Applied Mathematics - MATH-0504
Exercise statements and solutions

December 11, 2025

Note: The difficulty level evaluation (easy: 🌶️, medium: 🌶️🌶️, or hard: 🌶️🌶️🌶️) is not fixed. It may evolve, for example based on your possible feedback. The exam will contain all three difficulty levels.

Contents

1 Characteristic lines

1.1 Characteristic line 1 [Strauss 1.2, Ex.3]

Consider the following equation: $(1 + x^2)u_x + u_y = 0$.

(a) Find the general solution of this equation.

(b) Draw some of the characteristic lines.

Solution

(a) The directional derivative of u in the direction of $(1 + x^2, 1)$ must always be 0. Therefore, the characteristic lines, along which u is constant, satisfy the relation

$$\frac{dy}{dx} = \frac{1}{1 + x^2}, \tag{1.1.1}$$

which can be rewritten as

$$dy = \frac{1}{1 + x^2}\, dx. \tag{1.1.2}$$

Direct integration of both sides yields, with the integration constant $C \in \mathbb{R}$,

$$\begin{aligned} y &= \arctan(x) + C \\ \Leftrightarrow C &= y - \arctan(x). \end{aligned} \tag{1.1.3}$$

Knowing that u is constant on these characteristic lines, we have the following general solution:

$$\boxed{u(x, y) = f\big(y - \arctan(x)\big),} \tag{1.1.4}$$

where f is an arbitrary function.

(b) In order to draw the characteristics, it is more convenient to formulate the equation according to the origin.
Thus,

$$y(x) = \arctan(x) + C, \tag{1.1.5}$$

$$\Rightarrow \quad y(0) = C. \tag{1.1.6}$$

If one defines $y_0 \triangleq y(0)$,

$$C = y_0. \tag{1.1.7}$$

And then,

$$y(x) = y_0 + \arctan(x) \tag{1.1.8}$$

Depending on the value y_0, some characteristic lines are represented in Fig. 1.

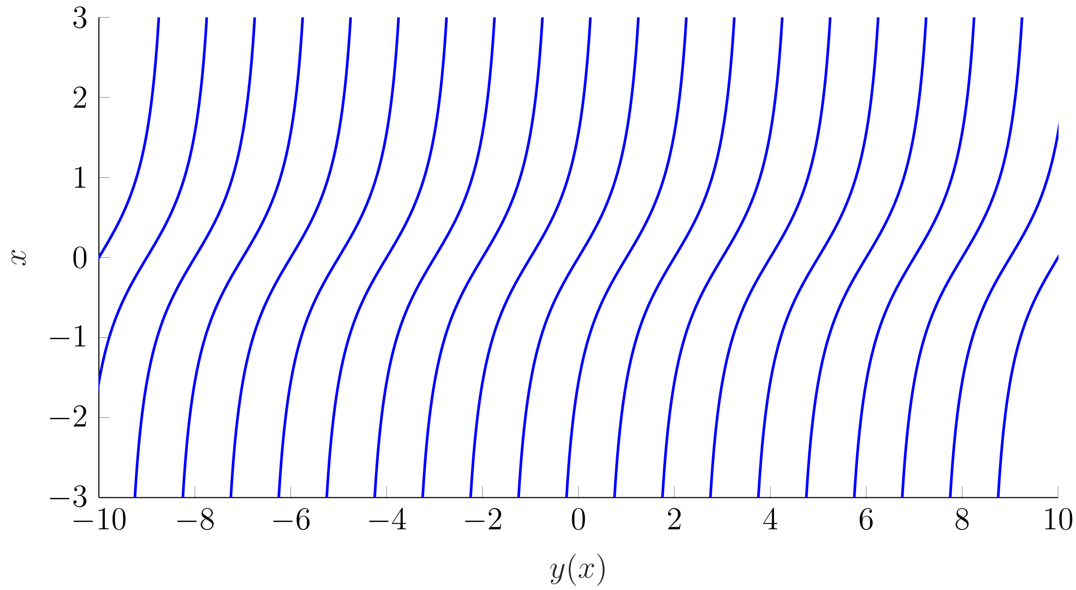

Figure 1: Characteristic curves $y - \arctan(x) = C$.

1.2 Characteristic line 2 [Strauss 1.2, Ex.7] 🌶️🌶️

Consider the following problem: $yu_x + xu_y = 0$ with $u(0, y) = \exp(-y^2)$.

(a) Solve this problem.

(b) Where is the solution uniquely determined in the xy-plane?

(c) Sketch the characteristic lines.

Solution

(a) The directional derivative of u along the vector (y, x) must be 0. As long as $y \neq 0$, the equation of the characteristic lines is therefore

$$\frac{dy}{dx} = \frac{x}{y}. \tag{1.2.1}$$

The characteristic lines satisfy

$$y^2 = x^2 + C \tag{1.2.2}$$

such that

$$C = y^2 - x^2. \tag{1.2.3}$$

Along these lines u is constant. Thus u is only a function of the parameter C. Then using Eq.(1.2.3) yields

$$u(x, y) = f(y^2 - x^2). \tag{1.2.4}$$

Thanks to the additional condition, it is possible to determine the shape of the function f. Indeed, at $x = 0$

$$f(y^2) = \exp\{-y^2\}. \tag{1.2.5}$$

Thus one can write

$$f(C) = \exp\{-C\} \tag{1.2.6}$$

and finally, from Eq.(1.2.3), one has

$$u(x, y) = \exp\{x^2 - y^2\}. \tag{1.2.7}$$

(b) The additional condition is valid for $x = 0$. The characteristic lines satisfying $C < 0$ never go through $x = 0$, therefore the condition does not apply in that case. The two following cases must be distinguished in the final answer

$$\begin{cases} \text{if } y^2 - x^2 < 0 \text{ then } u(x, y) = f(y^2 - x^2) \\ \text{if } y^2 - x^2 \geq 0 \text{ then } u(x, y) = \exp\{x^2 - y^2\}. \end{cases} \tag{1.2.8}$$

Furthermore, the initial condition is also impossible to verify when $y = 0$.

(c) In the case $C \geq 0$, the characteristic curves are depicted in Fig. 2. The lack of characterictics in certain regions is induced by the condition $y^2 - x^2 \geq 0$. There is also an intersection of the straight lines in $y = 0$, which is consistent as the solution is not defined in $y = 0$.
If one defines $y_0 \triangleq y(0)$, it is possible to write the characteristic curves as

$$y(x) = \pm\sqrt{x^2 + y_0^2} \tag{1.2.9}$$

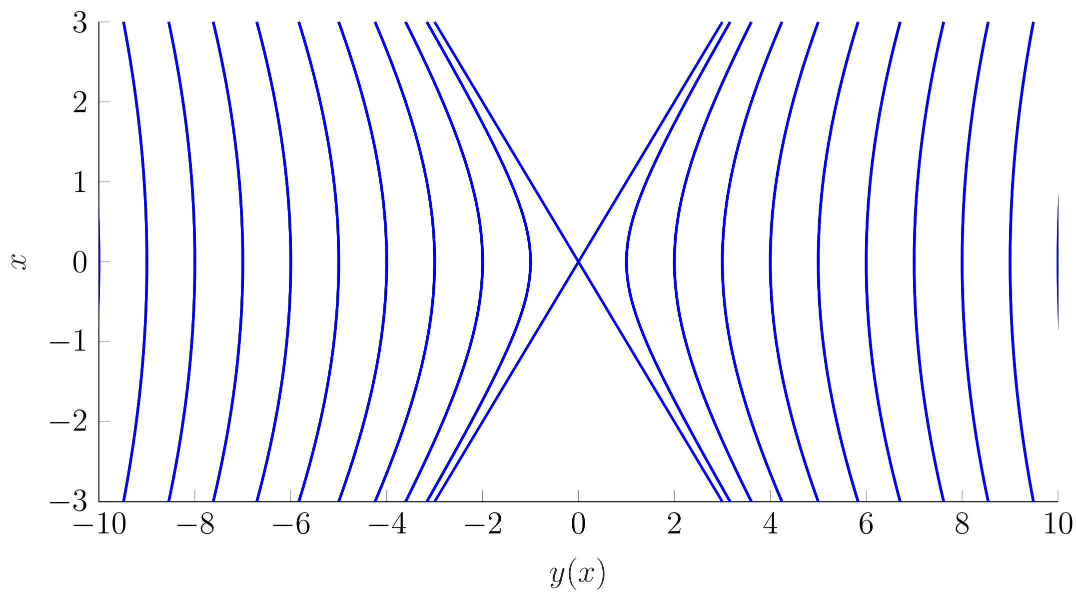

Figure 2: Characteristic curves for $C \geq 0$.

1.3 Characteristic line 3 [Strauss 1.5, Ex. 6]

Find the general solution of the equation $u_x + 2xy^2u_y = 0$.

Solution

The equation can be rewritten as

$$\boldsymbol{a} \cdot \nabla u = 0 \tag{1.3.1}$$

where $\boldsymbol{a} = (1, 2xy^2)$.
The characteristic curves thus satisfy the equation

$$\frac{dy}{dx} = \frac{2xy^2}{1}. \tag{1.3.2}$$

Therefore

$$\frac{dy}{y^2} = 2xdx, \tag{1.3.3}$$

i.e., upon integration,

$$-\frac{1}{y} + C = x^2 \tag{1.3.4}$$

$$\Rightarrow \quad y = \frac{1}{C - x^2}, \tag{1.3.5}$$

$$\Rightarrow \quad C = x^2 + \frac{1}{y}. \tag{1.3.6}$$

$u(x, y)$ is constant on the characteristic curves

$$y = \frac{1}{C - x^2}. \tag{1.3.7}$$

Indeed

$$\frac{du}{dx}\left(x, \frac{1}{C - x^2}\right) = \frac{\partial u}{\partial x} + \frac{2x}{(C - x^2)^2}\frac{\partial u}{\partial y} = u_x + 2xy^2u_y = 0. \tag{1.3.8}$$

Hence, $u(x, y) = f(C)$, *i.e.*

$$u(x, y) = f\left(x^2 + \frac{1}{y}\right), \tag{1.3.9}$$

where f is an arbitrary function.

1.4 Transport with decay [Strauss 1.2, Ex. 8]

Find the solution of the problem considering the following equation

$$au_x + bu_y + cu = 0,$$

with the general initial conditions

$$u(0, y) = f(y).$$

Solution

The equation can be written as

$$\boldsymbol{a} \cdot \nabla u = -cu \tag{1.4.1}$$

where $\boldsymbol{a} = (a, b)$. The directional derivative along (a, b) is not 0. Therefore u is not constant along the lines

$$\frac{dy}{dx} = \frac{b}{a} \tag{1.4.2}$$

$$\Rightarrow \qquad y(x) = \frac{b}{a}x + C. \tag{1.4.3}$$

Along these lines however, u satisfies

$$\frac{du}{dx} = -\frac{c}{a}u. \tag{1.4.4}$$

Indeed

$$\frac{d}{dx}\left(u\left(x, \frac{b}{a}x + C\right)\right) = u_x + \frac{b}{a}u_y = -\frac{c}{a}u. \tag{1.4.5}$$

Solving the Eq.(1.4.4) for u yields:

$$u = K\exp\left(-\frac{c}{a}x\right) \qquad K \in \mathbb{R}. \tag{1.4.6}$$

The K is constant on its characteristic. It means it could be different from one charactistic curve to the other. That is why K can be determined according to the origin of these curves and then expressed as $K = f(y(0))$. If one defines $y_0 \triangleq y(x = 0)$, the characteristic lines at the initial conditions can be written as

$$y(0) = \frac{b}{a}0 + C, \tag{1.4.7}$$

$$\Rightarrow \qquad C = y_0. \tag{1.4.8}$$

Then,

$$y(x) = \frac{b}{a}x + y_0, \tag{1.4.9}$$

$$\Rightarrow \qquad y_0 = y - \frac{b}{a}x. \tag{1.4.10}$$

Considering the initial condition, the K constant can also be determined

$$u(0, y(0)) = u(0, y_0) = K = f(y(0)) = f(y_0). \tag{1.4.11}$$

Therefore, the solution depending on any initial condition is given by

$$u(x, y) = f\left(y_0\right)\exp\left(-\frac{c}{a}x\right), \tag{1.4.12}$$

$$\Rightarrow \qquad u(x, y) = f\left(y - \frac{b}{a}x\right)\exp\left(-\frac{c}{a}x\right). \tag{1.4.13}$$

1.5 Characteristic line 4 [Olver 2.2, Example 2.4] 🌶️

Find the general solution of the equation $u_t + \frac{1}{x^2+1}u_x = 0$.

Solution

The general solution is

$$u(x,t) = f(\frac{1}{3}x^3 + x - t). \tag{1.5.1}$$

1.6 Characteristic line 5 [Olver 2.2, Example 2.5] 🌶️🌶️🌶️

Consider the equation

$$u_t + (x^2 - 1)u_x = 0, \quad \forall (x,t) \in \mathbb{R}\times]0,\infty[.$$

with the initial condition

$$u(x,0) = \exp\left(-x^2\right).$$

(a) Find the characteristic curves. Draw these curves.

(b) Which of these curves intersect the x-axis?

(c) Does the initial condition determine uniquely the solution of the problem? Where is the solution not defined?

(d) Give the solution to the problem in the region where it is uniquely defined.

Solution

(a) The characteristic curves are solutions of

$$\frac{dx}{dt} = x^2 - 1 \tag{1.6.1}$$

i.e

$$\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| = t + C, \quad C \in \mathbb{R} \qquad (\text{or} \qquad x = \pm 1). \tag{1.6.2}$$

The characteristic curves can be expressed explicitly as

$$x(t) = \begin{cases} \dfrac{1 + A\exp(2t)}{1 - A\exp(2t)} & \text{and} \quad x < -1 \\ \\ \dfrac{1 - A\exp(2t)}{1 + A\exp(2t)} & \text{and} \quad 1 < x < -1 \\ \\ \dfrac{1 + A\exp(2t)}{1 - A\exp(2t)} & \text{and} \quad x > 1 \end{cases} \tag{1.6.3}$$

where $A \triangleq \exp(2C)$ and thus $A > 0$.
These curves are given in Figure 3.

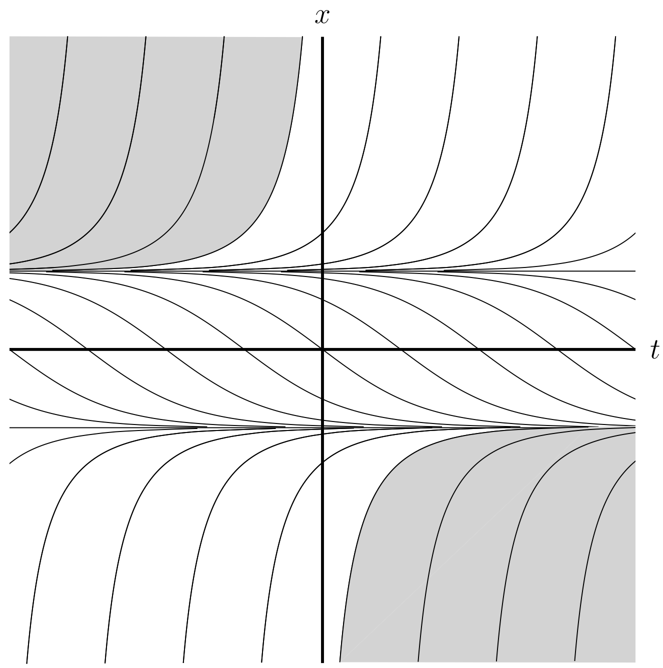

Figure 3: Characteristic curves.

(b)
- For $x < -1$, the possible intersections are given by

$$x_0 = \frac{1+A}{1-A}. \tag{1.6.4}$$

Because $x_0 < -1$, only the curves for which $A > 1$ intersect the axis.
- For $-1 < x < 1$, the possible intersections are given by

$$x_0 = \frac{1-A}{1+A}. \tag{1.6.5}$$

All ($\forall A > 0$) these curves intersect the axis.
- For $x > 1$, the possible intersections are given by

$$x_0 = \frac{1+A}{1-A}. \tag{1.6.6}$$

Because $x_0 > 1$, only the curves for which $A < 1$ intersect the axis.

(c) The solution is constant along each characteristic curve. The solution is therefore completely known provided the initial condition allows to determine this constant for each curve. The initial condition specifies the solution on the x-axis. Therefore, the constant value is fixed for the curves that crosses the x-axis. For the other curves, the value of the constant can not be determined and thus the solution is thus not unique.
The curves

$$x(t) = \frac{1 + A\exp(2t)}{1 - A\exp(2t)} \quad \text{and} \quad x < -1 \tag{1.6.7}$$

do not cross the x-axis when $A < 1$. From Figure 3, it appears that this region is equivalent to

$$x \leq \frac{1+\exp(2t)}{1-\exp(2t)}, \quad t > 0. \tag{1.6.8}$$

Similarly, the curves

$$x(t) = \frac{1+A\exp(2t)}{1-A\exp(2t)} \quad \text{and} \quad x > 1 \tag{1.6.9}$$

do not cross the x-axis when $A > 1$. From Figure 3, it appears that this region is equivalent to

$$x \geq \frac{1+\exp(2t)}{1-\exp(2t)}, \quad t < 0. \tag{1.6.10}$$

(d) On the regions where the characteristic curves intersect the x-axis, the constant A can be expressed in terms of the intersection coordinate x_0, *i.e*

- For $x < -1$, $A = \frac{x_0-1}{x_0+1}$
- For $-1 < x < 1$, $A = -\frac{x_0-1}{x_0+1}$
- For $x > 1$, $A = \frac{x_0-1}{x_0+1}$

thus

$$x(t) = \frac{(x_0+1)+(x_0-1)\exp(2t)}{(x_0+1)-(x_0-1)\exp(2t)} \tag{1.6.11}$$

and

$$x_0 = \frac{(x+1)+(x-1)\exp(-2t)}{(x+1)-(x-1)\exp(-2t)}. \tag{1.6.12}$$

Along the characteristic curves, the equation writes

$$\frac{d}{dt}u(x(t),t) = 0 \tag{1.6.13}$$

thus

$$u(x(t),t) = k = u(x(0),0) = \exp\left(-x_0^2\right) \tag{1.6.14}$$

and finally

$$u(x,t) = \exp\left(-\left[\frac{(x+1)+(x-1)\exp(-2t)}{(x+1)-(x-1)\exp(-2t)}\right]^2\right) \quad \text{for} \quad x > \frac{1+\exp(2t)}{1-\exp(2t)}, \quad t > 0. \tag{1.6.15}$$

1.7 Characteristic line 6 [Strauss 14.1, Ex.2] 🌶

Solve $(1+t)u_t + xu_x = 0$. Then solve it with the auxiliary condition $u(x,0) = x^5$ for $t > 0$.

