LIÈGE université
Sciences Appliquées

Lecture 5b Approximations of waves

Mathématiques appliquées (MATH0504-1)

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Learning objectives of this lecture

Understand key differences between the numerical approximations of diffusion and wave equations

Apply the von Neumann stability analysis to finite difference approximations of the wave equation

Outline

Approximation of waves, and initial conditions

Stability analysis

1 – Approximations of waves

Finite differences for the wave equation

Let's use centered differences for both terms of the one-dimensional wave equation:

$$u_{tt} = c^2 u_{xx}$$

We get

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{(\Delta t)^2} = c^2 \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}$$

This is an explicit scheme, which can be written as

$$u_j^{n+1} = s\left(u_{j+1}^n + u_{j-1}^n\right) + 2(1-s)u_j^n - u_j^{n-1}$$

with $s = c^2\,(\Delta t)^2\,/\,(\Delta x)^2$.

Finite differences for the wave equation

The value at the $(n + 1)^{\text{st}}$ time step depends on the *two* previous steps:

$$u_j^{n+1} = s\left(u_{j+1}^n + u_{j-1}^n\right) + 2(1 - s)u_j^n - u_j^{n-1}$$

Compare with the diffusion equation $u_t = k\, u_{xx}$, with a forward difference for u_t and a centered difference for u_{xx} , which led to:

$$u_j^{n+1} = s\left(u_{j+1}^n + u_{j-1}^n\right) + (1 - 2s)u_j^n$$

with $s = k\,\Delta t \,/\, (\Delta x)^2$.

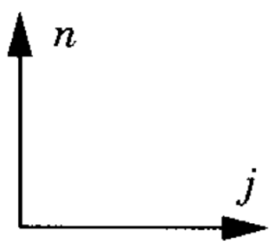
Example 1: Let's set the value of s to 2 ...

With $s = 2$,

$$u_j^{n+1} = s\left(u_{j+1}^n + u_{j-1}^n\right) + 2(1-s)u_j^n - u_j^{n-1}$$

the schemes simplifies into

$$u_j^{n+1} = 2\left(u_{j+1}^n + u_{j-1}^n - u_j^n\right) - u_j^{n-1}$$



8	−12	4	−13	−22	13	4	−12	8	$n = 4$
	4	−2	−3	6	−3	−2	4		$n = 3$
		2	1	−2	1	2			$n = 2$
			1	2	1				$n = 1$
			1	2	1				$n = 0$

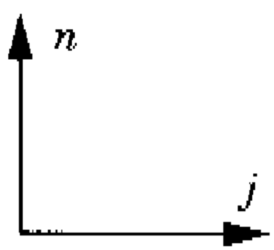
This is clearly unstable!

Example 2: What if the value of s is set to 1 ?

For $s = 1$, we have $\Delta x = c\,\Delta t$, which leads to:

$$u_j^{n+1} = u_{j+1}^{n} + u_{j-1}^{n} - u_j^{n-1}$$

With the same initial conditions as in the previous example, we get:



1	1	0	0	0	0	0	1	1	$n = 4$
	1	1	0	0	0	1	1		$n = 3$
		1	1	0	1	1			$n = 2$
			1	2	1				$n = 1$
			1	2	1				$n = 0$

Seems like good approximation of the true solution!

Remember: $u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)]$

2 – Initial conditions

How should we handle initial conditions?

We need to approximate

$$u(x,0) = \phi(x) \qquad u_t(x,0) = \psi(x)$$

To maintain an $O(\Delta t)^2$ truncation error,
we use a *centered* finite difference in time:

$$u_j^0 = \phi(j\Delta x), \qquad \frac{u_j^1 - u_j^{-1}}{2\Delta t} = \psi(j\Delta x).$$

In contrast, if a simpler approximation is used, the larger error on the initial condition would contaminate the overall solution.

How do we handle initial conditions?

For $n = 0$,

$$u_j^{n+1} = s\left(u_{j+1}^n + u_{j-1}^n\right) + 2(1-s)u_j^n - u_j^{n-1}$$

becomes

$$u_j^1 + u_j^{-1} = s\left(u_{j+1}^0 + u_{j-1}^0\right) + 2(1-s)u_j^0.$$

This gives the initial values (since $u_j^{-1} = u_j^1 - 2\Delta t\,\psi_j$)

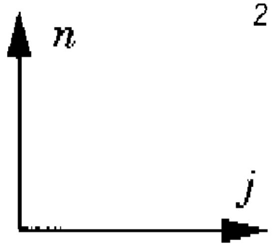
$$u_j^0 = \phi_j,$$

$$u_j^1 = \frac{s}{2}(\phi_{j+1} + \phi_{j-1}) + (1-s)\phi_j + \psi_j\,\Delta t$$

Example

Consider the same $\phi(x)$ as before, and $\psi(x) \equiv 0$.

