Lecture 5 Approximations of Waves

Mathématiques appliquées (MATH0504-1)

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Learning objectives of this lecture

Understand key differences between the numerical approximations of diffusion and wave equations

Apply von Neumann stability analysis to finite difference approximations of the wave equation

Outline

Approximation of waves

Initial conditions

Stability analysis

1 –Approximations of waves

Finite differences for the wave equation

Let's use centered differences for both terms of the one-dimensional wave equation

$$u_{tt} = c^2 u_{xx}$$

We get

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{(\Delta t)^2} = c^2 \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}$$

This is an explicit scheme, which can be written as

$$u_j^{n+1} = s\left(u_{j+1}^n + u_{j-1}^n\right) + 2(1-s)u_j^n - u_j^{n-1}$$

with $s = c^2(\Delta t)^2/(\Delta x)^2$

Finite differences for the wave equation

The value at the $(n+1)$st time step depends on the *two* previous steps:

$$u_j^{n+1} = s\left(u_{j+1}^n + u_{j-1}^n\right) + 2(1-s)u_j^n - u_j^{n-1}$$

Compare with the diffusion equation $u_t = u_{xx}$ with a forward difference for u_t and a centered difference for u_{xx}, which led to:

$$u_j^{n+1} = s\left(u_{j+1}^n + u_{j-1}^n\right) + (1-2s)u_j^n$$

with $s = \Delta t / (\Delta x)^2$

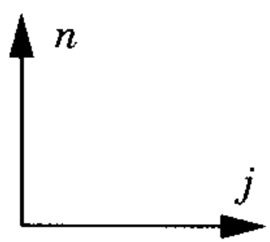
Example: $s = 2$

With $s = 2$

$$u_j^{n+1} = s\left(u_{j+1}^n + u_{j-1}^n\right) + 2(1 - s)u_j^n - u_j^{n-1}$$

the schemes simplifies into

$$u_j^{n+1} = 2\left(u_{j+1}^n + u_{j-1}^n - u_j^n\right) - u_j^{n-1}$$



8	–12	4	–13	–22	13	4	–12	8	$n = 4$
	4	–2	–3	6	–3	–2	4		$n = 3$
		2	1	–2	1	2			$n = 2$
			1	2	1				$n = 1$
			1	2	1				$n = 0$

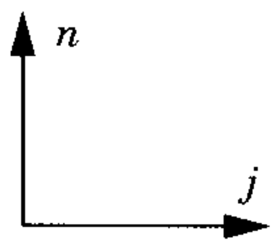
This is clearly unstable!

Example: $s = 1$

For $s = 1$ we have $\Delta x = c\,\Delta t$, which leads to:

$$u_j^{n+1} = u_{j+1}^n + u_{j-1}^n - u_j^{n-1}$$

With the same initial conditions as in the previous example we get:



1	1	0	0	0	0	0	1	1	$n = 4$
	1	1	0	0	0	1	1		$n = 3$
		1	1	0	1	1			$n = 2$
			1	2	1				$n = 1$
			1	2	1				$n = 0$

This is a pretty good approximation of the true solution!

Remember: $u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)]$

2 – Initial conditions

How should we handle initial conditions?

We need to approximate

$$u(x,0) = \phi(x) \qquad \partial u/\partial t(x,0) = \psi(x)$$

To maintain an $O(\Delta t)^2$ truncation error we use a *centered* finite difference in time:

$$u_j^0 = \phi(j\Delta x), \qquad \frac{u_j^1 - u_j^{-1}}{2\Delta t} = \psi(j\Delta x)$$

Using a simpler approximation, the larger error on the initial condition would contaminate the overall solution

How do we handle initial conditions?

For $n = 0$,

$$u_j^{n+1} = s\left(u_{j+1}^n + u_{j-1}^n\right) + 2(1-s)u_j^n - u_j^{n-1}$$

becomes

$$u_j^1 + u_j^{-1} = s\left(u_{j+1}^0 + u_{j-1}^0\right) + 2(1-s)u_j^0$$

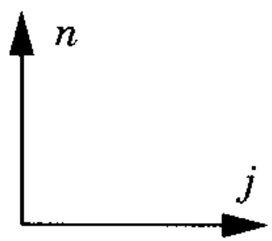
which gives the initial values (since $u_j^{-1} = u_j^1 - 2\Delta t\,\psi_j$)

$$u_j^0 = \phi_j,$$

$$u_j^1 = \frac{s}{2}(\phi_{j+1} + \phi_{j-1}) + (1-s)\phi_j + \psi_j\,\Delta t$$

Example

Consider the same $\phi(x)$ as before, and $\psi(x) \equiv 0$. This leads to



$\frac{1}{2}$	1	$\frac{1}{2}$	0	0	0	0	0	0	0	$\frac{1}{2}$	1	$\frac{1}{2}$	$n = 5$
	$\frac{1}{2}$	1	$\frac{1}{2}$	0	0	0	0	0	$\frac{1}{2}$	1	$\frac{1}{2}$		$n = 4$
		$\frac{1}{2}$	1	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	1	$\frac{1}{2}$			$n = 3$
			$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$				$n = 2$
				$\frac{1}{2}$	1	1	1	$\frac{1}{2}$					$n = 1$
					1	2	1						$n = 0$

This is an even better approximation to the true solution

3 – Stability

Von Neumann analysis

Remember the general procedure:

- separate the variables
- evaluate the amplification factor $\xi(k)$ as a function of the wave number k
- use the *von Neuman* stability condition:

$$|\xi(k)| \leq 1 \qquad \text{for all } k.$$

Von Neumann analysis

Let's plug the separated solution

$$u_j^n = \left(e^{ik\Delta x}\right)^j \left[\xi(k)\right]^n$$

into the difference equation

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{(\Delta t)^2} = c^2 \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}$$