Solution

The PDE can be rewritten as

$$u_t + \frac{x}{1+t}u_x = 0. \tag{1.7.1}$$

Therefore, the caracterictic curves statisfy the equation [1]

$$\begin{aligned} \frac{dx}{dt} &= \frac{x}{1+t}, && (1.7.2)\\ \Rightarrow \qquad x &= C(1+t) \quad C \in \mathbb{R}, && (1.7.3)\\ \Rightarrow \qquad C &= \frac{x}{1+t}. && (1.7.4) \end{aligned}$$

The solution thus has the following form

$$u(x,t) = f\left(\frac{x}{1+t}\right). \tag{1.7.5}$$

If the initial conditon is

$$u(x,0) = f(x) = x^5, \tag{1.7.6}$$

therefore the solution is

$$u(x,t) = \left(\frac{x}{1+t}\right)^5. \tag{1.7.7}$$

The characteristic lines are depicted in Fig. 4.

1.8 Characteristic line in 2D 🌶🌶🌶

Find the solution of the equation $yu_x + (-x)u_y + u_t = 0$ with $u(x,y,0) = \exp\big(-((x-1)^2 + 4y^2)\big)$.

Solution

The characteristic curves are defined as

$$\begin{cases} \dfrac{dx}{ds} = y \\ \dfrac{dy}{ds} = -x \\ \dfrac{dt}{ds} = 1 \end{cases} \tag{1.8.1}$$

with

$$x(0) = x_0, \quad y(0) = y_0 \quad \text{and} \quad t(0) = 0 \tag{1.8.2}$$

[1] Direct integration yields the solution $|x| = C\,|1+t|$, $C \in \mathbb{R}_0^+$. For a given value of C there is thus two lines. The solution written as $x = C'(1+t)$, $C' \in \mathbb{R}$ associates a unique line to a unique value of C'.

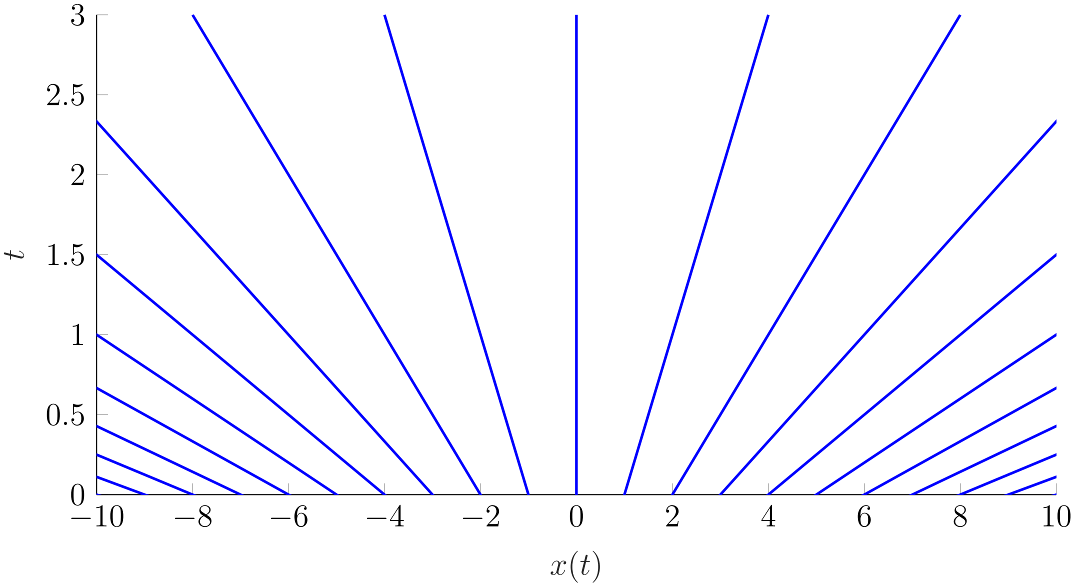

Figure 4: Characteristic curves.

thus

$$\begin{cases} x(s) = x_0 \cos s + y_0 \sin s, \\ y(s) = y_0 \cos s - x_0 \sin s, \\ t(s) = s. \end{cases} \tag{1.8.3}$$

The equation can then be written

$$\begin{pmatrix} y & -x & 1 \end{pmatrix} \cdot \begin{pmatrix} u_x & u_y & u_t \end{pmatrix} = 0 \tag{1.8.4}$$

$$\Leftrightarrow \quad \begin{pmatrix} \frac{dx}{ds} & \frac{dy}{ds} & \frac{dt}{ds} \end{pmatrix} \cdot \begin{pmatrix} u_x & u_y & u_t \end{pmatrix} = 0 \tag{1.8.5}$$

$$\Leftrightarrow \quad \frac{d}{ds} u(x(s), y(s), t(s)) = 0 \tag{1.8.6}$$

$$\Leftrightarrow \quad u(x(s), y(s), t(s)) = C. \tag{1.8.7}$$

The field u is therefore constant along any characteristic curve. With the constant C of course depending on the curve. From the initial data, one finds that the constant C for the curve passing through the point (x_0, y_0) at $t = 0$ is

$$C = u(x_0, y_0, 0) = \exp\left(- \left((x_0 - 1)^2 + 4y_0^2 \right) \right). \tag{1.8.8}$$

Finally using the form

$$\begin{cases} x_0 = x \cos t - y \sin t, \\ y_0 = y \cos t + x \sin t \end{cases} \tag{1.8.9}$$

of the characteristic curve, the solution is given by

$$u(x, y, t) = C = \exp\left(- \left[(x \cos t - y \sin t - 1)^2 + 4(y \cos t + x \sin t)^2 \right] \right). \tag{1.8.10}$$

2 Equation classification and conservation laws

Classification of second order linear PDEs

Second order PDEs are classified as elliptic, parabolic or hyperbolic depending on their coefficients. This classification has a major impact on the equation's behavior.

- *Elliptic* equations, such as the Laplace equation ($u_{xx} + u_{yy} = 0$).
- *Parabolic* equations, such as the Diffusion equation ($u_t - u_{xx} = 0$).
- *Hyperbolic* equations, such as the Wave equation ($u_{tt} = u_{xx}$).

These names come from a geometric analogy: if we replace derivatives by powers of the independent variable ($u_x x$ becomes x^2, $u_x y$ becomes xy, ...), then equation describes a conical. The classification of the PDE is then the same as the one of the conical. For instance, the heat equation is parabolic because $t = x^2$ represents a parabola in the tx plane.

If the equation has the form $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$, then a simple criterion is the sign of $B^2 - 4AC$. If it is negative, the equation is elliptic. When it is zero, the equation is parabolic, and a positive value represents a hyperbolic equation.

This criterion can be derived by considering that $Ax^2 + Bxy + Cy^2$ is a quadratic form $\boldsymbol{x}^T K \boldsymbol{x}$ with K a coefficient matrix:

$$K = \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix}.$$

Using this formulation, the conical described is elliptic, parabolic or hyperbolic respectively if the eigenvalues of K are all the same sign, if one is zero, or if they have different signs. Since the determinant $AC - \frac{B^2}{4}$ is the product of the eigenvalues, it can be used to classify the equation without explicit computation of the eigenvalues.

With more independent variables, a similar classification can be achieved: *elliptic* for a matrix whose eigenvalues have all the same sign, *parabolic* for a singular matrix and *hyperbolic* otherwise.

Conservation laws

A conservation law is an equation of the form, see [Olver, Definition 2.7] or [Lecture 3, Slide 48]

$$\frac{\partial c}{\partial t} + \nabla \cdot (\boldsymbol{f}) = 0.$$

The function c is known as the *conserved density*, while $\boldsymbol{f}$ is the associated *flux*. Indeed, let Ω be a subset of $\mathbb{R}^3$ whose boundary is denoted by $\partial\Omega$ then

$$\begin{aligned} & \int_\Omega \left(\frac{\partial c}{\partial t} + \nabla \cdot (\boldsymbol{f}) \right) d\Omega && = 0 \\ \Rightarrow \quad & \frac{\partial}{\partial t} \int_\Omega c \, d\Omega + \int_\Omega \nabla \cdot (\boldsymbol{f}) \; d\Omega = 0 \\ \Rightarrow \quad & \frac{\partial}{\partial t} \int_\Omega c \, d\Omega + \int_{\partial\Omega} \boldsymbol{f} \cdot \hat{\boldsymbol{n}} \, d\partial\Omega = 0. \end{aligned}$$

Defining

$$C \triangleq \int_{\Omega} c \, d\Omega \quad \text{and} \quad F \triangleq \int_{\partial\Omega} \boldsymbol{f} \cdot \hat{\boldsymbol{n}} \, d\partial\Omega$$

gives

$$\frac{\partial C}{\partial t} = -F$$

such that when the flux is, on average, going outside (resp. inside) of the volume Ω then the conserved quantity C decreases (resp. increases).

2.1 Maxwell's equations 🌶️🌶️

Consider the continuum media Maxwell's equations

$$\nabla \times (\boldsymbol{h}) = \boldsymbol{j} + \boldsymbol{d}_t \quad \text{(Ampère-Maxwell)}, \tag{$\diamond$}$$

$$\nabla \times (\boldsymbol{e}) = -\boldsymbol{b}_t \quad \text{(Faraday)}, \tag{$*$}$$

$$\nabla \cdot (\boldsymbol{b}) = 0 \quad \text{(Gauss)}, \tag{$\dagger$}$$

$$\nabla \cdot (\boldsymbol{d}) = \rho \quad \text{(Coulomb)}, \tag{$\ddagger$}$$

where $\boldsymbol{h}$ [A/m] is the magnetic field, $\boldsymbol{e}$ [V/m] is the electric field, $\boldsymbol{b}$ [W/m^2] is the magnetic flux density, $\boldsymbol{d}$ [C/m^2] is the electric displacement field, $\boldsymbol{j}$ [A/m^2] is the current density and ρ [C/m^3] is the charge density. (Because these equations have too many unknowns (*i.e* $\boldsymbol{h}$, $\boldsymbol{e}$, $\boldsymbol{b}$, $\boldsymbol{d}$, $\boldsymbol{j}$ and ρ), they must be complemented with constitutive equations.)

In the following exercises, the vector identity $\nabla \times (\nabla \times (\boldsymbol{u})) = \nabla(\nabla \cdot \boldsymbol{u}) - \nabla^2(\boldsymbol{u})$ (the curl of the curl of a field is the gradient of its divergence, minus its Laplacian) could be helpful.

(a) Consider empty space *i.e*

$$\boldsymbol{d} = \epsilon_0 \boldsymbol{e}, \quad \boldsymbol{b} = \mu_0 \boldsymbol{h}, \quad \rho = 0$$

where ϵ_0, μ_0 are constants and that $\boldsymbol{j}$ is a known function.

- Making the assumption of transverse electric (TE) fields, *i.e* $\boldsymbol{e} = u(x,t)\hat{\boldsymbol{z}}$ and $\boldsymbol{j} = j(x,t)\hat{\boldsymbol{z}}$, show that the electric field verifies

  $$u_{tt} - c^2 u_{xx} = f.$$

  Determine c and f. Classify this equation.

- Making the assumption of transverse magnetic (TM) fields, *i.e* $\boldsymbol{h} = u(x,t)\hat{\boldsymbol{z}}$ and $\nabla \times (\boldsymbol{j}) = m(x,t)\hat{z}$, show that the magnetic field verifies

  $$u_{tt} - c^2 u_{xx} = f.$$

  Determine c and f. Classify this equation.

(b) At steady state (*i.e.* $\frac{\partial}{\partial t} = 0$) in linear media ($\boldsymbol{b} = \mu \boldsymbol{h}$ and $\boldsymbol{d} = \epsilon \boldsymbol{e}$, ϵ and μ being constants), show that the vector potential $\boldsymbol{a}$, defined such that $\nabla \times (\boldsymbol{a}) = \boldsymbol{b}$ (with $\nabla \cdot (\boldsymbol{a}) = 0$ imposed) and the scalar potential ϕ, defined such that $-\nabla(\phi) = \boldsymbol{e}$ verifies

$$\Delta(\phi) = f_\phi,$$

and

$$\Delta(\boldsymbol{a}) = \boldsymbol{f}_{\boldsymbol{a}}.$$

Determine $\boldsymbol{f}_{\boldsymbol{a}}$ and f_ϕ. Classify these equations.

(c) Using the magnetodynamics hypothesis $\boldsymbol{d}_t = 0$ in linear media ($\boldsymbol{b} = \mu\boldsymbol{h}$, $\boldsymbol{d} = \epsilon\boldsymbol{e}$ and $\boldsymbol{j} = \sigma\boldsymbol{e}$, ϵ, μ and σ being constants) show that the magnetic field verifies

$$\boldsymbol{h}_t - \alpha\Delta(\boldsymbol{h}) = \boldsymbol{f}.$$

Determine $\boldsymbol{f}$ and α. Classify this equation.
The magnetic constitutive law is replaced $\boldsymbol{b}_t = \mu(\boldsymbol{h}_t + \omega_0\boldsymbol{h})$. Update the previous equations. Classify.

Solution

(a)

- The derivative w.r.t t of the Ampère-Maxwell equation Eq.($\diamond$) is

$$\nabla\times(\boldsymbol{h}_t) = \boldsymbol{j}_t + \epsilon_0\boldsymbol{e}_{tt}. \tag{2.1.1}$$

Using the Maxwell-Faraday equation Eq.($*$) yields

$$\boldsymbol{e}_{tt} + \frac{1}{\epsilon_0\mu_0}\nabla\times(\nabla\times(\boldsymbol{e})) = -\frac{1}{\epsilon_0}\boldsymbol{j}_t. \tag{2.1.2}$$

The double curl simplifies because of the Gauss law as

$$\begin{aligned}\nabla\times(\nabla\times(\boldsymbol{e})) &= \nabla(\nabla\cdot(\boldsymbol{e})) - \nabla\cdot(\nabla(\boldsymbol{e})) && (2.1.3)\\ &= -\nabla\cdot(\nabla(\boldsymbol{e})) && (2.1.4)\\ &= -\Delta(\boldsymbol{e}) && (2.1.5)\\ &= -\Delta(u)\,\hat{\boldsymbol{z}}. && (2.1.6)\end{aligned}$$

The Ampère-Maxwell equation Eq.($\diamond$) then finally writes

$$u_{tt} - \frac{1}{\epsilon_0\mu_0}\Delta(u) = -\frac{1}{\epsilon_0}j_t \tag{2.1.7}$$

thus

$$c = \frac{1}{\sqrt{\epsilon_0\mu_0}} \quad \text{and} \quad f = -\frac{1}{\epsilon_0}j_t \tag{2.1.8}$$

- The derivative w.r.t t of the Maxwell-Faraday equation Eq.($*$) is

$$\nabla\times(\boldsymbol{e}_t) = -\mu_0\boldsymbol{h}_{tt}. \tag{2.1.9}$$

Using the Ampère-Maxwell equation Eq.($\diamond$) yields

$$\boldsymbol{h}_{tt} + \frac{1}{\epsilon_0\mu_0}\nabla\times(\nabla\times(\boldsymbol{h})) = \frac{1}{\mu_0\epsilon_0}\nabla\times(\boldsymbol{j}). \tag{2.1.10}$$

The Maxwell-Faraday equation Eq.($*$) then finally writes

$$u_{tt} - \frac{1}{\epsilon_0\mu_0}\Delta(u) = \frac{1}{\mu_0\epsilon_0}m \tag{2.1.11}$$

thus

$$c = \frac{1}{\sqrt{\epsilon_0\mu_0}} \quad \text{and} \quad f = \frac{1}{\mu_0\epsilon_0}m. \tag{2.1.12}$$

These equations are

  - non homogeneous
  - linear
  - second order in space and time
  - hyperbolic

  Indeed, they can be written (for a single spatial dimension)

$$Au_{tt} + Bu_{tx} + Cu_{xx} + Du_t + Eu_x + Fu = G \tag{2.1.13}$$
$$1u_{tt} + 0\ u_{tx} - c^2 u_{xx} + 0\ u_t + 0\ u_x + 0\ u = f \tag{2.1.14}$$

  thus

$$B^2 - 4AC = 4c^2 > 0, \tag{2.1.15}$$

  and the analog geometric curve is a hyperbola.

