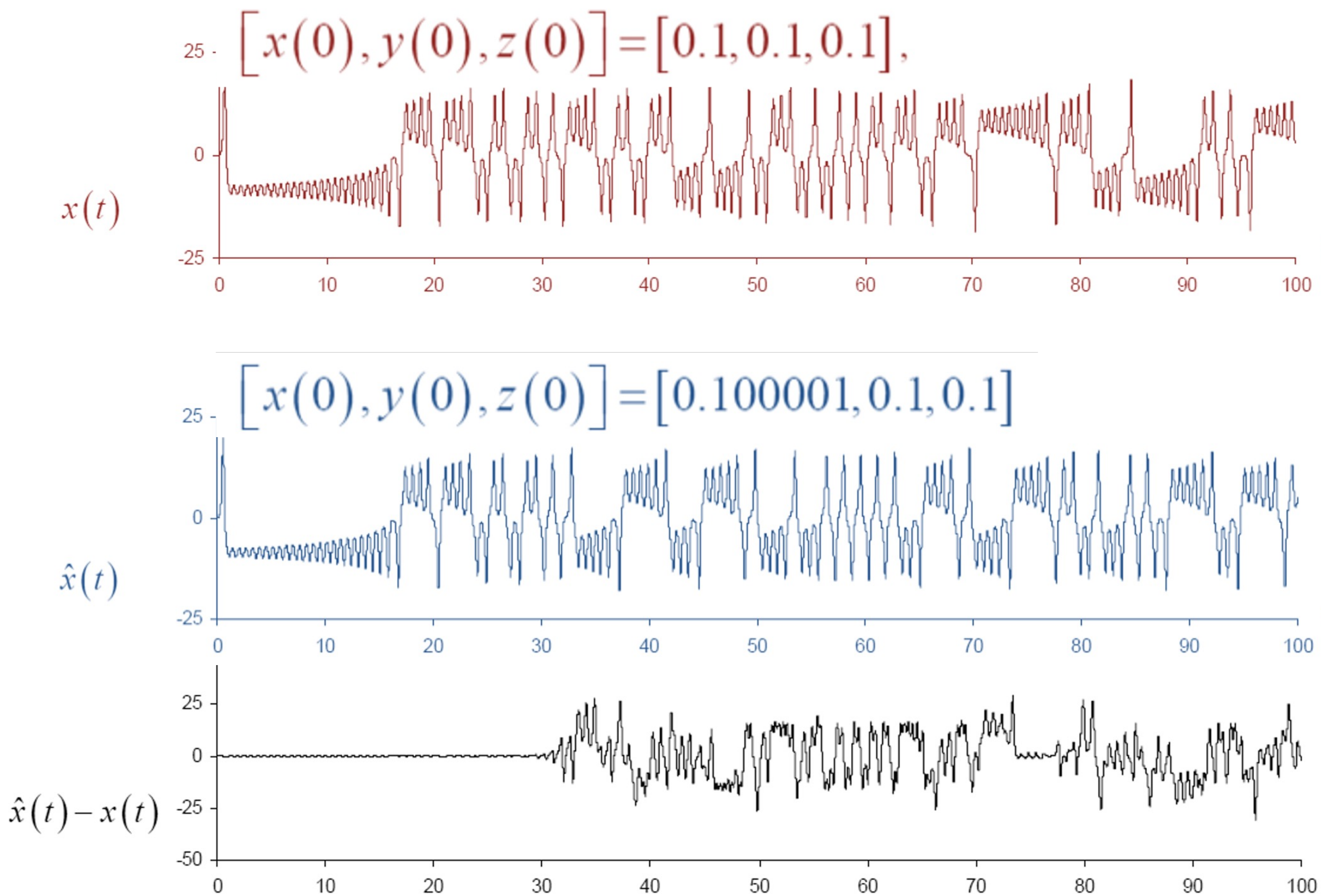
Well-posed problems

Well-posed problems consist of a PDE in a domain with a set of initial and/or boundary conditions (or other auxiliary conditions) that enjoy the following properties:

This is crucial because you can never measure the input data with perfect precision; but only up to some level of accuracy

1. *existence:* there exists at least one solution $u(x, t)$ satisfying all these conditions.
2. *uniqueness:* there is at most one solution.
3. *stability:* the unique solution $u(x, t)$ depends in a stable manner on the data of the problem. This means that if the data are changed a little, the corresponding solution changes only a little.

