Applied Mathematics - MATH-0504
Exam statements and solutions

January 14, 2019

Nom(s) : Prénom : Matricule :

N'oubliez pas d'indiquer vos ***nom(s) et prénom*** *sur chaque feuille rendue. Veuillez vérifier dès le début de l'examen que vous disposez de l'entièreté du questionnaire, qui compte 9 pages numérotées de 1 à 9 divisées en 3 questions et 6 pages de brouillon. Veuillez rendre à la fin de l'examen l'ensemble des 9 pages* ***agrafées****.* ***Essayez toutes les sous-questions****, beaucoup sont indépendantes. Les calculatrices sont* ***interdites****. L'examen dure 2 heures.*

Question I (10 Points)

Considérer l'équation d'onde à une dimension sur un domaine borné

$$u_{tt} - c^2 u_{xx} = 0 \quad \forall x \in]0,1[\tag{$\diamond$}$$

avec les conditions initiales

$$u(x,0) = \phi(x) \quad \text{et} \quad u_t(x,0) = \psi(x) \quad \forall x \in]0,1[$$

et des conditions aux limites en $x = 0$ et $x = 1$ (toutes ces conditions seront spécifiées plus tard).

(a) (4 Points) En utilisant la séparation $u(x,t) = w(t)v(x)$, trouver toutes les solutions séparables de l'Eq. ($\diamond$).

(b) (4 Points) Montrer que la solution à l'Eq. ($\diamond$) qui satisfait la condition à la limite homogène de Dirichlet $u(0,t) = 0$, $\forall t \geq 0$ et la condition à la limite homogène de Neumann $u_x(1,t) = 0$, $\forall t \geq 0$ peut être écrite comme

$$u(x,t) = \sum_{n=0}^{\infty} \left[A_n \cos(c\omega_n t) + B_n \sin(c\omega_n t)\right] \sin(\omega_n x). \tag{$\dagger$}$$

Donner l'expression des ω_n, $\forall n \in \mathbb{N}$.

(c) (2 Points) En partant de la solution générale ($\dagger$) du problème aux conditions aux limites, donner la solution du problème pour les conditions initiales suivantes :

$$\phi(x) = 2\sin\frac{\pi x}{2} \quad \text{et} \quad \psi(x) = 3c\pi \sin\frac{3\pi x}{2}.$$

Indice : Identifier les coefficients directement (et donc ne pas utiliser la formule générale des séries de Fourier).

Solution

(a) Using the ansatz $u = wv$, the wave equation writes as

$$w''v - c^2 wv'' = 0 \tag{1}$$

$$\Rightarrow \frac{w''}{c^2 w} = \frac{v''}{v}. \tag{2}$$

Since the right hand side depends only on x while the left hand side depends only on t, both sides must be equal to a constant (named λ), *i.e*

$$v'' - \lambda v = 0 \quad \text{and} \quad w'' - c^2 \lambda w = 0. \tag{3}$$

Spatial dependence Depending on the sign of λ, three solutions arise, *i.e*

$$\text{if } \lambda = 0 \quad \Rightarrow v = Ax + B, \tag{4}$$
$$\text{if } \lambda = \omega^2 > 0 \quad \Rightarrow v = C\exp(-\omega x) + D\exp(\omega x), \tag{5}$$
$$\text{if } \lambda = -\omega^2 < 0 \quad \Rightarrow v = E\cos(\omega x) + F\sin(\omega x). \tag{6}$$

Time dependence Depending on the sign of λ, three solutions arise, *i.e*

$$\text{if } \lambda = 0 \quad \Rightarrow v = Gt + H, \tag{7}$$
$$\text{if } \lambda = \omega^2 > 0 \quad \Rightarrow v = I\exp(-c\omega t) + J\exp(c\omega t), \tag{8}$$
$$\text{if } \lambda = -\omega^2 < 0 \quad \Rightarrow v = K\cos(c\omega t) + L\sin(c\omega t). \tag{9}$$

(b) Among all the eigensolutions found in the previous sub-question (*i.e* for any value of λ), only those that satisfy the homogeneous boundary conditions are kept.