(b) The Maxwell-Faraday Eq.($*$) and the magnetic Gauss' law Eq.(†) are trivially satisfied by the potentials indeed

$$\nabla \times (\boldsymbol{e}) = -\nabla \times (\nabla (\phi)) = 0 \tag{2.1.16}$$

and

$$\nabla \cdot (\boldsymbol{b}) = \nabla \cdot (\nabla \times (\boldsymbol{a})) = 0 \tag{2.1.17}$$

while the electric Gauss' law Eq.(†) becomes

$$\nabla \cdot (\boldsymbol{d}) = \rho \tag{2.1.18}$$
$$\nabla \cdot (\epsilon \boldsymbol{e}) = \rho \tag{2.1.19}$$
$$\nabla \cdot (-\nabla (\phi)) = \frac{\rho}{\epsilon} \tag{2.1.20}$$
$$\Delta (\phi) = -\frac{\rho}{\epsilon} \tag{2.1.21}$$

and the Ampère-Maxwell Eq.($\diamond$) becomes

$$\nabla \times (\boldsymbol{h}) = \boldsymbol{j} \tag{2.1.22}$$
$$\nabla \times (\nabla \times (\boldsymbol{a})) = \mu \boldsymbol{j} \tag{2.1.23}$$
$$\nabla (\nabla \cdot (\boldsymbol{a})) - \nabla \cdot (\nabla (\boldsymbol{a})) = \mu \boldsymbol{j} \tag{2.1.24}$$
$$\Delta (\boldsymbol{a}) = -\mu \boldsymbol{j} \tag{2.1.25}$$

These equations are

- non homogeneous
- linear
- second order in space and independent of time
- elliptic

Indeed, it can be written (for a two spatial dimensions) (either $u = \phi$, either $u = \boldsymbol{a}$)

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \tag{2.1.26}$$
$$1u_{xx} + 0\ u_{xy} + 1\ u_{yy} + 0\ u_x + 0\ u_y + 0\ u = G \tag{2.1.27}$$

thus

$$B^2 - 4AC = -4 < 0. \tag{2.1.28}$$

(c) The Ampère-Maxwell law Eq.(◇) becomes

$$\nabla \times (\boldsymbol{h}) = \sigma \boldsymbol{e} \tag{2.1.29}$$

and the Maxwell-Faraday's law Eq.($*$) can then be written

$$\frac{1}{\sigma}\nabla \times (\nabla \times (\boldsymbol{h})) = -\mu \left(\boldsymbol{h}_t + \omega_0 \boldsymbol{h}\right) \tag{2.1.30}$$

or

$$\boldsymbol{h}_t + \frac{1}{\mu\sigma}\nabla \times (\nabla \times (\boldsymbol{h})) + \omega_0 \boldsymbol{h} = 0 \tag{2.1.31}$$

$$\boldsymbol{h}_t - \frac{1}{\mu\sigma}\Delta (\boldsymbol{h}) + \omega_0 \boldsymbol{h} = 0 \tag{2.1.32}$$

This equation is

- homogeneous
- linear
- second order in space and first order in time
- parabolic

Indeed, it can be written (for a single spatial dimension)

$$A\boldsymbol{h}_{tt} + B\boldsymbol{h}_{tx} + C\ \ \boldsymbol{h}_{xx} + D\boldsymbol{h}_t + E\boldsymbol{h}_x + Fu = G \tag{2.1.33}$$

$$0\boldsymbol{h}_{tt} + 0\ \boldsymbol{h}_{tx} - \frac{1}{\mu\sigma}\boldsymbol{h}_{xx} + 1\ \boldsymbol{h}_t + 0\ \boldsymbol{h}_x + 0\ u = 0 \tag{2.1.34}$$

thus

$$B^2 - 4AC = 0. \tag{2.1.35}$$

In the general (2D or 3D) case, the quadratic form that expresses the second order partial derivatives is $K = \operatorname{diag}(0, -\frac{1}{\mu\sigma}, -\frac{1}{\mu\sigma}, -\frac{1}{\mu\sigma})$ (for the ordering t, x, y, z), which is singular. Thus, the problem is parabolic.

2.2 Navier's equation 🌶️🌶️

Consider the linear momentum conservation (for small displacements)

$$\rho \boldsymbol{u}_{tt} - \nabla \cdot (\boldsymbol{\sigma}) = \rho \boldsymbol{b}$$

with the isotropic Hooke's law

$$\sigma = \lambda \nabla \cdot (\boldsymbol{u})\, \boldsymbol{I} + 2\mu \boldsymbol{e}, \quad \boldsymbol{e} = \frac{1}{2}\left(\nabla (\boldsymbol{u}) + \nabla (\boldsymbol{u})^T\right)$$

where $\boldsymbol{u}$ [m] is the displacement field, ρ [kg/m^3] is the density, $\boldsymbol{\sigma}$ [N/m^2] is the stress tensor, $\boldsymbol{b}$ [N/kg] are the external forces and λ and μ [N/m^2] are the Lamé parameters.

(a) Show that for constant ρ, λ and μ, the conservation of momentum can be written

$$\boldsymbol{u}_{tt} - \frac{\lambda + 2\mu}{\rho}\nabla\left(\nabla\cdot(\boldsymbol{u})\right) + \frac{\mu}{\rho}\nabla\times\left(\nabla\times(\boldsymbol{u})\right) = \boldsymbol{b}.$$

Hint: The divergence of $\nabla(\boldsymbol{u})^T$ is $\nabla\left(\nabla\cdot(\boldsymbol{u})\right)$.

(b) Making the assumptions that $\nabla\cdot(u) = 0$ (shear only) or that $\nabla\times(u) = 0$ (pressure only), show that the conservation of momentum writes

$$\boldsymbol{u}_{tt} - c^2\Delta(\boldsymbol{u}) = \boldsymbol{f}.$$

Determine c^2 and $\boldsymbol{f}$ for both assumptions. Classify this equation.

Solution

(a) The divergence of the stress tensor is given by (using the summation convention)($\boldsymbol{u} = \begin{pmatrix} u^x & u^y & u^z \end{pmatrix}$)

$$\begin{aligned}
\nabla\cdot\left(\lambda\nabla\cdot(\boldsymbol{u})\,\boldsymbol{I} + 2\mu\boldsymbol{e}\right) &= \partial_i\left(\lambda u^k_k\delta^{ij} + \mu\left(u^j_i + u^i_j\right)\right) && (2.2.1)\\
&= \lambda u^k_{kj} + \mu\left(u^j_{ii} + u^i_{ji}\right) && (2.2.2)\\
&= \lambda u^k_{kj} + \mu\left(u^j_{ii} + u^i_{ij}\right) && (2.2.3)\\
&= \lambda u^k_{kj} + \mu\left(u^j_{ii} + u^k_{kj}\right) && (2.2.4)\\
&= (\lambda+\mu)u^k_{kj} + \mu u^j_{ii} && (2.2.5)\\
&= (\lambda+\mu)\nabla\left(\nabla\cdot(\boldsymbol{u})\right) + \mu\Delta(\boldsymbol{u}) && (2.2.6)
\end{aligned}$$

then using the vector identity

$$\nabla\times\left(\nabla\times(\boldsymbol{u})\right) = \nabla\left(\nabla\cdot(\boldsymbol{u})\right) - \nabla\cdot\left(\nabla(\boldsymbol{u})\right) \tag{2.2.7}$$

one finds

$$\nabla\cdot(\boldsymbol{\sigma}) = (\lambda+2\mu)\nabla\left(\nabla\cdot(\boldsymbol{u})\right) - \mu\nabla\times\left(\nabla\times(\boldsymbol{u})\right) \tag{2.2.8}$$

such that the linear momentum conservation writes

$$\boldsymbol{u}_{tt} - \frac{\lambda + 2\mu}{\rho}\nabla\left(\nabla\cdot(\boldsymbol{u})\right) + \frac{\mu}{\rho}\nabla\times\left(\nabla\times(\boldsymbol{u})\right) = \boldsymbol{b}. \tag{2.2.9}$$

(b) If $\nabla\cdot(u) = 0$ then the momentum conservation writes (using the identity Eq.(2.2.7))

$$\begin{aligned}
\boldsymbol{u}_{tt} + \frac{\mu}{\rho}\nabla\times\left(\nabla\times(\boldsymbol{u})\right) &= \boldsymbol{b} && (2.2.10)\\
\boldsymbol{u}_{tt} - \frac{\mu}{\rho}\nabla\cdot\left(\nabla(\boldsymbol{u})\right) &= \boldsymbol{b} && (2.2.11)\\
\boldsymbol{u}_{tt} - \frac{\mu}{\rho}\Delta(\boldsymbol{u}) &= \boldsymbol{b} && (2.2.12)
\end{aligned}$$

thus $c^2 = \frac{\mu}{\rho}$ and $\boldsymbol{f} = \boldsymbol{b}$.
Similarly, if $\nabla \times (u) = 0$ then the momentum conservation writes (using the identity Eq.(2.2.7))

$$\boldsymbol{u}_{tt} - \frac{\lambda + 2\mu}{\rho} \nabla \left(\nabla \cdot (\boldsymbol{u}) \right) = \boldsymbol{b} \tag{2.2.13}$$

$$\boldsymbol{u}_{tt} - \frac{\lambda + 2\mu}{\rho} \nabla \cdot \left(\nabla (\boldsymbol{u}) \right) = \boldsymbol{b} \tag{2.2.14}$$

$$\boldsymbol{u}_{tt} - \frac{\lambda + 2\mu}{\rho} \Delta (\boldsymbol{u}) = \boldsymbol{b} \tag{2.2.15}$$

thus $c^2 = \frac{\lambda+2\mu}{\rho}$ and $\boldsymbol{f} = \boldsymbol{b}$.

2.3 Stokes' equation 🌶🌶

Consider the mass and linear momentum conservation

$$\begin{cases} \dfrac{D\rho}{Dt} + \rho \nabla \cdot (\boldsymbol{v}) = 0 \\ \rho \dfrac{D\boldsymbol{v}}{Dt} - \nabla \cdot (\boldsymbol{\sigma}) = \rho \boldsymbol{b} \end{cases}$$

($\frac{D}{Dt} \triangleq \partial_t + \boldsymbol{v} \cdot \nabla ()$ is the material derivative)
for Newtonian fluids

$$\sigma = -p\boldsymbol{I} + 2\mu \boldsymbol{e}, \quad \boldsymbol{e} = \frac{1}{2} \left(\nabla (\boldsymbol{v}) + \nabla (\boldsymbol{v})^T \right)$$

where $\boldsymbol{v}$ [m/s] is the velocity field, p is the pressure [N/m^2], ρ [kg/m^3] is the density, $\boldsymbol{\sigma}$ [N/m^2] is the stress tensor, $\boldsymbol{b}$ [N/kg] are the external forces and μ [Ns/m^2] is the viscosity.

(a) For incompressible fluids, *i.e* $\frac{D\rho}{Dt} = 0$, show that these equations become

$$\begin{cases} \nabla \cdot (\boldsymbol{v}) = 0 \\ \rho \boldsymbol{v}_t + \rho \boldsymbol{v} \cdot \nabla (\boldsymbol{v}) - \mu \Delta (\boldsymbol{v}) + \nabla (p) = \rho \boldsymbol{b} \end{cases}$$

Classify the second equation of this system considering only the velocity field is unknown.

(b) The stream function $\boldsymbol{\psi}$ is such that $\boldsymbol{v} = \nabla \times (\boldsymbol{\psi})$. Determine $\boldsymbol{v}$ for $\boldsymbol{\psi} = \psi(x, y)\hat{z}$.

(c) At low Reynolds numbers, the terms $\rho \boldsymbol{v}_t$ and $\rho \boldsymbol{v} \cdot \nabla (\boldsymbol{v})$ can be neglected. In that case and if $\boldsymbol{b} = 0$, show that the curl of the momentum conservation can then be written

$$\Delta (\Delta (\psi)) = 0.$$

Classify this equation.

Solution

(a) The divergence of the stress tensor is given by (using the sommation convention)($\boldsymbol{v} = \begin{pmatrix} v^x & v^y & v^z \end{pmatrix}$)

$$\begin{aligned}
\nabla \cdot (-p\boldsymbol{I} + 2\mu\boldsymbol{e}) &= \partial_i \left(-p\delta^{ij} + \mu\left(v_i^j + v_j^i\right)\right) && (2.3.1)\\
&= -p_j + \mu\left(v_{ii}^j + v_{ji}^i\right) && (2.3.2)\\
&= -p_j + \mu\left(v_{ii}^j + v_{ij}^i\right) && (2.3.3)\\
&= -\nabla(p) + \mu\Delta(\boldsymbol{v}) + \mu\nabla(\nabla\cdot(\boldsymbol{v})) && (2.3.4)\\
&= -\nabla(p) + \mu\Delta(\boldsymbol{v}). && (2.3.5)
\end{aligned}$$

The system is

- non homogeneous ($\rho\boldsymbol{b} \neq 0$)
- non linear ($\rho\boldsymbol{v}\cdot\nabla(\boldsymbol{v})$)
- second order in space ($\mu\Delta(\boldsymbol{v})$) and first order in time ($\rho\boldsymbol{v}_t$).

(b)

$$\boldsymbol{v} = \nabla\times(\boldsymbol{\psi}) = \nabla\times(\psi(x,y)\hat{\boldsymbol{z}}) = \psi_y\hat{\boldsymbol{x}} - \psi_x\hat{\boldsymbol{y}} \qquad (2.3.6)$$

Thus $\boldsymbol{v}$ is in the $x-y$ plane and does not depend on z.

(c) The Laplacien of the velocity vector is given by

$$\begin{aligned}
\Delta(\boldsymbol{v}) &= \Delta\left(v^x\hat{\boldsymbol{x}} + v^y\hat{\boldsymbol{y}}\right) && (2.3.7)\\
&= \left(v_{xx}^x + v_{yy}^x\right)\hat{\boldsymbol{x}} + \left(v_{xx}^y + v_{yy}^y\right)\hat{\boldsymbol{y}}. && (2.3.8)
\end{aligned}$$

The curl of a vector in the $x-y$ plane that does not depend on z, *i.e.* $\boldsymbol{a} = a^x(x,y)\hat{\boldsymbol{x}} + a^y(x,y)\hat{\boldsymbol{y}}$, is given by

$$\nabla\times(\boldsymbol{a}) = \hat{\boldsymbol{z}}\left(a_x^y - a_y^x\right). \qquad (2.3.9)$$

Then combining these two results gives

$$\begin{aligned}
\nabla\times(\Delta(\boldsymbol{v})) &= \hat{\boldsymbol{z}}\left(\partial_x\left(v_{xx}^y + v_{yy}^y\right) - \partial_y\left(v_{xx}^x + v_{yy}^x\right)\right) && (2.3.10)\\
&= \hat{\boldsymbol{z}}\left(v_{xxx}^y + v_{yyx}^y - v_{xxy}^x - v_{yyy}^x\right). && (2.3.11)
\end{aligned}$$

Finally the curl of the momentum conservation is

$$\begin{aligned}
&\nabla\times(\mu\Delta(\boldsymbol{v})) &&= \nabla\times(\nabla(p)) && (2.3.12)\\
\Rightarrow &\hat{\boldsymbol{z}}\left(v_{yyx}^y + v_{xxx}^y - v_{xxy}^x - v_{yyy}^x\right) &&= 0 && (2.3.13)\\
\Rightarrow &-\psi_{xyyx} - \psi_{xxxx} - \psi_{yxxy} - \psi_{yyyy} &&= 0 && (2.3.14)\\
\Rightarrow &\left(\partial_{xx} + \partial_{yy}\right)\left(\psi_{xx} + \psi_{yy}\right) &&= 0 && (2.3.15)\\
\Rightarrow &\Delta(\Delta(\psi)) &&= 0 && (2.3.16)
\end{aligned}$$

The system is

- homogeneous
- linear
- fourth order in space and zeroth order in time.

2.4 Heat equation 🌶️🌶️

Consider the energy conservation

$$\rho \frac{De}{Dt} - \boldsymbol{\sigma} : \nabla (\boldsymbol{v}) + \nabla \cdot (\boldsymbol{q}) - \rho s = 0 \tag{2.4.1}$$

($\frac{D}{Dt} \triangleq \partial_t + \boldsymbol{v} \cdot \nabla\,()$ is the material derivative)
where $\boldsymbol{v}$ [m/s] is the velocity field, ρ [kg/m^3] is the density, $\boldsymbol{\sigma}$ [N/m^2] is the stress tensor, s [W/kg] are the external sources, e [J/kg] is the internal energy and $\boldsymbol{q}$ [W/m^2] is the heat flux.

(a) Considering that the internal energy is proportional to the temperature, ***i.e*** $e = cT$ while the heat flux is proportional to the temperature gradient, ***i.e*** $\boldsymbol{q} = -k\nabla (T)$ (c and k are constants) and that the velocity field and the stress tensor are known. Show that the conservation of energy becomes

$$T_t - \alpha \Delta (T) + \boldsymbol{\beta} \cdot \nabla (T) = f. \tag{2.4.2}$$

Determine α, $\boldsymbol{\beta}$ and f. Classify this equation.

Solution

(a) With the assumptions, the conservation of energy becomes

$$\rho \frac{De}{Dt} - \boldsymbol{\sigma} : \nabla (\boldsymbol{v}) + \nabla \cdot (\boldsymbol{q}) - \rho s = 0 \tag{2.4.3}$$

$$\rho c T_t + \rho c \boldsymbol{v} \cdot \nabla (T) - \boldsymbol{\sigma} : \nabla (\boldsymbol{v}) - k \nabla \cdot (\nabla (T)) - \rho s = 0 \tag{2.4.4}$$

$$T_t - \frac{k}{\rho c} \Delta (T) + \boldsymbol{v} \cdot \nabla (T) = \frac{\boldsymbol{\sigma} : \nabla (\boldsymbol{v})}{\rho c} + \frac{s}{c} \tag{2.4.5}$$

thus $\alpha = \frac{k}{\rho c}$, $\boldsymbol{\beta} = \boldsymbol{v}$ and $f = \frac{\boldsymbol{\sigma}:\nabla(\boldsymbol{v})}{\rho c} + \frac{s}{c}$.
This equation is

- non homogeneous
- linear
- second order in space and first order in time
- parabolic

Indeed, it can be written (for a single spatial dimension)

$$A T_{tt} + B T_{tx} + C\ T_{xx} + D T_t + E\,T_x + F u = G \tag{2.4.6}$$

$$0 T_{tt} + 0\ T_{tx} - \frac{k}{\rho c} T_{xx} + 1\ T_t + v^x T_x + 0\ u = G \tag{2.4.7}$$

thus

$$B^2 - 4AC = 0. \tag{2.4.8}$$

2.5 Nonlinear transport [Olver, Section 2.3] 🌶

Find the conserved density and the flux for the nonlinear transport equation

$$u_t + uu_x = 0.$$

Solution

$$c = u \quad \text{and} \quad f = \frac{1}{2}u^2. \tag{2.5.1}$$

2.6 Third order equation [Olver, Exercice 8.5.7] 🌶🌶

Consider the third order equation

$$u_t + u_{xxx} = 0.$$

(a) Find a trivial conserved quantity and its associated flux.

(b) Show that $c = u^2$ is a conserved quantity. Give the associated flux.

(c) For both couple, give a boundary condition such that C is constant.