An example based on ODEs



(Pope 2000)

Time series from the resolution of the so-called Lorenz equations for two slightly different initial conditions [$x(0)$, $y(0)$, $z(0)$]

Outline

Initial and boundary conditions

Well-posed problems

Types of second-order PDEs

Types of systems of first-order PDEs

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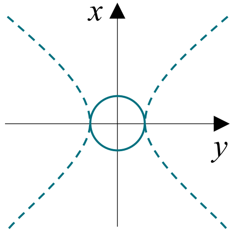
2 – Types of second-order equations

Types of second order equations

Laplace, wave, and diffusion equations are in some sense typical among all second-order PDEs.

However, these three equations are **radically different** from each other, in terms of represented physics, analytical features and numerical schemes.



It is natural that the Laplace equation $u_{xx} + u_{yy} = 0$ and the wave equation $u_{xx} - u_{yy} = 0$ should have very different properties: after all, the *algebraic* equation $x^2 + y^2 = 1$ represents a circle, whereas the equation $x^2 - y^2 = 1$ represents a hyperbola. The parabola is somehow in between.

Types of Second Order Equations

Let's consider the second order PDE in two variables

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0.$$

Theorem 1. By a linear transformation of the independent variables, the equation can be reduced to one of three forms:

(i) *Elliptic case:* If $a_{12}^2 < a_{11}a_{22}$, it is reducible to

$$u_{\xi\xi} + u_{\eta\eta} + \cdots = 0$$

(where $\cdots$ denotes terms of order 1 or 0)

Types of Second Order Equations

(ii) *Hyperbolic* *case:* If $a_{12}^2 > a_{11}a_{22}$, it is reducible to

$$u_{\xi\xi} - u_{\eta\eta} + \cdots = 0$$

(iii) *Parabolic* *case*: if $a_{12}^2 = a_{11}a_{22}$, it is reducible to

$$u_{\xi\xi} + \cdots = 0$$

(unless $a_{11} = a_{12} = a_{22} = 0 \longrightarrow$ 1st order PDE)

We will come back to this classification (and a generalization) later in the course.

Geometric analogy

The key quantity that determines the type of such a PDE is its discriminant:

$$\Delta = (2a_{12})^2 - 4\,\mathrm{a}_{11}\, a_{22}\ .$$

This reminds the discriminant of a quadratic equation

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + a_1 x + a_2 y + a_0 = 0$$

whose solutions trace out a plane curve.

The discriminant fixes its geometric type:

(i) *an ellipse:* If $\Delta < 0$

(ii) *a hyperbola:* If $\Delta > 0$

(iii) *a parabola*: If $\Delta = 0$

3 – Types of systems of first-order PDEs

i. 1^{st} order quasi-linear PDEs
ii. Introductory example
iii. General theory
iv. Application to a simple example

Here, we focus on systems of 1st order quasi-linear PDEs, with two independent variables

The motivation for studying systems of quasi-linear 1st order PDEs is twofold:

- a broad range of processes in engineering may be described by a set of 1st order PDEs;
- some higher order PDEs may be transformed into a system of 1st order PDEs.

Definition A quasi-linear PDE is a PDE in which the derivatives of highest order with respect to each independent variable appear linearly.

Quasi-linear higher order PDEs may be transformed into a system of 1st order PDEs

Let us consider as an example this 2nd order PDE:

$$u_x u_{xx} + u_y^3 u_{xy} + (\tan u) u_{yy} = f(u, u_x, u_y)$$

Define $p = u_x$ and $q = u_y$, so that the original PDE is equivalent to this system of 1st order PDEs:

$$\begin{cases} p\, p_x + q^3 (p_y + q_x)/2 + (\tan u) q_y = f(u, p, q) \\ q_x - p_y = 0 \\ u_x - p = 0 \end{cases}$$

Aim: combine the equations so that you end up with a set of ODEs, instead of PDEs

Let us consider now this simpler example:

$$u_{tt} - \varepsilon\, a^2\, u_{xx} = b$$

where, in general, a and b can be functions of x, t, u_x and u_t (not u). Notation ε refers to $+1$ or -1.

The wave and Laplace equations are particular cases.

Defining $p = u_x$ and $q = u_t$, the 2nd order PDE is equivalent to this system of 1st order PDEs:

$$\begin{cases} q_t - \varepsilon \left[a(x,t,p,q) \right]^2 p_x = b(x,t,p,q) \\ p_t - q_x = 0 \end{cases}$$

Aim: combine the equations so that you end up with a set of ODEs, instead of PDEs

$$\begin{cases} q_t - \varepsilon\left[a(x,t,p,q)\right]^2 p_x = b(x,t,p,q) & (1) \\ p_t - q_x = 0 & (2) \end{cases}$$

Linearly combining Eqs (1) and (2):

$$\sigma\,(1) + \lambda\,(2),$$

with σ and λ coefficients to be determined, leads to:

$$\sigma\,\partial_t q - \lambda\,\partial_x q - \sigma\varepsilon\left[a(x,t,p,q)\right]^2 \partial_x p + \lambda\,\partial_t p = \sigma b(x,t,p,q)$$