Stationary eigensolutions, *i.e*: $\lambda = 0$ Applying the boundary conditions to $v = Ax + B$ gives

$$\begin{cases} v(0) & = B & = 0 \\ v_x(1) & = A & = 0 \end{cases} \quad \Rightarrow A = B = 0 \tag{10}$$

such that there is no stationary eigensolution satisfying the homogeneous boundary conditions.

Evanescent eigensolutions, *i.e*: $\lambda > 0$ Applying the boundary conditions to $v = C\exp(-\omega x) + D\exp(\omega x)$ gives

$$\begin{cases} v(0) & = C + D & = 0 \\ v_x(1) & = -\omega C\exp(-\omega) + \omega D\exp(\omega) & = 0 \end{cases} \quad \Rightarrow \begin{pmatrix} 1 & 1 \\ -\omega\exp(-\omega) & \omega\exp(\omega) \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{11}$$

provided that $\omega > 0$, the unique solution to Eqs(11) is

$$C = D = 0 \tag{12}$$

thus there is no time growing eigensolution satisfying the homogeneous boundary condition neither.

Propagating eigensolutions, *i.e*: $\lambda < 0$ Applying the boundary conditions to $v = E\cos(\omega x) + F\cos(\omega x)$ gives

$$\begin{cases} v(0) & = E & = 0 \\ v_x(1) & = -\omega E\sin(\omega) + \omega F\cos(\omega) & = 0 \end{cases} \Rightarrow F\cos(\omega) = 0 \quad \Rightarrow \omega \to \omega_n = (n{+}1/2)\pi, \quad n = 0, 1, 2, 3, \dots \tag{13}$$

One should be carefull that the values $\omega_n = n\pi$ for $n < 0$ are not considered because they yield the same eigenvalue $\lambda_n = -\omega_n^2$.
The only eigensolutions compatible with the boundary conditions are therefore

$$v_n(x) = F_n \sin(\omega_n x), \quad \forall n = 0, 1, 2, 3, \dots \tag{14}$$

and the most general solution compatible with boundary condition is then

$$u(x,t) = \sum_{n=0}^{\infty} \left[K_n \cos(c\omega t) + L_n \sin(c\omega t)\right] F_n \sin(\omega_n x). \tag{15}$$

or

$$u(x,t) = \sum_{n=0}^{\infty} \left[A_n \cos(c\omega_n t) + B_n \sin(c\omega_n t)\right] \sin(\omega_n x). \tag{16}$$

if $A_n \triangleq K_n F_n$ and $B_n \triangleq L_n F_n$.

(c) Direct identification yields

$$A_0 = 2, \quad A_n = 0\ \forall n \in \mathbb{N}_0, \quad B_1 = 2, \quad B_n = 0\ \forall n \in \mathbb{N}\backslash\{1\}, \tag{17}$$

such that the final solution writes

$$u(x,t) = 2\cos\left(\frac{c\pi t}{2}\right)\sin\left(\frac{\pi x}{2}\right) + 2\sin\left(\frac{3c\pi t}{2}\right)\sin\left(\frac{3\pi x}{2}\right). \tag{18}$$

Question II (10 Points)

Considérer

$$A = \begin{pmatrix} 1 & 2 & 0 \\ -2 & -1 & 0 \end{pmatrix}$$

(a) (7 Points) Calculer une décomposition en valeurs singulières $A = U\Sigma V^T$. Donner les matrices U, Σ et V.

(b) (3 Points) Représenter dans $\mathbb{R}^3$

- les vecteurs singuliers à droite v_1, v_2, v_3,
- la sphere unité.

Tracer dans l'espace image

- les vecteurs singuliers à gauche pondérés $\sigma_1 u_1$ et $\sigma_2 u_2$, *i.e.*, l'image des vecteurs singuliers à droite,
- l'image par A de la sphère unité.