Solution

$$c = u \quad \text{and} \quad f = u_{xx} \quad \text{or} \quad c = u^2 \quad \text{and} \quad f = 2uu_{xx} - u_x^2. \tag{2.6.1}$$

A suitable boundary condition is that the flux cancels on the boundaries, *i.e*

$$f = 0 \quad \text{on} \quad \partial\Omega. \tag{2.6.2}$$

Another suitable boundary condition is that the ingoing and the outgoing part of the flux cancels, *i.e*

$$f = \text{Cst} \quad \text{on} \quad \partial\Omega. \tag{2.6.3}$$

Indeed for both cases, one has

$$F = \int_{\partial\Omega} f \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{n}} \, d\partial\Omega = 0 \quad \Rightarrow \quad \frac{\partial C}{\partial t} = 0. \tag{2.6.4}$$

2.7 Korteweg-de Vries equation [Olver, Exercice 8.5.18] 🌶🌶🌶

Consider the Korteweg-de Vries equation

$$u_t + u_{xxx} + uu_x = 0.$$

Show that $c = u$, $c = u^2$, $c = u_x^2 + \mu u^3$ (for a suitable constant μ) are conserved densities.

Solution

Easy for $c = u$, rather easy for $c = u^2$. For the third conserved quantity

$$\partial_t c = 2u_x u_{xt} + \mu 3u^2 u_t \tag{2.7.1}$$

then using the KdV equation and its derivative w.r.t x,

$$\begin{cases} u_t = -u_{xxx} - uu_x \\ u_{xt} = -u_{xxxx} - u_x^2 - uu_{xx} \end{cases} \tag{2.7.2}$$

it becomes

$$\partial_t c = -2u_x(u_{xxxx} + u_x^2 + uu_{xx}) - \mu 3u^2(u_{xxx} + uu_x). \tag{2.7.3}$$

The three following identities

$$\begin{cases} u_x u_{xxxx} = \partial_x(u_x u_{xxx} - \frac{1}{2}u_{xx}^2) \\ u_x^3 = \partial_x(uu_x^2) - 2uu_x u_{xx} \\ u^3 u_x = \partial_x(\frac{1}{4}u^4) \end{cases} \tag{2.7.4}$$

further simplify the derivative of the conserved density to

$$\partial_t c = -2\partial_x(u_x u_{xxx} - \frac{1}{2}u_{xx}^2) - 2\partial_x(uu_x^2) + 4uu_x u_{xx} - 2uu_x u_{xx} - \mu 3u^2 u_{xxx} - \mu 3\partial_x(\frac{1}{4}u^4). \tag{2.7.5}$$

The only term that is not yet expressed as a derivative of x is

$$2uu_x u_{xx} - \mu 3u^2 u_{xxx} \tag{2.7.6}$$

and provided $\mu = -1/3$ it can be written

$$2uu_x u_{xx} + u^2 u_{xxx} = \partial_x(u^2 u_{xx}). \tag{2.7.7}$$

Finally

$$\partial_t c = \partial_x(-2u_x u_{xxx} + u_{xx}^2 - 2uu_x^2 + u^2 u_{xx} + \frac{1}{4}u^4). \tag{2.7.8}$$

and the associated flux is

$$f = -\frac{1}{4}u^4 - u^2 u_{xx} + 2uu_x^2 + 2u_x u_{xxx} - u_{xx}^2. \tag{2.7.9}$$

2.8 Modified Korteweg-de Vries equation [Olver, Exercice 8.5.19] 🌶️🌶️

Consider the modified Korteweg-de Vries equation

$$u_t + u_{xxx} + u^2 u_x = 0.$$

Show that $c = u$ and $c = \frac{1}{2}u^2$ are two conserved densities and give the associated flux.

Solution

$$c = u \text{ and } f = u_{xx} + \frac{1}{3}u^3. \tag{2.8.1}$$

$$c = \frac{1}{2}u^2 \text{ and } f = uu_{xx} - \frac{1}{2}u_x^2 + \frac{1}{4}u^4. \tag{2.8.2}$$

2.9 Benjamin-Bona-Mahony equation [Olver, Exercice 8.5.19] 🌶️🌶️

Consider the Benjamin-Bona-Mahony equation

$$u_t - u_{xxt} - uu_x = 0.$$

Show that $c = u$, $c = \frac{1}{2}(u^2 + u_x^2)$ and $\frac{1}{3}u^3$ are three conserved densities and give the associated flux.

Solution

$$c = u \text{ and } f = -u_{xt} - \frac{1}{2}u^2. \tag{2.9.1}$$

$$c = \frac{1}{2}(u^2 + u_x^2) \text{ and } f = -uu_{xt} - \frac{1}{3}u^3. \tag{2.9.2}$$

$$c = \frac{1}{3}u^3 \text{ and } f = u_t^2 - u_{xt}^2 - u^2u_{xt} - \frac{1}{4}u^4. \tag{2.9.3}$$

2.10 Damped string [Strauss 2.2, Ex. 5] ([Lecture 3, Slide 48] revisited)🌶️

The equation of motion for a damped string is given by

$$u_{tt} - c^2u_{xx} + ru_t = 0, \quad r > 0.$$

The kinetic energy is defined here as

$$K = \frac{1}{2}\int_{\mathbb{R}} (u_t)^2 \, dx$$

while the potential is defined as

$$P = \frac{1}{2}\int_{\mathbb{R}} c^2 \left(u_x^2\right) \, dx.$$

(a) Show that, in this case, and assuming u and its derivatives vanish at infinity, the energy $E = K + P$ decreases.

(b) Instead of the total amount of energy, we study the energy density $e \triangleq \frac{1}{2}\left(u_t^2 + c^2u_x^2\right)$. If $d \triangleq \int_{t_0}^{t} ru_t^2(x,\tau)\, d\tau$, show that $e + d$ is a conserved quantity, *i.e.* show that there exists a flux f (and give its expression) such that $\frac{\partial(e+d)}{\partial t} + \frac{\partial f}{\partial x} = 0$. Give a physical interpretation of d and the parameter r.

Solution

(a) The energy decreases with time provided that

$$\frac{dE}{dt} < 0. \tag{2.10.1}$$

By definition, this derivative is given by

$$\begin{aligned}
\frac{dE}{dt} &= \frac{dK}{dt} + \frac{dP}{dt} && (2.10.2)\\
&= \int_{\mathbb{R}} u_t u_{tt}\, dx + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} c^2 u_x^2\, dx && (2.10.3)\\
&= \int_{\mathbb{R}} u_t c^2 u_{xx}\, dx - \int_{\mathbb{R}} r u_t u_t\, dx + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} c^2 u_x^2\, dx && (2.10.4)\\
&= -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} c^2 u_x^2\, dx + c^2\left[u_x u_t\right]_{-\infty}^{\infty} - \int_{\mathbb{R}} r u_t^2\, dx + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} c^2 u_x^2\, dx && (2.10.5)\\
&= -\int_{\mathbb{R}} r u_t^2\, dx < 0. && (2.10.6)
\end{aligned}$$

(b) First, we notice that, by construction, $\frac{\partial d}{\partial t} = r u_t^2$. We then compute $\frac{\partial e}{\partial t}$:

$$\begin{aligned}
\frac{\partial e}{\partial t} &= u_t u_{tt} + c^2 u_x u_{xt} && (2.10.7)\\
&= u_t\left(c^2 u_{xx} - r u_t\right) + c^2 u_x u_{xt} && (2.10.8)\\
&= \left(c^2 u_t u_{xx} + c^2 u_x u_{xt}\right) - \frac{\partial d}{\partial t} && (2.10.9)\\
&= \frac{\partial\left(c^2 u_t u_x\right)}{\partial x} - \frac{\partial d}{\partial t}. && (2.10.10)
\end{aligned}$$

This yields the requested equation with $f = -c^2 u_t u_x$.

2.11 Diffusion equation

Consider the diffusion equation

$$u_t - \alpha u_{xx} = 0 \quad (x,t) \in]0,1[\times\mathbb{R}^+$$

(a) Find a trivial conserved density and the associated flux.

(b) Give a physical interpretation of the conservation law when either u is a temperature field ([K]) or u a concentration field ([kg/m]).

(c) Consider now the boundary conditions

$$u_x(0) = -a \qquad u_x(1) = b.$$

Give the physical interpretation of a and b.

Solution

(a) This diffusion equation can be written

$$u_t - \alpha u_{xx} = \partial_t(u) + \partial_x(-\alpha u_x) \tag{2.11.1}$$

thus u is a conserved density whose associated flux is $-\alpha u_x$.

(b) **Concentration** The conserved quantity is

$$\int_0^1 u\,dx. \tag{2.11.2}$$

Its dimensions are $[u][dx]$ = [kg/m][m] = [kg]. This conserved quantity thus indicate the conservation of mass.

Temperature The conserved quantity is

$$\int_0^1 u\,dx \tag{2.11.3}$$

Multiplying this quantity by a constant volumetric thermal capacity c ([J/K/m]) then also gives a conserved quantity which writes

$$\int_0^1 cu\,dx \tag{2.11.4}$$

and whose dimensions are $[c][u][dx]$ = [J/K/m][K][m] = [J]. These two equivalent conserved quantities thus indicate the conservation of energy. (Indeed the energy density writes $e = cu$).

(c) The conservation law writes

$$\frac{\partial C}{\partial t} = -F \tag{2.11.5}$$

with, in this case

$$C = \int_0^1 u\,dx \tag{2.11.6}$$

and

$$\begin{aligned} F &= \int_0^1 \partial_x f\,dx && (2.11.7)\\ &= [f]_0^1 && (2.11.8)\\ &= [-\alpha u_x]_0^1 && (2.11.9)\\ &= (-\alpha u_x(1)) - (-\alpha u_x(0)) && (2.11.10)\\ &= \alpha(u_x(0) - u_x(1)). && (2.11.11) \end{aligned}$$

The boundary parameters a and b then controls the flux of mass (resp. energy) going in or out of the interval $[0, 1]$. Indeed

$$\begin{aligned} F &= \alpha(u_x(0) - u_x(1)) && (2.11.12)\\ &= \alpha(-a - b) && (2.11.13)\\ &= -\alpha(a + b). && (2.11.14) \end{aligned}$$

When $a > 0$ mass (resp. energy) enter the interval at $x = 0$. At the opposite, when $a < 0$ mass (resp. energy) leave the interval at $x = 0$. The interpretation for b is the same, but for $x = 1$.

2.12 Schrödinger equation 🌶️🌶️

Consider the homogeneous Schrodinger equation for one particle in one-dimension

$$u_t - iu_{xx} = 0.$$

(a) Show that $c = u^* u$ is a conserved quantity (* denotes the complex conjugate). Give the associated flux.

(b) Give a physical interpretation of the conservation law when u is a quantum wave function ([m$^{-1/2}$]).

Solution

Let us first compute $\partial_t (u^* u)$:

$$\begin{aligned}
\partial_t (u^* u) &= (u_t^* u) + (u^* u_t) && (2.12.1)\\
&= -i u_{xx}^* u + i u^* u_{xx} && (2.12.2)\\
&= -i \left(\partial_x (u u_x^*) - (u_x u_x^*)\right) + i \left(\partial_x (u_x u^*) - (u_x u_x^*)\right) && (2.12.3)\\
&= i \partial_x (u_x u^* - u u_x^*) && (2.12.4)
\end{aligned}$$

The associated flux is thus

$$f = -i(u^* u_x - u u_x^*). \qquad (2.12.5)$$

Physically speaking, the quantity uu^* is a probability density. Its integral must be 1 at all time, which implies it should be conserved.

3 Von Neumann stability analysis

3.1 Advection equation 🌶️

Consider the advection equation

$$u_t + au_x = 0.$$

(a) Using forward finite differences in time and space, *i.e*

$$u_x \approx \frac{u_{j+1}^n - u_j^n}{\Delta x} \text{ and } u_t \approx \frac{u_j^{n+1} - u_j^n}{\Delta t},$$

show that the scheme is explicit by giving the update equation $u_j^{n+1} = f(u_{j+1}^n, u_j^n)$.

(b) Establish a stability criterion for this scheme using Von Neumann analysis.

(c) Why does the stability criterion depend on the sign of a? What happens if a downward space difference, *i.e*

$$u_x \approx \frac{u_j^n - u_{j-1}^n}{\Delta x},$$

is used instead?
Hint: Remember that the general solution of the transport equation is $f(x - at)$.

(d) To remove the dependency on the sign of a, we try a symmetrical scheme using centered differences:

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right).$$

Establish a stability criterion using Von Neumann analysis.

(e) To improve the previous scheme, we replace the initial value by the average of the neighboring values. This scheme is called the Lax-Friedrichs scheme, whose update equation is

$$u_j^{n+1} = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) - \frac{a\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right).$$

Establish a stability criterion using Von Neumann analysis.

(f) Finally, consider the so-called Lax-Wendroff scheme, whose update equation is

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{1}{2}\left(\frac{a\Delta t}{\Delta x}\right)^2\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right).$$

Establish a stability criterion using Von Neumann analysis.

Solution

(a) Using finite differences approximation, the transport equation writes as

$$u_j^{n+1} = (1+\alpha)\, u_j^n - \alpha u_{j+1}^n \tag{3.1.1}$$

where $\alpha = \frac{a\Delta t}{\Delta x}$.

(b) Introducing any error mode

$$\epsilon_k\left(x,t\right) = \hat{\epsilon}(k,t)\exp\left(ikx\right) \tag{3.1.2}$$

in the discrete equation gives

$$\hat{\epsilon}(k,t_{n+1}) = \left(1+\alpha\right)\hat{\epsilon}(k,t_n) - \alpha\hat{\epsilon}(k,t_n)\exp\left(ik\Delta x\right). \tag{3.1.3}$$

Hence the amplification factor is given by

$$\xi_1 = 1+\alpha\left(1-\exp\left(ik\Delta x\right)\right) \tag{3.1.4}$$

such that the condition on α is

$$\begin{aligned}
& \left|\xi_1\right|^2 && \leq 1 && (3.1.5)\\
\Rightarrow\quad & \left|1+\alpha\left(1-\exp\left(ik\Delta x\right)\right)\right|^2 && \leq 1 && (3.1.6)\\
\Rightarrow\quad & 1+2\alpha\left(\alpha+1\right)\left(1-\cos\left(k\Delta x\right)\right) && \leq 1 && (3.1.7)\\
\Rightarrow\quad & 2\alpha\left(\alpha+1\right)\left(1-\cos\left(k\Delta x\right)\right) && \leq 0 && (3.1.8)\\
\Rightarrow\quad & 2\alpha\left(\alpha+1\right) && \leq 0 && (3.1.9)\\
\Rightarrow\quad & -1\leq\alpha\leq 0. && && (3.1.10)
\end{aligned}$$

(c) This is because the numerical domain of dependence must not be disjoint from the real domain of dependence.

(d) Introducing any error mode

$$\epsilon_k\left(x,t\right) = \hat{\epsilon}(k,t)\exp\left(ikx\right) \tag{3.1.11}$$

in the discrete equation gives

$$\frac{\hat{\epsilon}(k,t_{n+1})}{\hat{\epsilon}(k,t_n)} = \xi_1 = 1-\frac{\alpha}{2}\left[\exp(ik\Delta x)-\exp(-ik\Delta x)\right] \tag{3.1.12}$$

$$= 1-i\alpha\sin(k\Delta x) \tag{3.1.13}$$

thus the norm squared is given by

$$\left|\xi_1\right|^2 = \cos^2(k\Delta x)+\alpha^2\sin^2(k\Delta x) \tag{3.1.14}$$

$$= 1+\alpha^2\sin^2(k\Delta x). \tag{3.1.15}$$

Hence, the centered differences scheme is unconditionally unstable!

(e) Introducing any error mode

$$\epsilon_k\left(x,t\right) = \hat{\epsilon}(k,t)\exp\left(ikx\right) \tag{3.1.16}$$

in the discrete equation gives

$$\frac{\hat{\epsilon}(k,t_{n+1})}{\hat{\epsilon}(k,t_n)} = \xi_1 = \frac{1}{2}\left[\exp(ik\Delta x)+\exp(-ik\Delta x)\right]-\frac{\alpha}{2}\left[\exp(ik\Delta x)-\exp(-ik\Delta x)\right] \tag{3.1.17}$$

$$= \cos(k\Delta x)-i\alpha\sin(k\Delta x) \tag{3.1.18}$$

thus the norm squared is given by

$$|\xi_1|^2 = \cos^2(k\Delta x) + \alpha^2 \sin^2(k\Delta x) \tag{3.1.19}$$
$$= 1 + \left[\alpha^2 - 1\right] \sin^2(k\Delta x). \tag{3.1.20}$$

Hence, the Lax-Friedrichs scheme is stable if

$$-1 \leq \frac{a\Delta t}{\Delta x} \leq 1. \tag{3.1.21}$$

(f) Introducing any error mode

$$\epsilon_k\,(x,t) = \hat{\epsilon}(k,t)\exp\left(ikx\right) \tag{3.1.22}$$

in the discrete equation gives

$$\frac{\hat{\epsilon}(k,t_{n+1})}{\hat{\epsilon}(k,t_n)} = \xi_1 = 1 - \frac{\alpha}{2}\left(\exp\left(-ik\Delta x\right) - \exp\left(ik\Delta x\right)\right) + \frac{\alpha^2}{2}\left(\exp\left(-ik\Delta x\right) - 2 + \exp\left(ik\Delta x\right)\right) \tag{3.1.23}$$

$$= 1 + i\alpha \sin(k\Delta x) + \alpha^2\left(\cos(k\Delta x) - 1\right). \tag{3.1.24}$$

thus the norm squared is given by

$$|\xi_1|^2 = 1 + \alpha^4\left(\cos(k\Delta x) - 1\right)^2 + 2\alpha^2\left(\cos(k\Delta x) - 1\right) + \alpha^2\sin(k\Delta x)^2 \tag{3.1.25}$$
$$= 1 + \alpha^4\left(\cos(k\Delta x) - 1\right)^2 + \alpha^2\left(2\cos(k\Delta x) - 2\right) + \alpha^2\left(1 - \cos(k\Delta x)^2\right) \tag{3.1.26}$$
$$= 1 + \alpha^4\left(\cos(k\Delta x) - 1\right)^2 - \alpha^2\left(\cos(k\Delta x)^2 - 2\cos(k\Delta x) + 1\right) \tag{3.1.27}$$
$$= 1 + \alpha^4\left(\cos(k\Delta x) - 1\right)^2 - \alpha^2\left(\cos(k\Delta x) - 1\right)^2 \tag{3.1.28}$$
$$= 1 + \alpha^2\left(\alpha^2 - 1\right)\left(\cos(k\Delta x) - 1\right)^2. \tag{3.1.29}$$

Hence, the Lax-Wendroff scheme is stable if

$$-1 \leq \frac{a\Delta t}{\Delta x} \leq 1. \tag{3.1.30}$$

3.2 Diffusion equation in 2D 🌶🌶

Consider the 2D diffusion equation

$$u_t - a(u_{xx} + u_{yy}) = 0.$$

Using forward differences in time and central differences in space, establish a stability criterion using the Von Neumann stability analysis.