Aim: combine the equations so that you end up with a set of ODEs, instead of PDEs

$$\sigma\, \partial_t q - \lambda\, \partial_x q - \sigma\varepsilon \left[a\left(x, t, p, q\right) \right]^2 \partial_x p$$
$$+ \lambda\, \partial_t p = \sigma b\left(x, t, p, q\right)$$

or, by re-arranging the terms,

$$\underbrace{\left[\sigma\, \partial_t - \lambda\, \partial_x\right]}_{\sqrt{\sigma^2 + \lambda^2}\ \partial_{s_1}} q + \underbrace{\left[\lambda\, \partial_t - \sigma\, \varepsilon\, a^2\, \partial_x\right]}_{\sqrt{\lambda^2 + \sigma^2 a^4}\ \partial_{s_2}} p = \sigma b$$

curvilinear coordinates, with slopes

$$\frac{dx}{dt} = \ell_1 = -\frac{\lambda}{\sigma} \qquad \frac{dx}{dt} = \ell_2 = -\frac{\sigma\, \varepsilon\, a^2}{\lambda}$$

Aim: combine the equations so that you end up with a set of ODEs, instead of PDEs

To obtain ODEs (instead of PDEs), the derivation operators in the two terms should be the same (i.e. the slopes ℓ_1 and ℓ_2 of the curvilinear coordinates s_1 and s_2 should be equal):

$$\ell_1 = \ell_2 = \ell \qquad \Leftrightarrow \qquad -\frac{\lambda}{\sigma} = -\frac{\sigma\,\varepsilon\,a^2}{\lambda} = \ell$$

$$\Leftrightarrow \qquad \begin{pmatrix} \ell & 1 \\ \varepsilon\,a^2 & \ell \end{pmatrix} \begin{pmatrix} \sigma \\ \lambda \end{pmatrix} = 0$$

This leads to the compatibility condition $\ell^2 = \varepsilon\,a^2$.

Case 1: assume $\varepsilon = +1$ (and $a > 0$)

The compatibility equation $\ell^2 = a^2$ has two **real solutions**: $\ell = a$ and $\ell = -a$.

Hence, the considered system of two PDEs has two independent families of characteristic curves.

By definition, such a system is called **hyperbolic**.

Note that the slope of the characteristic curves depends only on a, the coefficient of the derivatives of highest order, not on b.

In other words, b does not influence the PDE type.

The slope of the characteristics are eigenvalues …

The considered system of 1st order PDEs

$$\begin{cases} q_t - \varepsilon\left[a\left(x,t,p,q\right)\right]^2 p_x = b\left(x,t,p,q\right) \\ p_t - q_x = 0 \end{cases}$$

may be written in matrix form:

$$\begin{pmatrix} p \\ q \end{pmatrix}_t + \underbrace{\begin{pmatrix} 0 & -1 \\ -\varepsilon\, a^2 & 0 \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} p \\ q \end{pmatrix}_x = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

Note that the eigenvalues of matrix $\mathbf{A}$ are nothing but the slopes of the characteristics: $\ell = \pm a$.

Case 2: assume $\varepsilon = -1$ (and $a > 0$)

The compatibility equation $\ell^2 = -a^2$ has two **complex solutions**: $\ell = i\,a$ and $\ell = -i\,a$.

Hence, the considered system of two PDEs has **no real families** of characteristic curves.

By definition, such a system is called **elliptic**.

Again, note that the solutions of the compatibility equation depend only on a, the coefficient of the derivatives of highest order, not on b.

Here also, b does not influence the type of PDE.

More general theory of characteristics for a system of 1st order PDEs

Consider now the most general system of 1st order PDEs, with N dependent variables / unknowns:

$$\mathbf{u}_t + \mathbf{A}(x, t, \mathbf{u})\, \mathbf{u}_x = \mathbf{h}(x, t, \mathbf{u})$$

with $\mathbf{u}$ the vector of N unknown functions,
$\mathbf{A}$ a N by N matrix and $\mathbf{h}$ a vector of dimension N.

Any PDE i of the system may be written as:

$$\partial_t u_i + \sum_{j=1}^{N} A_{ij} \partial_x u_j - h_i = 0$$

More general theory of characteristics for a system of 1st order PDEs

Let us look for a linear combination of the PDEs of the system:

$$\sum_{i=1}^{N} \sigma_i \, \partial_t u_i + \sum_{i=1}^{N} \sigma_i \sum_{j=1}^{N} A_{ij} \partial_x u_j - \sum_{i=1}^{N} \sigma_i \, h_i = 0$$

where σ_i are coefficients to be determined.