Assuming again $s = 1$, this leads to:



$$
\begin{array}{ccccccccccccccc}
\frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & & n = 5 \\
 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & & & n = 4 \\
 & & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & & & & n = 3 \\
 & & & \frac{1}{2} & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 1 & \frac{1}{2} & & & & & n = 2 \\
 & & & & \frac{1}{2} & 1 & 1 & 1 & \frac{1}{2} & & & & & & n = 1 \\
 & & & & & 1 & 2 & 1 & & & & & & & n = 0
\end{array}
$$

$$u_j^1 = \frac{s}{2}(\phi_{j+1} + \phi_{j-1})$$

This is an even better approximation of the true solution: $u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)]$

3 – Stability

Von Neumann analysis

Remember the general procedure:

1. inject the error mode in the difference equation;
2. simplify the exponentials, which leads to a recurrence relation;
3. solve the recurrence relation for the amplification factor(s) as a function of the wave number k ;
4. the scheme is stable if no mode diverges.

Von Neumann analysis

We start by injecting the error mode

$$\varepsilon_k(x_j, t_n) = \hat{\varepsilon}(k, t_n)e^{ikx_j} = \hat{\varepsilon}_n(e^{ik\Delta x})^j$$

in the difference equation

$$u_j^{n+1} = s\left(u_{j+1}^n + u_{j-1}^n\right) + 2(1-s)u_j^n - u_j^{n-1}$$

After simplifying the exponentials, we get:

$$\begin{aligned}\hat{\varepsilon}_{n+1} &= s\hat{\varepsilon}_n(e^{ik\Delta x} + e^{-ik\Delta x}) + 2(1-s)\hat{\varepsilon}_n - \hat{\varepsilon}_{n-1} \\ &= 2s\cos(k\Delta x)\hat{\varepsilon}_n + 2(1-s)\hat{\varepsilon}_n - \hat{\varepsilon}_{n-1}\end{aligned}$$

Von Neumann analysis

Posing $p = s\,[\cos(k\Delta x) - 1]$ and adjusting the time index, we get

$$\hat{\varepsilon}_{n+2} - 2(1+p)\hat{\varepsilon}_{n+1} + \hat{\varepsilon}_n = 0$$

The characteristic polynomial of this expression

$$\xi^2 - 2(1+p)\xi + 1 = 0$$

has the following roots:

$$\begin{aligned}
\xi_{1,2} &= \frac{2(1+p) \pm \sqrt{(2(1+p))^2 - 4}}{2} \\
&= 1 + p \pm \sqrt{(1+p)^2 - 1} \\
&= 1 + p \pm \sqrt{p^2 + 2p}
\end{aligned}$$

Von Neumann analysis

The solution of the recurrence equation has the form (warning, n is an exponent here!)

$$\hat{\varepsilon}_n = A\xi_1^n + B\xi_2^n$$

where A and B are two constants that depend on the initial error.

The solution is stable if no amplification factor is larger than one, i.e.

$$|\xi_1| \leq 1 \text{ and } |\xi_2| \leq 1$$

We need to analyse several cases, depending on the sign of the value under the square root in $\xi_{1,2}$.

Von Neumann analysis

Reminder: $p = s[\cos(k\Delta x) - 1] \leq 0$

Case 1: $p < -2$

- $p^2 + 2p > 0$
- there are two real roots: $\xi_{1,2} = 1 + p \pm \sqrt{p^2 + 2p}$
- one of them (ξ_2) is lower than -1
- thus $|\xi_2| > 1$ and the scheme is **unstable**

Von Neumann analysis

Case 2: $p > -2$

- $p^2 + 2p < 0$
- there are two complex roots:

$$\xi_{1,2} = 1 + p \pm \sqrt{p^2 + 2p}$$
$$= 1 + p \pm \sqrt{(-i^2)(p^2 + 2p)} = 1 + p \pm i\sqrt{-(p^2 + 2p)}$$

- so that

$$\left|\xi_{1,2}\right|^2 = (1+p)^2 - \left(p^2 + 2p\right) = 1$$

- hence, the scheme is **stable** (oscillations: wave 👍).