Solution

First, discretize the 2D diffusion equation with spatial step size Δx, Δy and temporal step size Δt.

- Forward in time:

$$u_t = \frac{u_{r,s}^{n+1} - u_{r,s}^n}{\Delta t}. \tag{3.2.1}$$

- Centered in space:

$$u_{xx} = \frac{u^n_{r+1,s} - 2u^n_{r,s} + u^n_{r-1,s}}{\Delta x^2}, \tag{3.2.2}$$
$$u_{yy} = \frac{u^n_{r,s+1} - 2u^n_{r,s} + u^n_{r,s-1}}{\Delta y^2}. \tag{3.2.3}$$

The 2D diffusion equation can thus be expressed as

$$\frac{u^{n+1}_{r,s} - u^n_{r,s}}{\Delta t} - a\left(\frac{u^n_{r+1,s} - 2u^n_{r,s} + u^n_{r-1,s}}{\Delta x^2} + \frac{u^n_{r,s+1} - 2u^n_{r,s} + u^n_{r,s-1}}{\Delta y^2}\right) = 0 \tag{3.2.4}$$

or as

$$u^{n+1}_{r,s} = u^n_{r,s} + \frac{a\Delta t}{\Delta x^2}\left(u^n_{r+1,s} - 2u^n_{r,s} + u^n_{r-1,s}\right) + \frac{a\Delta t}{\Delta y^2}\left(u^n_{r,s+1} - 2u^n_{r,s} + u^n_{r,s-1}\right). \tag{3.2.5}$$

Introducing

$$\beta_x = \frac{a\Delta t}{\Delta x^2} \quad \text{and} \quad \beta_y = \frac{a\Delta t}{\Delta y^2}, \tag{3.2.6}$$

the update equation is then

$$u^{n+1}_{r,s} = u^n_{r,s} + \beta_x\left(u^n_{r+1,s} - 2u^n_{r,s} + u^n_{r-1,s}\right) + \beta_y\left(u^n_{r,s+1} - 2u^n_{r,s} + u^n_{r,s-1}\right). \tag{3.2.7}$$

The stability criterion is obtained by injecting an arbitrary error mode

$$\epsilon^n_{rs} = \epsilon(x_r, y_s, t_n) = \hat{\epsilon}\left(k_x, k_y, t_n\right)\exp\left(i(k_x r\Delta x + k_y s\Delta y)\right) \tag{3.2.8}$$

into the discretized equation. For conciseness, $\hat{\epsilon}\left(k_x, k_y, t_n\right)$ is written as $\hat{\epsilon}_n$.
The evolution of the amplitude of any mode is then governed by

$$\begin{aligned}\hat{\epsilon}_{n+1}\exp\left(i(k_x r\Delta x + k_y s\Delta y)\right) &= \hat{\epsilon}_n\exp\left(i(k_x r\Delta x + k_y s\Delta y)\right)\\ &+ \beta_x\big[\hat{\epsilon}_n\exp\left(i(k_x(r+1)\Delta x + k_y s\Delta y)\right) - 2\hat{\epsilon}_n\exp\left(i(k_x r\Delta x + k_y s\Delta y)\right)\\ &\qquad + \hat{\epsilon}_n\exp\left(i(k_x(r-1)\Delta x + k_y s\Delta y)\right)\big]\\ &+ \beta_y\big[\hat{\epsilon}_n\exp\left(i(k_x r\Delta x + k_y(s+1)\Delta y)\right) - 2\hat{\epsilon}_n\exp\left(i(k_x r\Delta x + k_y s\Delta y)\right)\\ &\qquad + \hat{\epsilon}_n\exp\left(i(k_x r\Delta x + k_y(s-1)\Delta y)\right)\big].\end{aligned} \tag{3.2.9}$$

Dividing both sides by $\hat{\epsilon}_n\exp\left(i(k_x r\Delta x + k_y s\Delta y)\right)$ yields

$$\frac{\hat{\epsilon}_{n+1}}{\hat{\epsilon}_n} = \xi_1 = 1 + \beta_x\left(e^{ik_x\Delta x} - 2 + e^{-ik_x\Delta x}\right) + \beta_y\left(e^{ik_y\Delta y} - 2 + e^{-ik_y\Delta y}\right) \tag{3.2.10}$$
$$= 1 + 2\beta_x\left(\cos k_x\Delta x - 1\right) + 2\beta_y\left(\cos k_y\Delta y - 1\right) \tag{3.2.11}$$
$$= 1 - 4\beta_x\sin^2\frac{k_x\Delta x}{2} - 4\beta_y\sin^2\frac{k_y\Delta y}{2}. \tag{3.2.12}$$

Ensuring that no mode is divergent, *i.e.*

$$\left|\xi_1\right|^2 \leq 1 \quad \forall k_x, k_y \tag{3.2.13}$$

gives

$$\begin{aligned} -1 \le 1 - 4\beta_x \sin^2 \frac{k_x \Delta x}{2} - 4\beta_y \sin^2 \frac{k_y \Delta y}{2} &\le 1 \qquad (3.2.14)\\ \Rightarrow \quad \frac{1}{2} \ge \beta_x \sin^2 \frac{k_x \Delta x}{2} + \beta_y \sin^2 \frac{k_y \Delta y}{2} &\ge 0. \qquad (3.2.15) \end{aligned}$$

These two inequalities must be verified for any k_x, k_y. The most restrictive cases for the rightmost inequality are obtained by considering $\left\{\sin^2 \frac{k_x \Delta x}{2}, \sin^2 \frac{k_y \Delta y}{2}\right\} = \{1, 0\}$ and $\left\{\sin^2 \frac{k_x \Delta x}{2}, \sin^2 \frac{k_y \Delta y}{2}\right\} = \{0, 1\}$ which yields

$$\beta_x \ge 0 \quad \text{and} \quad \beta_y \ge 0. \qquad (3.2.16)$$

Then β_x and β_y being positive, the most restrictive case for the leftmost inequality is obtained considering $\left\{\sin^2 \frac{k_x \Delta x}{2}, \sin^2 \frac{k_y \Delta y}{2}\right\} = \{1, 1\}$ which yields

$$\frac{1}{2} \ge \beta_x + \beta_y = a\Delta t \left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right) \triangleq \frac{a\Delta t}{\Delta l^2}. \qquad (3.2.17)$$

or

$$\frac{\Delta t}{\Delta l^2} \le \frac{1}{2a}. \qquad (3.2.18)$$

3.3 Wave equation 🌶️🌶️

Consider the 1D wave equation

$$u_{tt} = c^2 u_{xx}.$$

Using the centered differences for both time and space, verify the stability of the numerical method using a Von Neumann analysis.

Solution

The discrete form of the wave equation can be written as

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{(\Delta t)^2} = c^2 \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}, \qquad (3.3.1)$$

forming the following update equation

$$u_j^{n+1} = s(u_{j+1}^n + u_{j-1}^n) + 2(1-s)u_j^n - u_j^{n-1}, \qquad (3.3.2)$$

with $s = c^2(\Delta t)^2/(\Delta x)^2$.
Introducing any error mode

$$\begin{aligned} \epsilon_k(x,t) &= \hat{\epsilon}(k,t)\ \exp(ikx), \qquad (3.3.3)\\ &= \hat{\epsilon}_n\ [\exp(ik\Delta x)]^j, \qquad (3.3.4) \end{aligned}$$

in the update equation gives

$$\hat{\epsilon}_{n+1} = s[\exp(ik\Delta x) + \exp(-ik\Delta x)]\ \hat{\epsilon}_n + 2(1-s)\ \hat{\epsilon}_n - \hat{\epsilon}_{n-1}, \tag{3.3.5}$$
$$= 2s\cos(k\Delta x)\ \hat{\epsilon}_n + 2(1-s)\ \hat{\epsilon}_n - \hat{\epsilon}_{n-1}. \tag{3.3.6}$$

If $p = s[\cos(k\Delta x) - 1]$, the recurrence relation can be rewritten equivalently as

$$\hat{\epsilon}_{n+2} - 2(1+p)\hat{\epsilon}_{n+1} + \hat{\epsilon}_n = 0. \tag{3.3.7}$$

The characteristic polynomial of this expression

$$\xi^2 - 2(1+p)\ \xi + 1 = 0, \tag{3.3.8}$$

which has two roots

$$\xi_{\frac{1}{2}} = 1 + p \pm \sqrt{p^2 + 2p}. \tag{3.3.9}$$

Provided both roots are distinct (*i.e* $\xi_1 \neq \xi_2$)), the solution of the recurrence relation in Eq.(3.3.7) has the form

$$\hat{\epsilon}_n = A\xi_1^n + B\xi_2^n, \quad ⚠\ n \text{ is an exponent} \tag{3.3.10}$$

where A and B are both constants that depend on the initial error.
Hence, the error does not diverge if modulus of the amplification factors verify simultaneously

$$|\xi_1| \leq 1 \quad \text{and} \quad |\xi_2| \leq 1. \tag{3.3.11}$$

Two different cases are taken into account according to the sign of the value inside the square root.
Reminder: The modulus of a complex number $|a + ib| = \sqrt{a^2 + b^2}$.

Case 1: $p^2 + 2p > 0 \qquad \Rightarrow p < -2$
There are two real roots, therefore both solutions must verify the relation

$$-1 \leq 1 + p \pm \sqrt{p^2 + 2p} \leq 1, \tag{3.3.12}$$
$$-2 - p \leq \pm\sqrt{p^2 + 2p} \leq -p. \tag{3.3.13}$$

It can be deduced that the solution ξ_2 cannot fulfil the condition

$$-\sqrt{p^2 + 2p} \geq -2 - p. \tag{3.3.14}$$

The left part of this inequality is always negative while the right part can only be positive as $p < -2$ in this case. Hence, $|\xi_2| > 1$ and is unstable. If only one of the modes diverges (in this case ξ_2), the whole solution is not stable.

Case 2: $p^2 + 2p < 0 \qquad \Rightarrow p > -2$
In both roots, the first term is real while the second one is imaginary and their moduli are equal to

$$|\xi_{\frac{1}{2}}|^2 = (1+p)^2 + (-p^2 - 2p) \tag{3.3.15}$$
$$= 1. \tag{3.3.16}$$

Then, the solution is always stable when $p > -2$ or equivalently when

$$s[\cos(k\Delta x) - 1] > -2, \tag{3.3.17}$$

$$\frac{2}{1-\cos(k\Delta x)} > s. \tag{3.3.18}$$

In the most restrictive case, the smallest value of the left part of the previous inequation is obtained when $\cos(k\Delta x) = -1$.
Thus, the method is stable when

$$c^2\frac{\Delta t^2}{\Delta x^2} < 1. \tag{3.3.19}$$

Case 3: $p^2 + 2p = 0 \qquad \Rightarrow p = -2$
If $\rho = 0$ the multiplicity of the root is two. The general solution of the recurrence relation Eq.(3.3.7) is then

$$\hat{\epsilon}_n = (An + B)\ (\xi_{1,2})^n. \tag{3.3.20}$$

In this particular case, one has

$$p = -2 \Rightarrow \xi_{1,2} = -1 \tag{3.3.21}$$

and thus the error evolves as

$$\hat{\epsilon}_n = (An + B)\ (-1)^n. \tag{3.3.22}$$

It implies that the errors only grow linearly with n. For the this reason, the solution is said to be "stable" when $p = -2$.

Note: Although the exercise is limited to stating the error "for all constants A and B," it is still possible to determine their expressions. Considering that the first two values of the recurrence relation are known, the constants A and B can easily be determined so that the error writes

$$\hat{\epsilon}_n = [-(\hat{\epsilon}_0 + \hat{\epsilon}_1)n + \hat{\epsilon}_0]\ (-1)^n. \tag{3.3.23}$$

Conclusion The numerical scheme is stable when

$$c^2\frac{\Delta t^2}{\Delta x^2} \leq 1. \tag{3.3.24}$$

3.4 Schrödinger equation 🌶🌶🌶

Consider the 1D and time-dependent Schrödinger equation of a free particle in quantum mechanics

$$u_t = \frac{i\hbar}{2m}u_{xx}.$$

The equation is similar to a diffusion equation with an imaginary coefficient. Show that this difference induces significant changes in the properties and solutions.

(a) Using the forward difference in time and centered in space, *i.e.*

$$u_{xx} \approx \frac{u^n_{j+1} - 2u^n_j + u^n_{j-1}}{(\Delta x)^2} \quad \text{and} \quad u_t \approx \frac{u^{n+1}_j - u^n_j}{\Delta t},$$

show that the scheme is explicit by giving the update equation $u^{n+1}_j = f(u^n_{j+1}, u^n_j, u^n_{j-1})$.

(b) Prove using Von Neumann analysis that this method is unstable.

(c) Consider now a new method using a centered difference in time and in space, whose update equation is

$$u_j^{n+1} = u_j^{n-1} + \frac{i\hbar}{m}\frac{\Delta t}{(\Delta x)^2}\,(u_{j+1}^n - 2u_j^n + u_{j-1}^n). \tag{†}$$

Establish a stability criterion using Von Neumann analysis.

Solution

(a) The update equation is

$$u_j^{n+1} = u_j^n + \frac{i\hbar}{2m}\frac{\Delta t}{\Delta x^2}\,(u_{j+1}^n - 2u_j^n + u_{j-1}^n). \tag{3.4.1}$$

(b) Introducing any error mode

$$\epsilon(x,t) = \hat{\epsilon}(k,t)\ \exp(ikx), \tag{3.4.2}$$

$$= \hat{\epsilon}_n\ [\exp(ik\Delta x)]^j, \tag{3.4.3}$$

and replace this expression in Eq.(3.4.1).

We obtain the recurrence relation

$$\hat{\epsilon}_{n+1} = \left[1 + \frac{i\hbar}{2m}\frac{\Delta t}{\Delta x^2}\,(\exp(ik\Delta x) - 2 + \exp(-ik\Delta x))\right]\hat{\epsilon}_n(k). \tag{3.4.4}$$

The solution of this equation is

$$\hat{\epsilon}_n = A\xi_1^n \quad ⚠\ n \text{ is an exponent} \tag{3.4.5}$$

with

$$\xi_1 = 1 + \frac{i\hbar}{2m}\frac{\Delta t}{\Delta x^2}\,(\exp(ik\Delta x) - 2 + \exp(-ik\Delta x)), \tag{3.4.6}$$

$$= 1 + \frac{i\hbar}{2m}\frac{\Delta t}{\Delta x^2}\,(2\cos(k\Delta x) - 2), \tag{3.4.7}$$

$$= 1 - i\,\frac{2\hbar}{m}\frac{\Delta t}{\Delta x^2}\,\sin^2\left(\frac{k\Delta x}{2}\right). \tag{3.4.8}$$

The Von Neumann stability is obtained when all modes are not divergent, *i.e.*

$$|\xi_1| \leq 1. \tag{3.4.9}$$

The modulus of ξ_1 is given by

$$|\xi_1| = \left[1 + \frac{4\hbar^2}{m^2}\frac{\Delta t^2}{\Delta x^4}\sin^4\left(\frac{k\Delta x}{2}\right)\right]^{1/2}. \tag{3.4.10}$$

Hence, using this expression in Eq.(3.4.9), the following stability condition is obtained

$$\sin^4\left(\frac{k\Delta x}{2}\right) \leq 0. \tag{3.4.11}$$

As the right part of the equation is always positive, the method is unstable.

(c) When the error mode in Eq.(3.4.3) is replaced in the discrete method in Eq.(†), it gives

$$\frac{\hat{\epsilon}_{n+1}-\hat{\epsilon}_{n-1}}{2\Delta t} = \frac{i\hbar}{2m}\frac{1}{\Delta x^2}\left[\exp(i\,k\Delta x) - 2 + \exp(-ik\Delta x)\right]\hat{\epsilon}_n. \tag{3.4.12}$$

This equation can be rewritten equivalently as

$$\frac{\hat{\epsilon}_{n+2}-\hat{\epsilon}_{n}}{2\Delta t} = \frac{i\hbar}{2m}\frac{1}{\Delta x^2}\left[\exp(i\,k\Delta x) - 2 + \exp(-ik\Delta x)\right]\hat{\epsilon}_{n+1}, \tag{3.4.13}$$

$$\Rightarrow \quad \hat{\epsilon}_{n+2}-\hat{\epsilon}_{n} = \frac{i\hbar}{m}\frac{\Delta t}{\Delta x^2}\left[2\cos(k\Delta x) - 2\right]\hat{\epsilon}_{n+1}, \tag{3.4.14}$$

$$\Rightarrow \quad \hat{\epsilon}_{n+2} + \left[i\,\frac{4\hbar}{m}\frac{\Delta t}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)\right]\hat{\epsilon}_{n+1} - \hat{\epsilon}_{n} = 0. \tag{3.4.15}$$

The characteristic polynomial of this recurrence relation

$$\xi^2 + \left[i\,\frac{4\hbar}{m}\frac{\Delta t}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)\right]\xi - 1 = 0, \tag{3.4.16}$$

admits two roots

$$\xi_{\frac{1}{2}} = -i\frac{2\hbar}{m}\frac{\Delta t}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right) \pm \sqrt{1 - \frac{4\hbar^2}{m^2}\frac{\Delta t^2}{\Delta x^4}\sin^4\left(\frac{k\Delta x}{2}\right)}, \tag{3.4.17}$$

$$= -i\frac{2\hbar}{m}\frac{\Delta t}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right) \pm \sqrt{\rho}, \tag{3.4.18}$$

with $\rho = 1 - \frac{4\hbar^2}{m^2}\frac{\Delta t^2}{\Delta x^4}\sin^4\left(\frac{k\Delta x}{2}\right)$.

Provided both roots are distinct (*i.e* $\xi_1 \neq \xi_2$)), the solution of the recurrence relation in Eq.(3.4.15) has the following form

$$\hat{\epsilon}_n = A\xi_1^n + B\xi_2^n, \qquad \text{⚠ } n \text{ is an exponent} \tag{3.4.19}$$

where A and B are both constants that depend on the initial error.