Solution

(a)

$$AA^T = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix} \tag{19}$$

thus $\sigma_1 = 3$, $\sigma_2 = 1$ and

$$u_1 = \pm \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} \text{ and } u_2 = \pm \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} \tag{20}$$

then

$$v_1 = \pm \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}, \quad v_2 = \pm \begin{pmatrix} \frac{-\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix} \text{ and } v_3 = \pm \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{21}$$

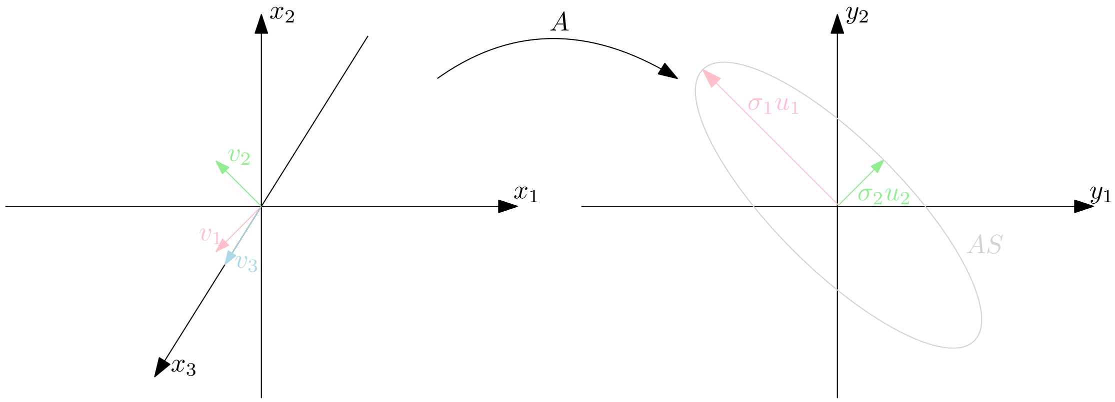
as

$$A^T A = \begin{pmatrix} 5 & 4 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{22}$$

such that a singular value decomposition is

$$U = \begin{pmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } V = \begin{pmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{23}$$

(b) The representation is given below


x_2
A
y_2
v_2
$\sigma_1 u_1$
x_1
$\sigma_2 u_2$
y_1
v_1
v_3
AS
x_3

Question III (10 Points)

Considérer l'équation suivante d'advection-diffusion pour la fonction u dans un domaine non-borné à une dimension (1D) :

$$u_t + \frac{c_0 t}{t_0} u_x - k u_{xx} = 0 \quad \text{pour} \quad (x,t) \in \mathbb{R}\times]0,+\infty[, \tag{$\star$}$$

avec k, c_0 et t_0 des constantes strictement positives, et la condition initiale, en $t = 0$:

$$u(x,0) = \phi(x) = A\,\exp\left(-\frac{x^2}{4k\tau}\right), \tag{$\odot$}$$

avec A et τ des constantes strictement positives.
Suivre les étapes successives ci-dessous pour résoudre ce problème.

(a) Considérer uniquement la partie advective de l'équation :

$$u_t + \frac{c_0 t}{t_0} u_x = 0 \quad \text{pour} \quad (x,t) \in \mathbb{R}\times]0,+\infty[.$$

- (2 Points) Trouver les courbes caractéristiques de cette équation.
- (1 Point) Donner la solution générale de cette équation.

(b) (1 Point) Simplifier l'équation de advection-diffusion (Eq. ($\star$)) en utilisant le changement de variables

$$\begin{cases} z = x - \dfrac{c_0 t^2}{2t_0}, \\ s = t. \end{cases}$$

Ecrire également la condition initiale (Eq. ($\odot$)) en terme des nouvelles variables (z,s).

(c) (4 Points) Résoudre l'équation de diffusion

$$u_s - k u_{zz} = 0 \quad \text{pour} \quad (z,s) \in \mathbb{R}\times]0,+\infty[,$$

avec la condition initiale

$$u(z,0) = \phi(z) = A\,\exp\left(-\frac{z^2}{4k\tau}\right).$$

Rappel: La fonction de Green S du problème 1D de diffusion dans un domaine non borné est

$$S(z,s) = \frac{1}{\sqrt{4\pi k s}} \exp\left(-\frac{z^2}{4ks}\right),$$

et l'intégrale de Poisson $\int_{\mathbb{R}} \exp(-\xi^2)\, d\xi = \sqrt{\pi}$.