Von Neumann analysis

Case 3: $p = -2$

- $p^2 + 2p = 0$
- roots are **multiple** :

$$\xi_{1,2} = 1 + p \pm \cancel{\sqrt{p^2 + 2p}} = -1$$

- so that the scheme is **‘stable’**
  provided that the initial error is constant in time.

Von Neumann analysis

Stability thus requires $p = s\left[\cos\left(k\Delta x\right) - 1\right] \geq -2$ for all k, i.e.

$$s \leq \frac{-2}{\cos\left(k\Delta x\right) - 1} = \frac{2}{1 - \cos\left(k\Delta x\right)}$$

In the most restrictive case, the smallest value on the right of the inequality is obtained when

$$\cos(k\Delta x) = -1$$

and thus

$$s = \frac{c^2 \Delta t^2}{\Delta x^2} \leq 1$$

Interpretation

At each time step Δt, the values of the numerical solution spread out by one unit Δx. So $\Delta x \,/\, \Delta t$ is the *propagation speed of the numerical scheme.*

The propagation speed *for the exact wave equation* is c.

So the stability condition

$$s = \frac{c^2 \Delta t^2}{\Delta x^2} \leq 1 \qquad \Leftrightarrow \qquad \left(\frac{\Delta x}{\Delta t}\right)^2 \geq c^2$$

requires the numerical propagation speed to be at least as large as the exact wave propagation speed.

Interpretation

$$-1 \le \boxed{\frac{c\Delta t}{\Delta x} \le 1}$$

Stable

$$\frac{c\Delta t}{\Delta x} \le 1$$

$$\frac{c}{\Delta x / \Delta t} \le 1$$

$$c \le \frac{\Delta x}{\Delta t}$$

Physical celerity $\le$ "Numerical celerity"

Unstable

$$\frac{c\Delta t}{\Delta x} > 1$$

$$\frac{c}{\Delta x / \Delta t} > 1$$

$$c > \frac{\Delta x}{\Delta t}$$

Physical celerity $>$ "Numerical celerity"

Interpretation

$$-1 \leq \boxed{\frac{c\Delta t}{\Delta x} \leq 1}$$

Stable

$$\frac{c\Delta t}{\Delta x} \leq 1$$

Physical celerity $\leq$ "Numerical celerity"