Hence, the error does not diverge if modulus of the amplification factors verify simultaneously

$$|\xi_1| \leq 1 \quad \text{and} \quad |\xi_2| \leq 1. \tag{3.4.20}$$

Reminder: The modulus of a complex number $|a + ib| = \sqrt{a^2 + b^2}$.

That's why two different situations will be considered as the modulus of $\xi_{\frac{1}{2}}$ will depend on the sign of ρ.

Case 1: $\rho < 0$

If ρ is negative, both terms of $\xi_{\frac{1}{2}}$ are imaginary and its modulus is equal to

$$\left|\xi_{\frac{1}{2}}\right| = \left|-\frac{2\hbar}{m}\frac{\Delta t}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right) \pm \sqrt{-\rho}\right| \tag{3.4.21}$$

$$= \left|-\sqrt{1-\rho} \pm \sqrt{-\rho}\right|. \tag{3.4.22}$$

The roots $\xi_{\frac{1}{2}}$ must verify the following condition to be stable

$$-1 \leq -\sqrt{1-\rho} \pm \sqrt{-\rho} \leq 1. \tag{3.4.23}$$

But, as $\rho < 0$, ξ_2 verifies

$$\xi_2 < -1. \tag{3.4.24}$$

The modes for which $\rho < 0$ are thus divergent.

Case 2: $\rho > 0$

If ρ is positive, the first term of $\xi_{\frac{1}{2}}$ is imaginary while the second one is real. Then, the solutions ξ_1 and ξ_2 have the same modulus which is equal to

$$|\xi_{\frac{1}{2}}| = \sqrt{\frac{4\hbar^2}{m^2}\frac{\Delta t^2}{\Delta x^4}\sin^4\left(\frac{k\Delta x}{2}\right) + \left(1 - \frac{4\hbar^2}{m^2}\frac{\Delta t^2}{\Delta x^4}\sin^4\left(\frac{k\Delta x}{2}\right)\right)}, \tag{3.4.25}$$

$$= 1. \tag{3.4.26}$$

The conditions on ξ_1 and ξ_2 in Eq.(3.4.20) are verified and all the modes for which $\rho > 0$ are not divergent.

Case 3: $\rho = 0$

If $\rho = 0$ the multiplicity of the root is two. The general solution of the recurrence relation Eq.(3.4.19) is then

$$\hat{\epsilon}_n = (An + B)\,(\xi_{1,2})^n. \tag{3.4.27}$$

In this particular case, one has

$$\rho = 0 \Rightarrow \xi_{1,2} = \pm i \tag{3.4.28}$$

and thus the error evolves as

$$\hat{\epsilon}_n = (An + B)\,(\pm i)^n. \tag{3.4.29}$$

It implies that the errors only grow linearly with n. For this reason, the solution is said to be "stable" when $\rho = 0$.

Note Although the exercise is limited to stating the error "for all constants A and B," it is still possible to determine their expressions. Considering that the first two values of the recurrence relation are known, the constants A and B can easily be determined. For example, the error writes

$$\hat{\epsilon}_n = \left[(i\hat{\epsilon}_1 - \hat{\epsilon}_0)n + \hat{\epsilon}_0\right](\pm i)^n. \tag{3.4.30}$$

Conclusion To obtain $\rho \geq 0$, one has to solve

$$\rho \geq 0 \tag{3.4.31}$$
$$\Rightarrow \quad 1 - \frac{4\hbar^2}{m^2}\frac{\Delta t^2}{\Delta x^4}\sin^4\left(\frac{k\Delta x}{2}\right) \geq 0, \tag{3.4.32}$$
$$\Rightarrow \quad \frac{2\hbar}{m}\frac{\Delta t}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right) \leq 1. \tag{3.4.33}$$

The worst case is obtained when $k\Delta x = \pm\pi/2$. The method is thus stable only when the following relation is respected

$$\frac{\Delta t}{\Delta x^2} \leq \frac{m}{2\hbar}. \tag{3.4.34}$$

3.5 Advection equation : Leapfrog scheme equation 🌶️🌶️🌶️

Consider the 1D advection equation

$$u_t + au_x = 0 \tag{$\diamond$}$$

with a a positive constant.
Using the leapfrog numerical scheme, *i.e.*

$$u_t \approx \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t}, \qquad u_x \approx \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

where Δt et Δx are respectively the time step and a spatial step and where $u_j^n \triangleq u(j\Delta x, n\Delta t)$.

(a) Show this method is explicit by giving the update equation.

(b) Establish a stability criterion using Von Neumann analysis.

Solution

(a) Using finite difference approximation, the update equation writes as

$$u_j^{n+1} = u_j^{n-1} - \alpha\,(u_{j+1}^n - u_{j-1}^n), \tag{3.5.1}$$

with $\alpha = \frac{a\Delta t}{\Delta x}$.

(b) Introducing any error mode

$$\epsilon(x,t) = \hat{\epsilon}(k,t)\ \exp(ikx), \tag{3.5.2}$$
$$= \hat{\epsilon}_n\,[\exp(ik\Delta x)]^j, \tag{3.5.3}$$

and replace this expression in Eq.(3.5.1).
We obtain the recurrence relation

$$\hat{\epsilon}_{n+1} = \hat{\epsilon}_{n-1} - \alpha\,[\exp(ik\Delta x) - \exp(-ik\Delta x)]\,\hat{\epsilon}_n. \tag{3.5.4}$$

This equation can be rewritten equivalently as

$$\hat{\epsilon}_{n+1} = \hat{\epsilon}_{n-1} - i\ 2\alpha \sin(k\Delta x)\ \hat{\epsilon}_n. \tag{3.5.5}$$

The characteristic polynomial of this recurrence relation

$$\xi^2 + i\ 2\alpha \sin(k\Delta x)\ \xi - 1 = 0, \tag{3.5.6}$$

admits two roots

$$\xi_{\frac{1}{2}} = -i\alpha \sin(k\Delta x) \pm \sqrt{1 - \alpha^2 \sin^2(k\Delta x)}, \tag{3.5.7}$$

$$= -i\alpha \sin(k\Delta x) \pm \sqrt{\rho}, \tag{3.5.8}$$

$$\tag{3.5.9}$$

with $\rho = 1 - \alpha^2 \sin^2(k\Delta x)$.
Provided both roots are distinct (*i.e* $\xi_1 \neq \xi_2$)), the solution of the recurrence relation in Eq.(3.5.5) has the following form

$$\hat{\epsilon}_n = A\xi_1^n + B\xi_2^n, \quad \text{⚠ } n \text{ is an exponent} \tag{3.5.10}$$

where A and B are both constants that depend on the initial error.
Hence, the error does not diverge if modulus of the amplification factors verify simultaneously

$$|\xi_1| \leq 1 \quad \text{and} \quad |\xi_2| \leq 1. \tag{3.5.11}$$

Two different situations will be considered as the modulus of $\xi_{\frac{1}{2}}$ will depend on the sign of ρ.

Case 1: $\rho < 0$

If ρ is negative, both terms of $\xi_{\frac{1}{2}}$ are imaginary and its modulus is equal to

$$\left|\xi_{\frac{1}{2}}\right| = \left|-\alpha \sin(k\Delta x) \pm \sqrt{-\rho}\right| \tag{3.5.12}$$

$$= \left|-\sqrt{1-\rho} \pm \sqrt{-\rho}\right|. \tag{3.5.13}$$

The roots $\xi_{\frac{1}{2}}$ must verify the following condition to be stable

$$-1 \leq -\sqrt{1-\rho} \pm \sqrt{-\rho} \leq 1. \tag{3.5.14}$$

But, as $\rho < 0$, ξ_2 verifies

$$\text{Im}(\xi_2) < -1. \tag{3.5.15}$$

The modes for which $\rho < 0$ are thus divergent.

Case 2: $\rho > 0$

If ρ is positive, the first term of $\xi_{\frac{1}{2}}$ is imaginary while the second one is real. Then, the solutions ξ_1 and ξ_2 have the same modulus which is equal to

$$|\xi_{\frac{1}{2}}| = \sqrt{\alpha^2 \sin^2(k\Delta x) + 1 - \alpha^2 \sin^2(k\Delta x)}, \tag{3.5.16}$$

$$= 1. \tag{3.5.17}$$

The conditions on ξ_1 and ξ_2 in Eq.(3.5.11) are verified and all the modes for which $\rho > 0$ are not divergent.

Case 3: $\rho = 0$
If $\rho = 0$ the multiplicity of the root is two. The general solution of the recurrence relation Eq.(3.5.5) is then

$$\hat{\epsilon}_n = (An + B)\,(\xi_{1,2})^n. \tag{3.5.18}$$

In this particular case, one has

$$\rho = 0 \Rightarrow \xi_{1,2} = \pm i \tag{3.5.19}$$

and thus the error evolves as

$$\hat{\epsilon}_n = (An + B)\,(\pm i)^n. \tag{3.5.20}$$

It implies the errors only grow linearly with n. For this reason, the solution is said to be "stable" when $\rho = 0$.

Note: Although the exercise is limited to stating the error "for all constants A and B," it is still possible to determine their expressions. Considering that the first two values of the recurrence relation are known, the constants A and B can easily be determined so that the error writes

$$\hat{\epsilon}_n = [(i\hat{\epsilon}_1 - \hat{\epsilon}_0)n + \hat{\epsilon}_0]\,(\pm i)^n. \tag{3.5.21}$$

Conclusion To obtain $\rho \geq 0$, one has to solve

$$\begin{aligned}
& \rho \geq 0 && (3.5.22)\\
\Rightarrow\quad & 1 - \alpha^2 \sin^2(k\Delta x) \geq 0, && (3.5.23)\\
\Rightarrow\quad & \alpha^2 \sin^2(k\Delta x) \leq 1, && (3.5.24)\\
\Rightarrow\quad & |\alpha| \leq 1. && (3.5.25)
\end{aligned}$$

The worst case is obtained when $k\Delta x = \pm\pi/2$. The method is thus stable only when the following relation is respected

$$|\alpha| \leq 1, \tag{3.5.26}$$

$$\left|\frac{a\Delta t}{\Delta x}\right| \leq 1. \tag{3.5.27}$$

It is interesting to highlight that the condition does not depend on the sign of a.

3.6 Diffusion equation : Dufort-Frankel scheme equation 🌶🌶

Consider the 1D diffusion equation

$$u_t - b u_{xx} = 0, \tag{$\diamond$}$$

with b a positive constant.
Using the Dufort-Frankel numerical scheme, *i.e.*

$$u_t \approx \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t}, \qquad u_{xx} \approx \frac{u_{j-1}^n - (u_j^{n-1} + u_j^{n+1}) + u_{j+1}^n}{\Delta x^2},$$

where Δt et Δx are respectively the time step and a spatial step and where $u_j^n \triangleq u(j\Delta x, n\Delta t)$.

(a) Show this method is explicit by giving the update equation.

(b) Establish a stability criterion using Von Neumann analysis.

Solution

(a) Using finite difference approximation, the update equation writes as

$$u_j^{n+1} = \frac{1-\alpha}{1+\alpha}\, u_j^{n-1} + \frac{\alpha}{1+\alpha}\, (u_{j-1}^n + u_{j+1}^n), \tag{3.6.1}$$

with $\alpha = 2b\frac{\Delta t}{\Delta x^2}$.

(b) Introducing any error mode

$$\begin{aligned} \epsilon(x,t) &= \hat{\epsilon}(k,t)\ \exp(ikx), &(3.6.2)\\ &= \hat{\epsilon}_n\ [\exp(ik\Delta x)]^j, &(3.6.3)\end{aligned}$$

and replace this expression in Eq.(3.6.1).
We obtain the recurrence relation

$$(1+\alpha)\ \hat{\epsilon}_{n+1} + (\alpha-1)\ \hat{\epsilon}_{n-1} - 2\alpha\cos(k\Delta x)\ \hat{\epsilon}_n = 0. \tag{3.6.4}$$

The characteristic polynomial of this recurrence relation

$$(1+\alpha)\ \xi^2 + (\alpha-1) - 2\alpha\cos(k\Delta x)\ \xi = 0, \tag{3.6.5}$$

admits two roots

$$\xi_{\frac{1}{2}} = \frac{\alpha\cos(k\Delta x) \pm \sqrt{1-\alpha^2\sin^2(k\Delta x)}}{1+\alpha}. \tag{3.6.6}$$

Provided both roots are distinct (*i.e* $\xi_1 \neq \xi_2$)), the solution of the recurrence relation in Eq.(3.6.4) has the following form

$$\hat{\epsilon}_n = A\xi_1^n + B\xi_2^n, \qquad ⚠\ n \text{ is an exponent} \tag{3.6.7}$$

where A and B are both constants that depend on the initial error.
Hence, the error does not diverge if modulus of the amplification factors verify simultaneously

$$|\xi_1| \leq 1 \quad \text{and} \quad |\xi_2| \leq 1. \tag{3.6.8}$$

Two different situations will be considered as the modulus of $\xi_{\frac{1}{2}}$ will depend on the sign of ρ.

Case 1: $\rho < 0$

If ρ is negative, the first term of $\xi_{\frac{1}{2}}$ is real while the second one is imaginary. Then, the solutions ξ_1 and ξ_2 have the same modulus which is equal to

$$\begin{aligned} \left|\xi_{\frac{1}{2}}\right| &= \sqrt{\frac{\alpha^2\left[\cos^2(k\Delta x) + \sin^2(k\Delta x)\right] - 1}{(1+\alpha)^2}} &(3.6.9)\\ &= \sqrt{\frac{\alpha^2 - 1}{(1+\alpha)^2}} &(3.6.10)\end{aligned}$$

As $\alpha > 0$, then

$$\left|\xi_{\frac{1}{2}}\right|^2 = \frac{\alpha^2-1}{(1+\alpha)^2} \leq 1 \tag{3.6.11}$$

In this case, the solution is always stable.

Case 2: $\rho > 0$

If ρ is positive, both terms are real. Then to be stable, if the most restrictive cases are considered, one has to verify:

$$\xi_1 = \frac{\alpha\cos(k\Delta x) + \sqrt{1-\alpha^2\sin^2(k\Delta x)}}{1+\alpha} \leq 1, \tag{3.6.12}$$

$$\Rightarrow \qquad \xi_1 \leq \frac{\alpha+1}{1+\alpha} = 1 \leq 1, \tag{3.6.13}$$

and,

$$\xi_2 = \frac{\alpha\cos(k\Delta x) - \sqrt{1-\alpha^2\sin^2(k\Delta x)}}{1+\alpha} \geq -1, \tag{3.6.14}$$

$$\Rightarrow \qquad \xi_2 \geq \frac{-\alpha-1}{1+\alpha} = -1 \geq -1, \tag{3.6.15}$$

The conditions on ξ_1 and ξ_2 in Eq.(3.6.8) are valid and all the modes for which $\rho > 0$ are not divergent.

Case 3: $\rho = 0$

If $\rho = 0$ multiplicity of the root is two, the solution of Eq.(3.6.7) is then given by

$$\hat{\epsilon}_n = (An + B)\,(\xi_{1,2})^n. \tag{3.6.16}$$

with

$$|\xi_{1,2}| = \left|\frac{\alpha\cos(k\Delta x)}{1+\alpha}\right| < 1 \tag{3.6.17}$$

which implies that the solution is also stable when $\rho = 0$.

Conclusion In all the cases, the error does not diverge. It means that this method is unconditionally stable.

3.7 Advection-diffusion equation

Consider the 1D advection-diffusion equation

$$u_t + wu_x - bu_{xx} = 0, \tag{$\diamond$}$$

where w is a constant and b a positive constant.

One numerical solution for this equation can be computed using the following finite differences approximations

$$u_t \approx \frac{u_j^{n+1} - u_j^n}{\Delta t}, \qquad u_x \approx \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \quad \text{et} \quad u_{xx} \approx \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2},$$

where Δt and Δx are respectively the time step and the spatial step, and where $u_j^n \triangleq u(j\Delta x, n\Delta t)$.

Study the Von Neumann stability of this solution by following the step hereunder:

(a) Give the update equation with respect to $\alpha \triangleq \frac{w\Delta t}{2\Delta x}$ and $\beta \triangleq \frac{b\Delta t}{\Delta x^2}$. Is this numerical scheme explicit or implicit?

(b) Prove that the squared modulus of the amplification factor can be written like

$$\|\xi\|^2 = 1 + X\left(16\alpha^2 - 8\beta + 16(\beta^2 - \alpha^2)X\right),$$

where $X(k) \triangleq \sin^2 \frac{k\Delta x}{2}$ and k is the wave number of the considered error mode.

(c) Give the stability criterion with respect to α and β.

(d) Using the result obtained in the previous question,

- can this scheme be stable when there is only diffusion?
- can this scheme be stable when there is only advection?

Justify.

Solution

(a) The update equation is

$$u_j^{n+1} = u_j^n + \alpha\left(u_{j-1}^n - u_{j+1}^n\right) + \beta\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right), \tag{3.7.1}$$

this scheme is then explicit.

(b) The error amplitude $\hat{\epsilon}^n(k)$ at a frequency k and a time step n verifies

$$\hat{\epsilon}^{n+1} = \hat{\epsilon}^n + \alpha\left(\hat{\epsilon}^n \exp\left(-ik\Delta x\right) - \hat{\epsilon}^n \exp\left(ik\Delta x\right)\right) + \beta\left(\hat{\epsilon}^n \exp\left(ik\Delta x\right) - 2\hat{\epsilon}^n - \hat{\epsilon}^n \exp\left(-ik\Delta x\right)\right). \tag{3.7.2}$$

The amplification factor ξ is then given by

$$\xi(k) \triangleq \frac{\hat{\epsilon}^{n+1}}{\hat{\epsilon}^n} = 1 - i2\alpha \sin k\Delta x + 2\beta\left(\cos k\Delta x - 1\right) \tag{3.7.3}$$

when its squared norm can be written like

$$\|\xi(k)\|^2 = 4\alpha^2 \sin^2 k\Delta x + \left(1 - 4\beta \sin^2 \frac{k\Delta x}{2}\right)^2, \tag{3.7.4}$$

$$= 16\alpha^2 \sin^2 \frac{k\Delta x}{2} \cos^2 \frac{k\Delta x}{2} + \left(1 - 4\beta \sin^2 \frac{k\Delta x}{2}\right)^2, \tag{3.7.5}$$

$$= 16\alpha^2 \sin^2 \frac{k\Delta x}{2} \cos^2 \frac{k\Delta x}{2} + 1 + 16\beta^2 \sin^4 \frac{k\Delta x}{2} - 8\beta \sin^2 \frac{k\Delta x}{2}, \tag{3.7.6}$$

$$= 16\alpha^2 X(1 - X) + 1 + 16\beta^2 X^2 - 8\beta X, \tag{3.7.7}$$

$$= 1 + X\left(16\alpha^2(1 - X) + 16\beta^2 X - 8\beta\right), \tag{3.7.8}$$

$$= 1 + X\left(16\alpha^2 - 8\beta + 16(\beta^2 - \alpha^2)X\right) \tag{3.7.9}$$

where $X(k) \triangleq \sin^2 \frac{k\Delta x}{2}$ spans from 0 to 1.