Indice : L'identité suivante peut être utile :

$$z^2 - 2yz + (1 + s/\tau)y^2 = \left(\frac{z}{\sqrt{1+(s/\tau)}} - y\sqrt{1+(s/\tau)}\right)^2 + z^2 - \frac{z^2}{1+(s/\tau)}.$$

(d) (2 Points) Utiliser les résultats des étapes précédentes pour donner la solution du problème complet d'advection-diffusion (Eq. ($\star$) with Eq. ($\odot$)). Esquisser la solution aux instants $t = 0$, $t = t_0$ et $t = 2t_0$.

Solution

(a)
- The convection equation asserts that the directional derivative of u in the direction of the vector $(1, c_0t/t_0)$ in the (t, x)-space is identical to zero.
  The curves in the (t, x)-space with tangent $(1, c_0t/t_0)$ have the slope
  $$\frac{dx}{dt} = \frac{c_0t}{t_0}, \tag{24}$$
  which yields, after integration, the characteristic curves
  $$C = x - \frac{c_0t^2}{2t_0}, \quad \forall C \in \mathbb{R}. \tag{25}$$
- The solution u is constant along the characteristic curves, therefore, u can be an arbitrary function of $x - c_0t^2/2t_0$ and the general solution writes, with f an arbitrary function of one variable,
  $$u(x, t) = f\left(x - \frac{c_0t^2}{2t_0}\right). \tag{26}$$

(b) The differential operators ∂_x, ∂_{xx} and ∂_t must be expressed in terms of the new variables z and s. Using the chain rule, it follows

$$\partial_x = \frac{\partial z}{\partial x}\partial_z + \frac{\partial s}{\partial x}\partial_s = 1\,\partial_z + 0\,\partial_s = \partial_z, \tag{27}$$
$$\partial_{xx} = \partial_x(\partial_x) = \partial_x(\partial_z) = \left(\frac{\partial z}{\partial x}\partial_z + \frac{\partial s}{\partial x}\partial_s\right)(\partial_z) = (\partial_z)(\partial_z) = \partial_{zz}, \tag{28}$$
$$\partial_t = \frac{\partial z}{\partial t}\partial_z + \frac{\partial s}{\partial t}\partial_s = -\frac{2c_0t}{2t_0}\,\partial_z + 1\,\partial_s = -\frac{c_0s}{t_0}\partial_z + \partial_s. \tag{29}$$

Substituting in the original equation Eq. ($\star$) gives

$$-\frac{c_0s}{t_0}u_z + u_s + \frac{c_0s}{t_0}u_z - ku_{zz} = 0 \quad \text{for} \quad (z, s) \in \mathbb{R}\times]0, +\infty[, \tag{30}$$

and thus the convection-diffusion equation takes the form of a simple diffusion equation in the convected axes,

$$u_s - ku_{zz} = 0 \quad \text{for} \quad (z, s) \in \mathbb{R}\times]0, +\infty[. \tag{31}$$

The initial condition Eq. ($\odot$) writes simply, for $s = 0$,

$$u(z, 0) = \phi(z) = A\ \exp\left(-\frac{z^2}{4k\tau}\right). \tag{32}$$

(c) The solution is the convolution of the Green's function and the initial condition. Successively,