We get

$$\frac{\xi - 2 + \xi^{-1}}{(\Delta t)^2} = c^2 \frac{e^{ik\Delta x} - 2 + e^{-ik\Delta x}}{(\Delta x)^2}$$

$$= c^2 \frac{2\cos(k\Delta x) - 2}{(\Delta x)^2}$$

Von Neumann analysis

Thus: $\xi + \xi^{-1} - 2 = 2s[\cos(k\Delta x) - 1]$

Posing $p = s[\cos(k\Delta x) - 1]$, we get

$$\xi^2 - 2(1+p)\xi + 1 = 0$$

which has the roots

$$\begin{aligned}\xi &= \frac{2(1+p) \pm \sqrt{(2(1+p))^2 - 4}}{2} \\ &= 1 + p \pm \sqrt{(1+p)^2 - 1} \\ &= 1 + p \pm \sqrt{p^2 + 2p}\end{aligned}$$

$$\xi = 1 + p \pm \sqrt{p^2 + 2p}$$

Von Neumann analysis

Note that $p = s[\cos(k\Delta x) - 1] \leq 0$

- If $p < -2$, $p^2 + 2p > 0$ and there are two real roots, one of which is less than -1, which means that $|\xi| > 1$, so that the scheme is unstable
- If $p > -2$, $p^2 + 2p < 0$ and there are two complex conjugate roots $1 + p \pm i\sqrt{-p^2 - 2p}$ which satisfy

  $$|\xi|^2 = (1 + p)^2 - p^2 - 2p = 1$$

  so $\xi = \cos\theta + i\sin\theta$ for some real number θ and the solution oscillates in time
- If $p = -2$, then $\xi = -1$

Von Neumann analysis

Stability thus requires $p = s[\cos(k\Delta x) - 1] \geq -2$ for all k, i.e.

$$s \leq \frac{2}{1 - \cos(k\,\Delta x)} \qquad \text{for all } k$$

And thus

$$s = c^2 \frac{(\Delta t)^2}{(\Delta x)^2} \leq 1$$

Interpretation

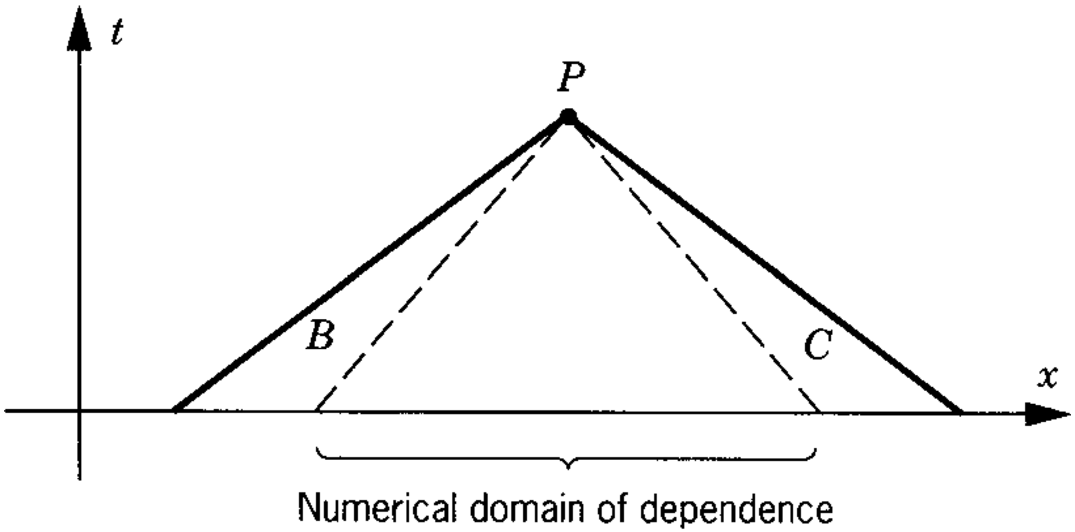
At each time step Δt the values of the numerical solution spread out by one unit Δx

So $\Delta x / \Delta t$ is the *propagation speed of the numerical scheme*

The propagation speed for the exact wave equation is c

So the stability condition $s = c^2 \frac{(\Delta t)^2}{(\Delta x)^2} \leq 1$ requires the numerical propagation speed to be at least as large as the continuous propagation speed

Interpretation



$$c = 1$$

$$\Delta t / \Delta x = 2$$

$$s = 4$$

The computed solution at the point P does not make use of the initial data in the regions B and C as it ought to.

Therefore, the scheme leads to entirely erroneous values of the solution.

Other schemes

- The scheme we just studied uses centered differences in space and time. If we apply it to “singular” initial data, the results are stable but not accurate – better schemes should be used in such cases

$$
\begin{array}{ccccccccccccccl}
1 & & -1 & & 1 & & -1 & & 1 & & -1 & & 1 & & n = 4 \\
 & 1 & & -1 & & 1 & & -1 & & 1 & & & & & n = 3 \\
 & & & & 1 & & -1 & & 1 & & & & & & n = 2 \\
 & & & & & & 1 & & & & & & & & n = 1 \\
 & & & & & & 1 & & & & & & & & n = 0
\end{array}
$$

- There are also implicit schemes for the wave equation (like the Crank– Nicolson scheme). They are less urgently needed here since the stability condition for the explicit scheme does not require the time step to be so much smaller than the spatial step.

Take-home messages

- Approximations of waves can be carried out in a similar way to what was done for diffusions
- The Neumann stability analysis can be performed in exactly the same way
- The stability criterion for the centered difference explicit scheme for the wave equation is less demanding on the time step than for diffusions