$$c\Delta t \leq \Delta x$$

Distance travelled by the physical wave $\leq$ Mesh size

Unstable

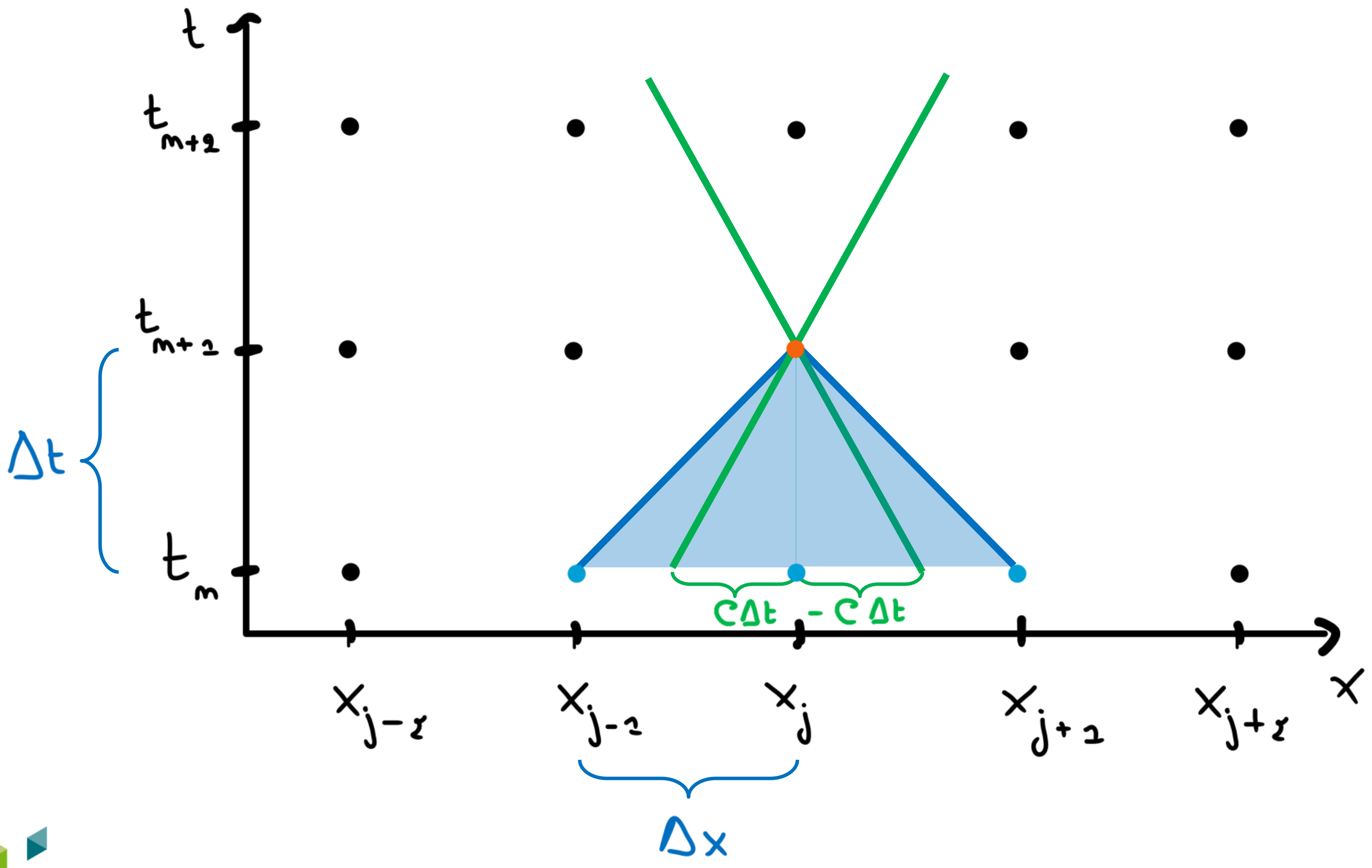
$$\frac{c\Delta t}{\Delta x} > 1$$

Physical celerity $>$ "Numerical celerity"