$$
\begin{aligned}
u(z,s) &= \int_{\mathbb{R}} S(z-y,s)\,\phi(y)\,dy &(33)\\
&= \int_{\mathbb{R}} \frac{A}{\sqrt{4\pi ks}} \exp\left(-\frac{(z-y)^2}{4ks}\right)\,\exp\left(-\frac{y^2}{4k\tau}\right)dy &(34)\\
&= \int_{\mathbb{R}} \frac{A}{\sqrt{4\pi ks}} \exp\left(-\frac{(z-y)^2+(s/\tau)y^2}{4ks}\right)dy, &(35)\\
&= \int_{\mathbb{R}} \frac{A}{\sqrt{4\pi ks}} \exp\left(-\frac{z^2-2yz+(1+s/\tau)y^2}{4ks}\right)dy, &(36)\\
&= \int_{\mathbb{R}} \frac{A}{\sqrt{4\pi ks}} \exp\left(-\frac{\left(\frac{z}{\sqrt{1+(s/\tau)}}-y\sqrt{1+(s/\tau)}\right)^2+z^2-\frac{z^2}{1+(s/\tau)}}{4ks}\right)dy, &(37)\\
&= \int_{\mathbb{R}} \frac{A}{\sqrt{4\pi ks}} \exp\left(-\frac{\left(\frac{z}{\sqrt{1+(s/\tau)}}-y\sqrt{1+(s/\tau)}\right)^2}{4ks}\right)\exp\left(-\frac{z^2-\frac{z^2}{1+(s/\tau)}}{4ks}\right)dy, &(38)\\
&= \frac{A}{\sqrt{4\pi ks}}\exp\left(-\frac{z^2}{4k(s+\tau)}\right)\int_{\mathbb{R}} \exp\left(-\frac{\left(\frac{z}{\sqrt{1+(s/\tau)}}-y\sqrt{1+(s/\tau)}\right)^2}{4ks}\right)dy. &(39)
\end{aligned}
$$

A change of variables $\xi = \left(z/\sqrt{1+(s/\tau)} - y\sqrt{1+(s/\tau)}\right)/\sqrt{4ks}$ gives

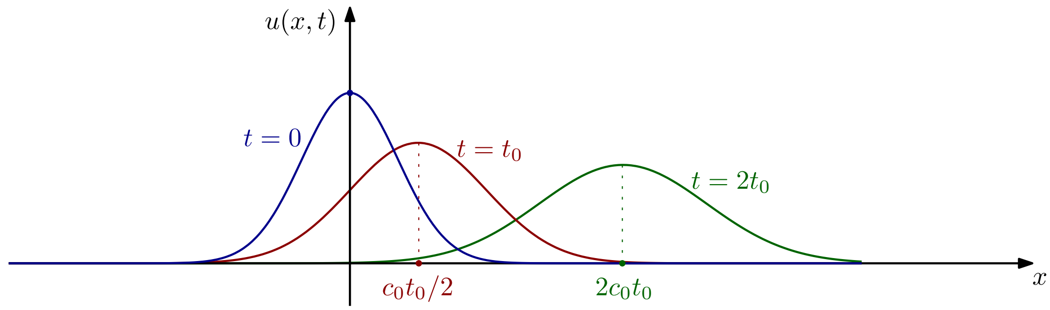
$$
\begin{aligned}
u(z,s) &= \frac{A}{\sqrt{4\pi ks}}\frac{\sqrt{4ks}}{\sqrt{1+(s/\tau)}}\exp\left(-\frac{z^2}{4k(s+\tau)}\right)\underbrace{\int_{\mathbb{R}}\exp\left(-\xi^2\right)d\xi}_{\sqrt{\pi}}, &(40)\\
&= \frac{A}{\sqrt{1+(s/\tau)}}\exp\left(-\frac{z^2}{4k(s+\tau)}\right). &(41)
\end{aligned}
$$

This is the solution to the diffusion equation with the initial condition.

(d) The problem solved in (c) is exactly the original problem expressed in terms of variables (z,s) found in (b). The solution in terms of (x,t) is therefore (Eq. (41) after substitution to (x,t)),

$$
u(x,t) = \frac{A}{\sqrt{1+(t/\tau)}}\exp\left(-\frac{(x-c_0t^2/2t_0)^2}{4k(t+\tau)}\right). \tag{42}
$$

The solution is sketched below. The Gaussian is convected in the positive x-direction with an increasing speed, in addition, its amplitude decreases and its width increases with time.


$u(x,t)$
$t=0$
$t=t_0$
$t=2t_0$
x
$c_0t_0/2$
$2c_0t_0$