(c) For this scheme to be stable in the sense of Von Neumann, the norm of the amplification factor must be lower or equal to one, which implies

$$\begin{aligned} &\|\xi(k)\|^2 &&\leq 1 \quad (3.7.10)\\ \Leftrightarrow &1 + X\left(16\alpha^2 - 8\beta + 16(\beta^2 - \alpha^2)X\right) &&\leq 1 \quad (3.7.11)\\ \Leftrightarrow &16\alpha^2 - 8\beta + 16(\beta^2 - \alpha^2)X &&\leq 0 \quad (3.7.12) \end{aligned}$$

and then

$$\begin{cases} 16\alpha^2 - 8\beta & \leq 0, \\ 16\beta^2 - 8\beta & \leq 0, \end{cases} \quad (3.7.13)$$

$$\Leftrightarrow \begin{cases} \dfrac{\alpha^2}{\beta} & \leq \dfrac{1}{2}, \\ \beta & \leq \dfrac{1}{2}. \end{cases} \quad (3.7.14)$$

Therefore this scheme is stable for diffusion only if $\beta \leq \frac{1}{2}$ but unconditionally unstable for only advection as $\beta = 0$ implies $\alpha = 0$.

4 Green's functions and general solutions

4.1 Spherical wave equation [Strauss 2.1, Ex. 8] 🌶️

The *spherical wave equation* reads

$$u_{tt} = c^2 \left(u_{rr} + \frac{2}{r} u_r \right).$$

(a) Using the substitution $v = ru$, show that the spherical wave equation can be rewritten as $v_{tt} = c^2 v_{rr}$.

(b) Find the general solution of the spherical wave equation.

(c) Give an expression for the solution corresponding to the initial conditions $u(r,0) = \phi(r)$ and $u_t(r,0) = \psi(r)$, for given functions ϕ and ψ.

(d) Deduce a Green's functions with respect to initial condition, then one for initial time derivatives, i.e. find $S_\phi(r,t,r')$ such that it is a solution of the PDE with $S_\phi(r,0,r') = \delta(r-r')$ and $\partial_t S_\phi(r,0,r') = 0$, as well as a function $S_\psi(r,t,r')$ such that $S_\psi(r,0,r') = 0$ and $\partial_t S_\psi(r,0,r') = \delta(r-r')$.

Solution

(a) Using the change of variables $v = ru$ yields

$$v_t = ru_t, \quad v_{tt} = ru_{tt}, \quad v_r = u + ru_r, \quad \text{and} \quad v_{rr} = 2u_r + ru_{rr}. \tag{4.1.1}$$

The equation thus becomes

$$\begin{aligned} \frac{v_{tt}}{r} &= c^2 \frac{v_{rr}}{r}, && (4.1.2) \\ \Rightarrow v_{tt} &= c^2 v_{rr}. && (4.1.3) \end{aligned}$$

(b)

$$\begin{aligned} v &= f(r+ct) + g(r-ct), && (4.1.4) \\ \Rightarrow u &= \frac{v}{r} = \frac{1}{r}\left[f(r+ct) + g(r-ct) \right]. && (4.1.5) \end{aligned}$$

(c) Let us define $\Phi(r) = r\phi(r)$ and $\Psi(r) = r\psi(r)$. Using the general solution of this initial value problem is

$$v(r,t) = \frac{1}{2}\left[\Phi(r+ct) + \Phi(r-ct) \right] + \frac{1}{2c} \int_{r-ct}^{r+ct} \Psi(s)\, ds. \tag{4.1.6}$$

Since $v = ru$, $\Psi(r) = r\psi(r)$ and $\Phi(r) = r\phi(r)$, the solution for $u(r,t)$ is given by

$$u(r,t) = \frac{1}{2r}\left[(r+ct)\phi(r+ct) + (r-ct)\phi(r-ct) \right] + \frac{1}{2cr} \int_{r-ct}^{r+ct} s\psi(s)\, ds. \tag{4.1.7}$$

(d) Taking $\phi(r) = \delta(r - r')$ and $\psi(r) = 0$ and plugging these in 4.1.7, we deduce

$$S_\phi(r,t,r') = \frac{1}{2r}\left[(r+ct)\delta(r+ct-r') + (r-ct)\delta(r-ct-r')\right]. \tag{4.1.8}$$

Similarly, when the Dirac is in ψ, we get

$$S_\psi(r,t,r') = \frac{1}{2cr}\int_{r-ct}^{r+ct} s\delta(r'-s)\,ds \tag{4.1.9}$$

$$= \frac{r'}{2cr}\mathbb{1}_{[r-ct,r+ct]}(r'), \tag{4.1.10}$$

where the indicator function $\mathbb{1}_A(x)$ is 1 if $x \in A$ and 0 elsewhere.

4.2 Wave equation factorization I [Strauss 2.1, Ex. 10]

Solve the equation $u_{xx} + (s_1 - s_2)u_{xt} - s_1 s_2 u_{tt} = 0$ using the initial conditions $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$.

Solution

The equation can be written

$$\left(\partial_{xx} + (s_1 - s_2)\partial_{xt} - s_1 s_2 \partial_{tt}\right) u = 0. \tag{4.2.1}$$

Factoring the operator yields

$$\left(\partial_x + s_1\partial_t\right)\left(\partial_x - s_2\partial_t\right) u = 0. \tag{4.2.2}$$

Let $v(x,t) = \left(\partial_x - s_2\partial_t\right) u$, the problem can be written as a system first order equations

$$\begin{cases} \left(\partial_x + s_1\partial_t\right) v = 0, & (4.2.3) \\ \left(\partial_x - s_2\partial_t\right) u = v, & (4.2.4) \\ u(x,0) = \phi(x), & (4.2.5) \\ u_t(x,0) = \psi(x). & (4.2.6) \end{cases}$$

From Eq.(4.3.3), it can be shown that v is constant along the characteristic lines of equations $t - s_1 x = C$. Therefore

$$v(x,t) = f(t - s_1 x). \tag{4.2.7}$$

Using Eq.(4.3.7), Eq.(4.3.4) can be written as:

$$\left(\partial_x - s_2\partial_t\right) u = f(t - s_1 x). \tag{4.2.8}$$

It can be verified by differentiation that a particular solution of Eq.(4.3.8) is given by $u^p(x,t) = h(t - s_1 x)$, where $h'(y) = -\frac{1}{s_1 + s_2} f(y)$.
Besides this, using the same reasoning as the one used for v, one can show that the general solution of the homogeneous part of Eq.(4.3.8) can be written as $u^h(x,t) = g(t + s_2 x)$.

As Eq.(4.3.8) is linear, the solution can be written as the sum of a particular solution and the general solution of the homogeneous PDE *i.e.*

$$u(x,t) = h(t - s_1 x) + g(t + s_2 x). \tag{4.2.9}$$

Then using the initial conditions Eq.(4.3.5) and Eq.(4.3.6) yields

$$\begin{cases} h(-s_1 x) + g(s_2 x) = \phi(x), & (4.2.10) \\ h'(-s_1 x) + g'(s_2 x) = \psi(x). & (4.2.11) \end{cases}$$

Deriving Eq.(4.3.10) with respect to x and using the chain rule, successively gives

$$\begin{cases} -s_1 h'(-s_1 x) + s_2 \quad g'(s_2 x) = \phi'(x), \\ \quad h'(-s_1 x) + \quad g'(s_2 x) = \psi(x). \end{cases} \tag{4.2.12}$$

$$\Leftrightarrow \begin{cases} (s_1 + s_2) g'(s_2 x) = \phi'(x) + s_1 \psi(x), \\ -(s_1 + s_2) h'(-s_1 x) = \phi'(x) - s_2 \psi(x). \end{cases} \tag{4.2.13}$$

$$\Leftrightarrow \begin{cases} g'(s_2 x) = \dfrac{1}{s_1 + s_2} \phi'(x) + \dfrac{s_1}{s_1 + s_2} \psi(x), \\ h'(-s_1 x) = -\dfrac{1}{s_1 + s_2} \phi'(x) + \dfrac{s_2}{s_1 + s_2} \psi(x). \end{cases} \tag{4.2.14}$$

$$\Leftrightarrow \begin{cases} g'(y) = \dfrac{1}{s_1 + s_2} \phi'(y/s_2) + \dfrac{s_1}{s_1 + s_2} \psi(y/s_2), \\ h'(y) = -\dfrac{1}{s_1 + s_2} \phi'(-y/s_1) + \dfrac{s_2}{s_1 + s_2} \psi(-y/s_1). \end{cases} \tag{4.2.15}$$

Let solve this system one equation at a time. Integrating both sides of the first equation of Eqs.(4.3.15) successively gives

$$\int_0^x g'(y) dy = \int_0^x \frac{1}{s_1 + s_2} \phi'(y/s_2) dy + \int_0^x \frac{s_1}{s_1 + s_2} \psi(y/s_2) dy, \tag{4.2.16}$$

$$= \int_0^{x/s_2} \frac{s_2}{s_1 + s_2} \phi'(\tilde{y}) d\tilde{y} + \int_0^{x/s_2} \frac{s_2 s_1}{s_1 + s_2} \psi(\tilde{y}) d\tilde{y}, \text{ with } \tilde{y} = y/s_2 \tag{4.2.17}$$

$$g(x) = \frac{s_2}{s_1 + s_2} \phi(x/s_2) + \frac{s_2 s_1}{s_1 + s_2} \int_0^{x/s_2} \psi(\tilde{y}) d\tilde{y} + A \tag{4.2.18}$$

where A is an integration constant. Then solving the second equation of Eqs.(4.3.15) gives

$$\int_0^x h'(y) dy = \int_0^x -\frac{1}{s_1 + s_2} \phi'(-y/s_1) dy + \int_0^x \frac{s_2}{s_1 + s_2} \psi(-y/s_1) dy, \tag{4.2.19}$$

$$= \int_0^{-x/s_1} \frac{s_1}{s_1 + s_2} \phi'(\tilde{y}) d\tilde{y} + \int_0^{-x/s_1} -\frac{s_2 s_1}{s_1 + s_2} \psi(\tilde{y}) d\tilde{y}, \text{ with } \tilde{y} = -y/s_1 \tag{4.2.20}$$

$$h(x) = \frac{s_1}{s_1 + s_2} \phi(-x/s_1) - \frac{s_2 s_1}{s_1 + s_2} \int_0^{-x/s_1} \psi(\tilde{y}) d\tilde{y} + B \tag{4.2.21}$$

where B is an integration constant. From Eq.(4.3.9), Eq.(4.3.17) and Eq.(4.3.19) it follows

$$u(x,t) = \frac{s_1}{s_1+s_2}\phi\left(\frac{-(t-s_1x)}{s_1}\right) + \frac{s_2}{s_1+s_2}\phi\left(\frac{(t+s_2x)}{s_2}\right) - \frac{s_1s_2}{s_1+s_2}\int_0^{\frac{-(t-s_1x)}{s_1}}\psi(s)ds + \frac{s_1s_2}{s_1+s_2}\int_0^{\frac{(t+s_2x)}{s_2}}\psi(s)ds + A + B. \quad (4.2.22)$$

From the first equation in Eqs.(4.3.10), Eq.(4.3.17) and Eq.(4.3.19), it can be shown that $A + B = 0$. Therefore, the solution is

$$u(x,t) = \frac{1}{s_1+s_2}\left[s_1\phi\left(x - \frac{t}{s_1}\right) + s_2\phi\left(x + \frac{t}{s_2}\right) + s_1s_2\int_{x-\frac{t}{s_1}}^{x+\frac{t}{s_2}}\psi(s)ds\right]. \quad (4.2.23)$$

4.3 Wave equation factorization II [Strauss 2.1, Ex. 10] 🌶

Solve the equation $u_{tt} + (c_1 - c_2)u_{xt} - c_1c_2u_{xx} = 0$ using the initial conditions $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$. Find the Green's functions S_ϕ and S_ψ that represent fundamental solutions for an impulsion as initial condition.

Solution

The equation can be written

$$\left(\partial_{tt} + (c_1 - c_2)\partial_{xt} - c_1c_2\partial_{xx}\right)u = 0. \quad (4.3.1)$$

Factoring the operator yields

$$\left(\partial_t + c_1\partial_x\right)\left(\partial_t - c_2\partial_x\right)u = 0. \quad (4.3.2)$$

Let $v(x,t) = \left(\partial_t - c_2\partial_x\right)u$, the problem can be written as a system first order equations

$$\begin{cases} \left(\partial_t + c_1\partial_x\right)v = 0, & (4.3.3)\\ \left(\partial_t - c_2\partial_x\right)u = v, & (4.3.4)\\ u(x,0) = \phi(x), & (4.3.5)\\ u_t(x,0) = \psi(x). & (4.3.6)\end{cases}$$

From Eq.(4.3.3), it can be shown that v is constant along the characteristic lines of equations $x - c_1t = C$. Therefore

$$v(x,t) = f(x - c_1t). \quad (4.3.7)$$

Using Eq.(4.3.7), Eq.(4.3.4) can be written as:

$$\left(\partial_t - c_2\partial_x\right)u = f(x - c_1t). \quad (4.3.8)$$

It can be verified by differentiation that a particular solution of Eq.(4.3.8) is given by $u^p(x,t) = h(x - c_1t)$, where $h'(y) = -\dfrac{1}{c_1+c_2}f(y)$.
Besides this, using the same reasoning as the one used for v, one can show that the general solution of the homogeneous part of Eq.(4.3.8) can be written as $u^h(x,t) = g(x + c_2t)$.

As Eq.(4.3.8) is linear, the solution can be written as the sum of a particular solution and the general solution of the homogeneous PDE *i.e.*

$$u(x,t) = h(x - c_1 t) + g(x + c_2 t). \tag{4.3.9}$$

Then using the initial conditions Eq.(4.3.5) and Eq.(4.3.6) yields

$$\begin{cases} h(x) + g(x) = \phi(x), & (4.3.10) \\ -c_1 h'(x) + c_2 g'(x) = \psi(x). & (4.3.11) \end{cases}$$

Deriving Eq.(4.3.10) with respect to x successively gives

$$\begin{cases} h'(x) + g'(x) = \phi'(x), \\ -c_1 h'(x) + c_2 g'(x) = \psi(x). \end{cases} \tag{4.3.12}$$

$$\Leftrightarrow \begin{cases} (c_1 + c_2) g'(x) = c_1 \phi'(x) + \psi(x), \\ -(c_1 + c_2) h'(x) = -c_2 \phi'(x) + \psi(x). \end{cases} \tag{4.3.13}$$

$$\Leftrightarrow \begin{cases} g'(x) = \dfrac{c_1}{c_1 + c_2} \phi'(x) + \dfrac{1}{c_1 + c_2} \psi(x), \\ h'(x) = \dfrac{c_2}{c_1 + c_2} \phi'(x) - \dfrac{1}{c_1 + c_2} \psi(x). \end{cases} \tag{4.3.14}$$

(4.3.15)

Let solve this system one equation at a time. Integrating both sides of the first equation of Eqs.(4.3.15) successively gives

$$\int_0^x g'(y)dy = \int_0^x \frac{c_1}{c_1 + c_2} \phi'(y)dy + \int_0^x \frac{1}{c_1 + c_2} \psi(y)dy, \tag{4.3.16}$$

$$g(x) = \frac{c_1}{c_1 + c_2} \phi(x) + \frac{1}{c_1 + c_2} \int_0^x \psi(y)dy + A \tag{4.3.17}$$

where A is an integration constant. Then solving the second equation of Eqs.(4.3.15) gives

$$\int_0^x h'(y)dy = \int_0^x \frac{c_2}{c_1 + c_2} \phi'(y)dy - \int_0^x \frac{1}{c_1 + c_2} \psi(y)dy, \tag{4.3.18}$$

$$h(x) = \frac{c_2}{c_1 + c_2} \phi(x) - \frac{1}{c_1 + c_2} \int_0^x \psi(y)dy + B \tag{4.3.19}$$

where B is an integration constant. From Eq.(4.3.9), Eq.(4.3.17) and Eq.(4.3.19) it follows

$$u(x,t) = \frac{c_2}{c_1 + c_2} \phi(x - c_1 t) + \frac{c_1}{c_1 + c_2} \phi(x + c_2 t) - \frac{1}{c_1 + c_2} \int_0^{x - c_1 t} \psi(s)ds + \frac{1}{c_1 + c_2} \int_0^{x + c_2 t} \psi(s)ds + A + B. \tag{4.3.20}$$

From the first equation in Eqs.(4.3.10), Eq.(4.3.17) and Eq.(4.3.19), it can be shown that $A + B = 0$. Therefore, the solution is

$$u(x,t) = \frac{1}{c_1 + c_2} \left[c_2 \phi(x - c_1 t) + c_1 \phi(x + c_2 t) + \int_{x - c_1 t}^{x + c_2 t} \psi(s)ds \right]. \tag{4.3.21}$$

From this, we can deduce the Green's functions:

$$S_\phi(x,t,x') = \frac{1}{c_1+c_2}\left[c_2\delta(x-c_1t-x') + c_1\delta(x+c_2t-x')\right]. \tag{4.3.22}$$

Similarly, when the Dirac is in ψ, we get

$$S_\psi(x,t,x') = \frac{1}{c_1+c_2}\int_{x-c_1t}^{x+c_2t}\delta(x'-s)\,ds \tag{4.3.23}$$
$$= \frac{1}{c_1+c_2}\mathbb{1}_{[x-c_1t,x+c_2t]}(x'). \tag{4.3.24}$$

4.4 Equation factorization [Olver Ex. 2.4.19]

Solve the equation $u_{tt}-2u_{tx}-3u_{xx}=0$ using the initial conditions $u(x,0)=x^2$ and $u_t(x,0)=\exp(x)$. Use the general solution from the previous exercise, and check that using a linear combination of fundamental solutions yields the same result.