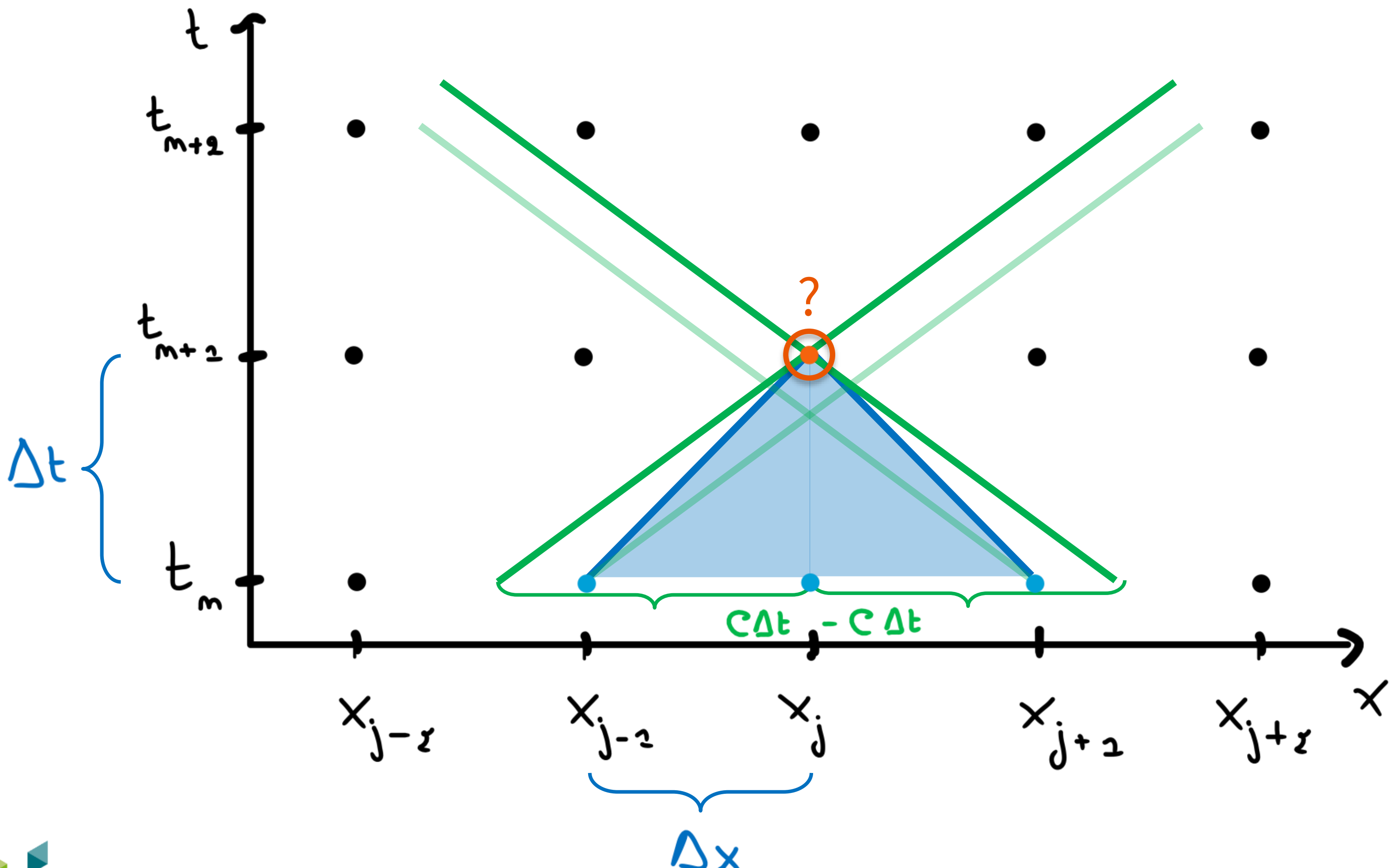
$$c\Delta t > \Delta x$$

Distance travelled by the physical wave $>$ Mesh size

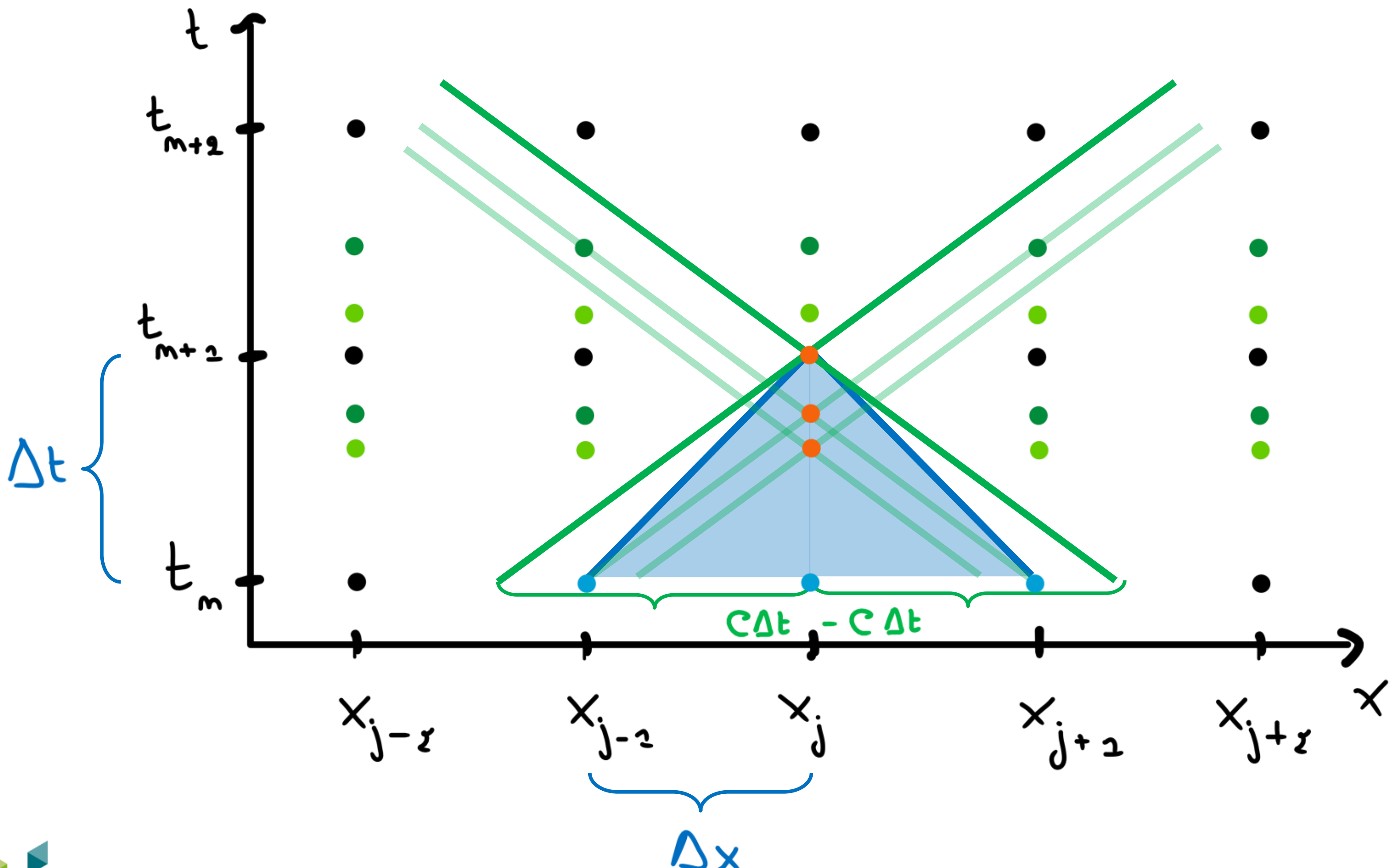
Interpretation



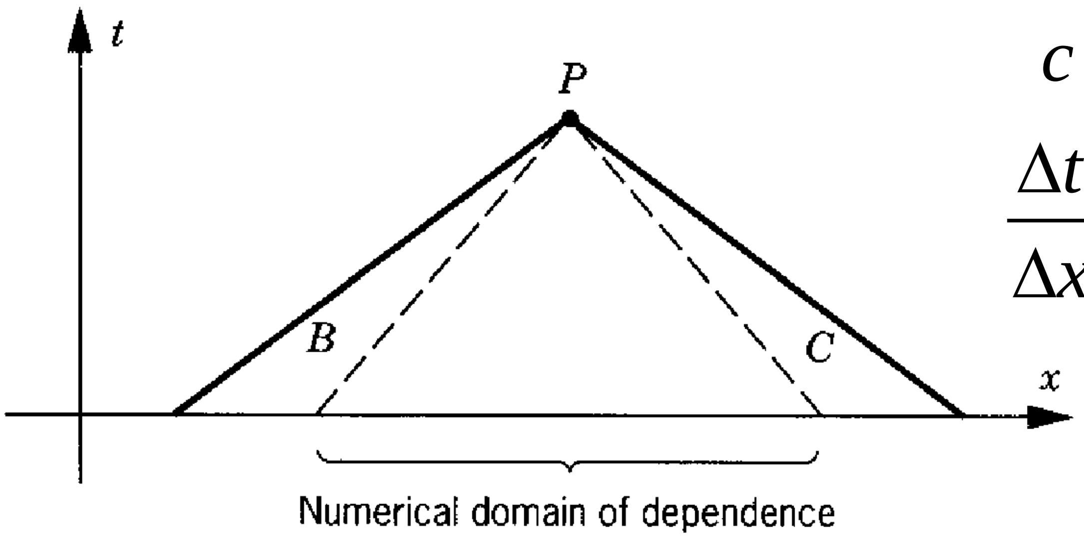
Interpretation



Interpretation



Interpretation



$$\left.\begin{array}{r} c = 1 \\ \dfrac{\Delta t}{\Delta x} = 2 \end{array}\right\} \quad \Rightarrow \quad s = 4$$

The computed solution at the point P does not make use of the initial data in the regions B and C as it ought to.

Therefore, the scheme leads to entirely erroneous values of the solution.

Other schemes

$$
\begin{array}{ccccccccc}
1 & -1 & 1 & -1 & 1 & -1 & 1 & & n = 4 \\
 & 1 & -1 & 1 & -1 & 1 & & & n = 3 \\
 & & 1 & -1 & 1 & & & & n = 2 \\
 & & & 1 & & & & & n = 1 \\
 & & & 1 & & & & & n = 0
\end{array}
$$

The scheme we just studied uses centered differences in space and time. If it is applied to "singular" initial data, the results are stable but not accurate – better schemes should be used in such cases.

Implicit schemes also exist for the wave equation (like the Crank– Nicolson scheme). They are less urgently needed here since the stability condition for the explicit scheme does not require the time step to be so much smaller than the spatial step.

Take-home messages

- Approximations of waves can be carried out in a similar way to what was done for diffusions
- Care in discretizing the initial (and boundary) conditions is essential to get accurate solutions
- The von Neumann stability analysis gives us a stability criterion for the centered difference explicit scheme
- The stability criterion for the wave equation is less demanding on the time step than for diffusions