LIÈGE université
Sciences Appliquées

Lecture 5 Von Neumann stability analysis

Mathématiques appliquées (MATH0504-1)

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Learning objective of this lecture

Generalize the procedure used in the last lecture to determine the stability of finite difference schemes applied to the diffusion equation

Outline

Error modes and Fourier decomposition

Von Neuman stability analysis

1 – Error modes and Fourier decomposition

Introduction

We consider a linear PDE whose solution is approximated with a time-stepping finite difference scheme.

The discretizations in space and time are assumed to be on regular grids x_j (with index j), and t_n (with index n), respectively, with

$$x_j = j\Delta x, \quad t_n = n\Delta t$$

The approximation of the solution u on this grid is denoted by

$$u(x_j, t_n) \sim u_j^n$$

Evolution of the error

Let us consider that at time step 0, an error

$$\varepsilon(x, t_0)$$

appears in the solution and let us assume that this error (function of the spatial variable x) is both integrable (→ Fourier transform) and square integrable (→ Parseval's theorem).

Evolution of the error

Therefore, approximation of the solution u can be expressed as the linear combination of the exact solution and any error, such as

$$u_j^n = u(x_j, t_n) + \varepsilon(x_j, t_n)\ .$$

Since the PDE is linear, and the solution already verifies the equation, the time evolution of the initial error will be the same as the one of the solution for the homogeneous problem:

$$\mathrm{L}\left[u(x_j, t_n) + \varepsilon(x_j, t_n)\right] = \mathrm{L}\underbrace{\left[u(x_j, t_n)\right]}_{=0} + \mathrm{L}\left[\varepsilon(x_j, t_n)\right]$$

= 0 → Verifies the PDE !

Example

For example, consider the transport equation

$$u_t + a u_x = f$$

with the finite difference approximation

$$u_t \approx \frac{u_j^{n+1} - u_j^n}{\Delta t}, \quad u_x \approx \frac{u_{j+1}^n - u_j^n}{\Delta x}$$

which leads to the difference equation:

$$u_j^{n+1} - (1 + a\frac{\Delta t}{\Delta x})u_j^n + a\frac{\Delta t}{\Delta x}u_{j+1}^n = \Delta t f_j^n$$

Example

At step $n+1$ the error satisfies

$$[u(x_j,t_{n+1})+\varepsilon(x_j,t_{n+1})]-(1+a\frac{\Delta t}{\Delta x})[u(x_j,t_n)+\varepsilon(x_j,t_n)]+$$

$$a\frac{\Delta t}{\Delta x}[u(x_{j+1},t_n)+\varepsilon(x_{j+1},t_n)]=\Delta t\ f_j^n$$

i.e.

$$\varepsilon(x_j,t_{n+1})-(1+a\frac{\Delta t}{\Delta x})\varepsilon(x_j,t_n)+a\frac{\Delta t}{\Delta x}\varepsilon(x_{j+1},t_n)=0$$

Stability

The finite difference scheme is said to be stable if the L^2-norm (in space) of the error does not diverge with time, i.e. if there exists a real constant C such that

$$\lim_{n\to\infty} ||\varepsilon(x,t_n)||^2 \leq C||\varepsilon(x,t_0)||^2$$

Sufficient condition

Using the spatial Fourier decomposition of the error (it exists because the error is integrable)

$$\varepsilon(x,t_n) = \int \hat{\varepsilon}(k,t_n) e^{ikx} dk$$

a sufficient condition for stability is that each mode does not diverge, i.e., that there exists a real constant C such that

$$\lim_{n\to\infty} |\hat{\varepsilon}(k,t_n)|^2 \leq C|\hat{\varepsilon}(k,t_0)|^2, \quad \forall k$$

Sufficient condition

Indeed, using Parseval's theorem:

$$
\begin{aligned}
\lim_{n\to\infty} ||\varepsilon(x,t_n)||^2 &= \lim_{n\to\infty} \int |\hat{\varepsilon}(k,t_n)|^2 dk \\
&\leq C \int |\hat{\varepsilon}(k,t_0)|^2 dk \\
&= C||\varepsilon(x,t_0)||^2
\end{aligned}
$$

(We assume uniform convergence with n of the suite $\hat{\varepsilon}(k,t_n)$ so we can pass the limit under the integration sign.)

Reminder: Parseval's theorem states that the power calculated in the "time" domain is the same as the power calculated in the "frequency" domain.

2 – Von Neumann stability analysis

Stability analysis procedure

By linearity of the equation, each error mode sampled on the spatial grid, defined as

$$\varepsilon_k(x_j, t_n) = \hat{\varepsilon}(k, t_n) e^{ikx_j}$$

can be studied individually. The procedure is thus:

1. Inject $\varepsilon_k(x_j, t_n)$ in the difference equation
2. Simplify the exponentials, which leads to a (linear) recurrence relation for $\hat{\varepsilon}(k, t_n)$ (usually simply denoted by $\hat{\varepsilon}_n$) with coefficients that depend on k
3. The scheme is stable if no mode diverges

Linear recurrence equations

The second order linear recurrence equation

$$a\hat{\varepsilon}_{n+2} + b\hat{\varepsilon}_{n+1} + c\hat{\varepsilon}_n = 0$$

with a, b and c coefficients that do not depend on n, admits solutions of the form

$$\hat{\varepsilon}_n = A\xi_1^n + B\xi_2^n \qquad \text{or} \qquad \hat{\varepsilon}_n = (A + Bn)\xi^n$$

if its characteristic polynomial $a\xi^2 + b\xi + c$ has distinct roots ξ_1 and ξ_2, or degenerate roots ξ, respectively, with A and B two constants that depend on the initial values.

Linear recurrence equations

The roots correspond to the *amplification factors.*

The solution is stable if no amplification factor is larger than one, i.e. for the second order equation:

$$|\xi_1| \leq 1 \ \text{ and } \ |\xi_2| \leq 1, \quad \text{or} \quad |\xi| \leq 1$$

This is exactly what we derived in the *ad hoc* procedure for the finite difference approximation of the diffusion equation in the last lecture.

Take-home messages

This Von Neumann stability analysis generalizes the procedure that we applied in the last lecture for the diffusion equation.

The solution of the recurrence relation for $\hat{\varepsilon}(k, t_n)$ will involve the amplification factors: if the modulus of any of these amplification factors is greater than one, the scheme diverges.

Next, we will apply this general stability analysis to the wave equation.

Take-home messages

- Von Neumann stability verifies that an error made at one time step does not dramatically grow with time
- Von Neumann analysis is a necessary but not sufficient stability criterion
- It only studies stability for
  - infinite domain (does not consider boundary conditions!)
  - linear PDE with constant coefficients