Using Kroenecker delta δ_{ij}, the equations write:

$$\sum_{j=1}^{N} \left[\left(\sum_{i=1}^{N} \sigma_i \, \delta_{ij} \right) \partial_t + \left(\sum_{i=1}^{N} \sigma_i A_{ij} \right) \partial_x \right] u_j = \sum_{i=1}^{N} \sigma_i \, h_i$$

More general theory of characteristics for a system of 1st order PDEs

The terms in the square brackets are all directional derivatives, which could be written in characteristic form, as follows:

$$\sum_{j=1}^{N}\left[\left(\sum_{i=1}^{N}\sigma_i\,\delta_{ij}\right)\partial_t+\left(\sum_{i=1}^{N}\sigma_i A_{ij}\right)\partial_x\right]u_j=\sum_{i=1}^{N}\sigma_i\,h_i$$

$$\sigma_j\left(\partial_t+\ell\,\partial_x\right)=\sigma_j\sqrt{1+\ell^2}\;\partial_s$$

where ℓ is the slope of the characteristic curves

… provided that compatibility conditions are verified!

More general theory of characteristics for a system of 1st order PDEs

The following algebraic equations need to be satisfied, for all j:

$$\frac{\sum_{i=1}^{N} \sigma_i A_{ij}}{\sum_{i=1}^{N} \sigma_i \, \delta_{ij}} = \ell \quad \Rightarrow \quad \sum_{i=1}^{N} \sigma_i A_{ij} - \ell \sum_{i=1}^{N} \sigma_i \, \delta_{ij} = 0$$

Identity matrix

$$\Rightarrow \quad \sum_{i=1}^{N} \left[\left(A^{\mathrm{T}} \right)_{ji} - \ell \delta_{ij} \right] \sigma_i = 0 \quad \Rightarrow \quad \left(\mathbf{A}^{\mathrm{T}} - \ell \, \mathbf{I} \right) \boldsymbol{\sigma} = 0$$

⇨ Compatibility condition: $\det(\mathbf{A}^{\mathrm{T}} - \ell \, \mathbf{I}) = 0.$

More general theory of characteristics for a system of 1st order PDEs

From the compatibility condition $\det(\mathbf{A}^{\mathrm{T}} - \ell\,\mathbf{I}) = 0$, the N possible characteristic slopes ℓ_k of a system of 1st order PDEs are the **eigenvalues of matrix $\mathbf{A}$**.

If all eigenvalues of $\mathbf{A}$ are **real** (and corresponding eigenvectors are independent, i.e. $\mathbf{A}$ is diagonalisable), then the system of PDEs is **hyperbolic**.

If all eigenvalues of $\mathbf{A}$ are **complex** (and corresponding eigenvectors are independent, i.e. $\mathbf{A}$ is diagonalisable), then the system of PDEs is **elliptic**.

More general theory of characteristics for a system of 1st order PDEs

The case where the eigenvectors are not independent, i.e. $\mathbf{A}$ is not diagonalisable, often corresponds to **parabolic** systems of PDEs.

If some eigenvalues of $\mathbf{A}$ are real and others are complex, then the system of PDEs is **hybrid**.

A simple example

We consider the case of the wave equation:

$$u_{tt} = c^2 \, u_{xx}$$

Let us define the following new unknowns:

$$q = u_t \qquad \text{and} \qquad p = u_x$$

Then, we have the *system* of 1st order PDEs:

$$q_t - c^2 \, p_x = 0$$

$$p_t - q_x = 0$$

A simple example

The system may be written in matrix form as

$$\begin{pmatrix} q \\ p \end{pmatrix}_t + \begin{pmatrix} 0 & -c^2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}_x = 0$$

The eigenvalues of the matrix are $\lambda = \pm c$, which correspond to the slopes of the characteristics.

The system of two 1st order PDEs has two families of characteristics, just like 2nd order wave equation.

Take-home messages

By definition, the solution of a "well-posed" problem *(i)* exists, *(ii)* is unique and *(iii)* is stable. This is achieved by prescribing suitable auxiliary conditions, such as initial and boundary conditions.

Depending on the sign of the coefficients of the highest derivatives, second-order PDEs are either *(i)* elliptic, *(ii)* hyperbolic, or *(iii)* parabolic.

For a system of 1st-order PDEs, the type of the system depends on the eigenvalues of the matrix.

The various types of PDEs have radically different properties → **next class**: the wave equation.

What’s next?

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Répartition entre les salles TD / TP

B5b S24 : TD1

B5b S34 : TD2

B5b S26 : TP1

B5b S28 : TP2

B5b S30 : TP3