Solution

The PDE can be put in the form $u_{tt}+(c_1-c_2)u_{tx}-3u_{xx}=0$ if we take[2] $c_1=1, c_2=3$. Then, noting ϕ the initial value and ψ the initial derivative w.r.t time, equation 4.3.21 yields

$$u(x,t) = \frac{1}{4}\left[3\phi\left(x-t\right) + 1\phi\left(x+3t\right) + \int_{x-t}^{x+3t}\psi(s)ds\right] \tag{4.4.1}$$
$$= \frac{1}{4}\left[3\left(x-t\right)^2 + 1\left(x+3t\right)^2 + \int_{x-t}^{x+3t}e^s\,ds\right] \tag{4.4.2}$$
$$= \frac{1}{4}\left[3\left(x-t\right)^2 + 1\left(x+3t\right)^2 + e^{x+3t} - e^{x-t}\right]. \tag{4.4.3}$$

To compute the solution with fundamental solutions, we compute linear combination of them. For simplicity, let us split $u=u_\phi+u_\psi$, with the first part having a null initial time derivative, and the second one a null initial value.
Then:

$$u_\phi(x,t) = \int_{-\infty}^{+\infty} S_\phi(x,t,x')\phi(x')dx' \tag{4.4.4}$$
$$= \int_{-\infty}^{+\infty}\frac{1}{4}\left[3\delta(x-t-x') + 1\delta(x+3t-x')\right]x'^2\,dx' \tag{4.4.5}$$
$$= \frac{1}{4}\left[3(x-t)^2 + (x+3t)^2\right]. \tag{4.4.6}$$

Then, the part related to ψ can be computed similarly:

[2] We can also take $c_1=-3, c_2=-1$, however positive values are easier to reason about.

$$\begin{aligned}
u_\psi(x,t) &= \int_{-\infty}^{+\infty} S_\psi(x,t,x')\psi(x')dx' &(4.4.7)\\
&= \frac{1}{4}\int_{-\infty}^{+\infty} 1_{[x-t,x+3t]}e^{x'}\,dx' &(4.4.8)\\
&= \frac{1}{4}\int_{x-t}^{x+3t} e^{x'}\,dx' &(4.4.9)\\
&= \frac{1}{4}\left[e^{x+3t}-e^{x-t}\right]. &(4.4.10)
\end{aligned}$$

Adding both parts of the solutions yields the expected result.

4.5 Inhomogeneous wave equation [Olver 2.4, Ex. 11] 🌶️🌶️

(a) Solve the initial value problem

$$\partial_{tt}u - c^2\partial_{xx}u = 0, \quad u(x,0) = \sin(x), \quad \partial_t u(x,0) = \cos(x), \quad -\infty < x < \infty, \quad t \geq 0. \tag{4.5.1}$$

(b) Is $u(x,t)$ a periodic function of t?

(c) Solve the forced initial value problem

$$\partial_{tt}u - c^2\partial_{xx}u = \cos(\omega t), \quad u(x,0) = \sin(x), \quad \partial_t u(x,0) = \cos(x), \quad -\infty < x < \infty, \quad t \geq 0. \tag{4.5.2}$$

(d) Does the solution exhibits resonance ?

(e) What would happen if the forcing function is $\sin(\omega t)$ instead of $\cos(\omega t)$?

Solution

(a) Starting from

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0, \tag{4.5.3}$$

define the characteristic coordinates

$$\xi = x + ct \quad \text{and} \quad \eta = x - ct. \tag{4.5.4}$$

The chain rule gives

$$\partial_x = \partial_\xi + \partial_\eta \quad \text{and} \quad \partial_t = c\partial_\xi - c\partial_\eta \tag{4.5.5}$$

such that the equation simplifies as

$$\begin{aligned}
& -4c^2(\partial_\xi)(\partial_\eta)u = 0 &(4.5.6)\\
\Rightarrow\quad & \partial_\xi\partial_\eta u = 0 &(4.5.7)\\
\Rightarrow\quad & u(x,t) = f(\xi)+g(\eta) = f(x+ct)+g(x-ct). &(4.5.8)
\end{aligned}$$

The initial conditions yields

$$\begin{cases} u(x,0) = f(x) + g(x) \quad\quad = \sin(x), & (4.5.9)\\ u'(x,0)) = cf'(x) - cg'(x) = \cos(x). & (4.5.10)\end{cases}$$

Differentiating the first equation yields

$$f'(x) + g'(x) = \cos(x) \tag{4.5.11}$$

Then solving for $f'(x)$ and $g'(x)$ gives

$$\begin{cases} f'(x) = \dfrac{c+1}{2c}\cos(x) & (4.5.12)\\ g'(x) = \dfrac{c-1}{2c}\cos(x) & (4.5.13)\end{cases}$$

Integrating Eq.(4.5.12) and Eq.(4.5.13) gives

$$\begin{cases} f(x) = \dfrac{c+1}{2c}\sin(x) + A & (4.5.14)\\ g(x) = \dfrac{c-1}{2c}\sin(x) + B & (4.5.15)\end{cases}$$

Since $f(x) + g(x) = \sin(x)$, $A + B = 0$. Finally, the solution $u(x,t) = f(x+ct) + g(x-ct)$ to the problem is given by

$$u(x,t) = \frac{c+1}{2c}\sin(x+ct) + \frac{c-1}{2c}\sin(x-ct). \tag{4.5.16}$$

(b) The function $u(x,t)$ is a periodic function of t if it is possible to find T such that

$$u(x, t+nT) = u(x,t) \tag{4.5.17}$$

for any $n \in \mathbb{N}$.
It can be checked that choosing $T = \frac{2\pi}{c}$ this condition is satisfied so that $u(x,t)$ is a periodic function of t.

(c) The equation can be written as

$$(\partial_t + c\partial_x)(\partial_t - c\partial_x)\, u = \cos(\omega t). \tag{4.5.18}$$

Define the characteristic coordinates

$$\xi = x + ct \quad \text{and} \quad \eta = x - ct. \tag{4.5.19}$$

The chain rule gives

$$\partial_x = \partial_\xi + \partial_\eta \quad \text{and} \quad \partial_t = c\partial_\xi - c\partial_\eta. \tag{4.5.20}$$

such that the equation simplifies as

$$-4c^2\left(\partial_\xi\right)\left(\partial_\eta\right)u = \cos\left(\omega\frac{\xi-\eta}{2c}\right) \tag{4.5.21}$$

$$\Rightarrow \quad \partial_{\xi\eta}u = -\frac{1}{4c^2}\cos\left(\omega\frac{\xi-\eta}{2c}\right) \tag{4.5.22}$$

$$\Rightarrow \quad u(\xi,\eta) = -\frac{1}{\omega^2}\cos\left(\omega\frac{\xi-\eta}{2c}\right) + f(\xi) + g(\eta) \tag{4.5.23}$$

$$\Rightarrow \quad u(x,t) = -\frac{1}{\omega^2}\cos(\omega t) + f(x+ct) + g(x-ct). \tag{4.5.24}$$

The initial conditions then give

$$\begin{cases} u(x,0) = -\frac{1}{\omega^2} + f(x) + g(x) = \sin(x), & (4.5.25)\\ u'(x,0)) = cf'(x) - cg'(x) \qquad = \cos(x). & (4.5.26)\end{cases}$$

Differentiating the first equation yields

$$f'(x) + g'(x) = \cos(x). \tag{4.5.27}$$

Then solving for $f'(x)$ and $g'(x)$ yields

$$\begin{cases} f'(x) = \frac{c+1}{2c}\cos(x) & (4.5.28)\\ g'(x) = \frac{c-1}{2c}\cos(x) & (4.5.29)\end{cases}$$

Integrating Eq.(4.5.29) gives

$$\begin{cases} f(x) = \frac{c+1}{2c}\sin(x) + A & (4.5.30)\\ g(x) = \frac{c-1}{2c}\sin(x) + B & (4.5.31)\end{cases}$$

Since $f(x) + g(x) = \sin(x) + \frac{1}{\omega^2}$, $A + B = \frac{1}{\omega^2}$ and the final solution is

$$u(x,t) = \frac{1}{\omega^2}(1-\cos(\omega t)) + \frac{c+1}{2c}\sin(x+ct) + \frac{c-1}{2c}\sin(x-ct). \tag{4.5.32}$$

(d) The solution Eq.(4.5.32) is bounded $\forall\omega > 0$, hence there is no resonance.

(e) In that case, the solutions is

$$u(x,t) = \frac{1}{\omega^2}(\omega t - \sin(\omega t)) + \frac{c+1}{2c}\sin(x+ct) + \frac{c-1}{2c}\sin(x-ct). \tag{4.5.33}$$

$u(x,t)$ grows with t which indicates resonance.

4.6 Diffusion of a window [Strauss 2.4, Ex. 1]

Solve the following initial value problem using the fundamental solution of the diffusion equation,

$$\begin{cases} u_t - k u_{xx} = 0 & \text{for} \quad (x,t) \in \mathbb{R}\times]0,+\infty[, \\ u(x,0) = 1 & \text{for} \quad |x| < l, \\ u(x,0) = 0 & \text{for} \quad |x| \geq l. \end{cases}$$

Write your answer in terms of $\operatorname{erf}(x)$.
Reminder:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\zeta^2} d\zeta.$$

Solution

The general solution is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\ell}^{\ell} \exp\left(-\left[\frac{x-y}{\sqrt{4kt}}\right]^2\right) dy. \tag{4.6.1}$$

Introducing the following change of variables

$$\xi = \frac{x-y}{\sqrt{4kt}} \tag{4.6.2}$$

gives

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_0^{\frac{x+\ell}{\sqrt{4kt}}} \exp(-\xi^2) d\xi - \frac{1}{\sqrt{\pi}} \int_0^{\frac{x-\ell}{\sqrt{4kt}}} \exp(-\xi^2) d\xi \tag{4.6.3}$$

$$= \frac{1}{2}\left[\operatorname{erf}\left(\frac{x+\ell}{\sqrt{4kt}}\right) - \operatorname{erf}\left(\frac{x-\ell}{\sqrt{4kt}}\right)\right]. \tag{4.6.4}$$

4.7 Diffusion of an exponential [Strauss 2.4, Ex. 3]

Solve the following initial value problem using the fundamental solution of the diffusion equation,

$$\begin{cases} u_t - k u_{xx} = 0 & \text{for} \quad (x,t) \in \mathbb{R}\times]0,+\infty[, \\ u(x,0) = \exp(3x) & \text{for} \quad (x,t) \in \mathbb{R} \times \{0\}. \end{cases}$$

Reminder:

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}} \qquad \text{(Gaussian integral).}$$

Solution

Replacing the given initial conditions in the general formula successively gives

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4kt\pi}} \int_{-\infty}^{+\infty} \exp\left(-\left[\frac{x-y}{\sqrt{4kt}}\right]^2\right) \exp(3y) dy \\ &= \frac{1}{\sqrt{4kt\pi}} \int_{-\infty}^{+\infty} \exp\left(\frac{-y^2+2xy-x^2}{4kt} + 3y\right) dy \\ &= \frac{1}{\sqrt{4kt\pi}} \int_{-\infty}^{+\infty} \exp\left(\frac{-y^2+2xy-x^2+12kty}{4kt}\right) dy. \end{aligned} \tag{4.7.1}$$

Completing the square appearing in the exponential, the solution writes

$$u(x,t) = \frac{1}{\sqrt{4kt\pi}} \int_{-\infty}^{+\infty} \exp\left(\frac{-y^2+2xy-x^2+12kty-36k^2t^2+36k^2t^2+12ktx-12ktx}{4kt}\right) dy \tag{4.7.2}$$

$$= \frac{1}{\sqrt{4kt\pi}} \int_{-\infty}^{+\infty} \exp\left(-\left[\frac{x-y+6kt}{\sqrt{4kt}}\right]^2 + \frac{36k^2t^2+12ktx}{4kt}\right) dy \tag{4.7.3}$$

$$= \frac{\exp(9kt+3x)}{\sqrt{4kt\pi}} \int_{-\infty}^{+\infty} \exp\left(-\left[\frac{x-y+6kt}{\sqrt{4kt}}\right]^2\right) dy. \tag{4.7.4}$$

Finally using the change of variable

$$\xi = \frac{x-y+6kt}{\sqrt{4kt}}, \tag{4.7.5}$$

the solution can be written as follows

$$u(x,t) = \frac{\exp(9kt+3x)}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp\left(-\xi^2\right) d\xi \tag{4.7.6}$$

$$= \exp\left(3(x+3kt)\right). \tag{4.7.7}$$

4.8 Diffusion with constant dissipation [Strauss 2.4, Ex. 16]

Solve the following diffusion problem with constant dissipation

$$\begin{cases} u_t - ku_{xx} + bu = 0 & \text{for} \quad (x,t) \in \mathbb{R}\times]0,+\infty[, \\ u(x,0) = \delta(x) & \text{for} \quad (x,t) \in \mathbb{R}\times\{0\}. \end{cases}$$

where $b \in \mathbb{R}_0^+$ is a constant parameter.
Reminder: The Dirac distribution δ has the following property

$$\int_{-\infty}^{+\infty} f(x)\delta(x)dx = f(0).$$

Hint: use the substitution $u(x,t) = e^{-bt}v(x,t)$.

Solution

Using the substitution $u(x,t) = e^{-bt}v(x,t)$, the terms in the equation become

$$u_t = -b\exp(-bt)\,v + \exp(-bt)\,v_t, \tag{4.8.1}$$
$$u_{xx} = \exp(-bt)\,v_{xx}. \tag{4.8.2}$$

Thus the equation takes the form

$$\exp(-bt)\,v_t - k\exp(-bt)\,v_{xx} = 0, \tag{4.8.3}$$

which can be simplified into an equation of the known form:

$$v_t - kv_{xx} = 0. \tag{4.8.4}$$

The initial condition writes $v(x,0) = \delta(x)$. The solution of this equation is then written as

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{+\infty}\exp\left(-\left[\frac{x-y}{\sqrt{4kt}}\right]^2\right)\delta(y)\mathrm{d}y \tag{4.8.5}$$
$$= \frac{1}{\sqrt{4\pi kt}}\exp\left(\frac{-x^2}{4kt}\right) \tag{4.8.6}$$

and finally

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\exp\left(\frac{-x^2}{4kt}\right)\exp(-bt). \tag{4.8.7}$$

4.9 Heat equation with convection [Strauss 2.4, Ex. 18]

Consider the following diffusion problem with convection

$$\begin{cases} u_t - ku_{xx} + Vu_x = 0 & \text{for} \quad (x,t) \in \mathbb{R}\times]0,+\infty[, \\ u(x,0) = \phi(x) & \text{for} \quad (x,t) \in \mathbb{R}\times\{0\}, \end{cases}$$

where V is a constant.

(a) Solve this problem for a general ϕ.
Hint: Consider a moving reference frame by using the change of variables $y = x - Vt$ and $z = t$.

(b) Compute the solution for the initial condition $\phi(x) = 1$ for $|x| \leq l$. Write your answer in terms of $\mathrm{erf}(x)$.

Solution

(a) Performing the proposed change of variable, the differential operators write

$$\partial_x = \frac{\partial y}{\partial x}\,\partial_y + \frac{\partial z}{\partial x}\,\partial_z = \partial_y, \tag{4.9.1}$$
$$\partial_t = \frac{\partial y}{\partial t}\,\partial_y + \frac{\partial z}{\partial t}\,\partial_z = -V\,\partial_y + \partial_z. \tag{4.9.2}$$

The equation becomes

$$-Vu_y + u_z - ku_{yy} + Vu_y = 0, \tag{4.9.3}$$

which simplifies as

$$u_z - ku_{yy} = 0. \tag{4.9.4}$$

The general solution of Eq.(4.9.4) is

$$u(y, z) = \frac{1}{\sqrt{4\pi kz}} \int_{-\infty}^{+\infty} \exp\left(-\left[\frac{y-\zeta}{\sqrt{4kz}}\right]^2\right) \phi(\zeta)\mathrm{d}\zeta, \tag{4.9.5}$$

for a general function ϕ.

(b) Writing Eq.(4.9.5) with the initial condition $\phi(x) = 1$ for $|x| \leq l$ gives

$$u(y, z) = \frac{1}{\sqrt{4\pi kz}} \int_{-\ell}^{+\ell} \exp\left(-\left[\frac{y-\zeta}{\sqrt{4kz}}\right]^2\right) \mathrm{d}\zeta. \tag{4.9.6}$$

Therefore, the solution writes

$$u(y, z) = \frac{1}{2}\left[\mathrm{erf}\left(\frac{y+l}{\sqrt{4kz}}\right) - \mathrm{erf}\left(\frac{y-l}{\sqrt{4kz}}\right)\right], \tag{4.9.7}$$

$$= \frac{1}{2}\left[\mathrm{erf}\left(\frac{x-Vt+l}{\sqrt{4kt}}\right) - \mathrm{erf}\left(\frac{x-Vt-l}{\sqrt{4kt}}\right)\right]. \tag{4.9.8}$$

4.10 Electric potential generated by a charged sphere 🌶🌶🌶

In the whole space of dimension 3, a sphere of radius $a > 0$ has a uniform surface charge σ on its surface, with no charge in its volume. Using the fundamental solution of the Laplace equation

$$S(\boldsymbol{x}) = \frac{-1}{4\pi \|\boldsymbol{x}\|},$$

give the expression of the electrostatic potential and electric field in $\mathbb{R}^3$. Laws of electrostatics are, for the electric field $\boldsymbol{e}$, the charge density ρ and the constant permittivity ε_0,

$$\nabla \times \boldsymbol{e} = \boldsymbol{0}, \qquad \nabla \cdot \boldsymbol{e} = \frac{\rho}{\varepsilon_0}.$$

(a) From the first equation (Faraday's law), one can define the electrostatic potential v such that $\boldsymbol{e} = -\nabla v$. Express the second equation (Gauss's law) in terms of this electrostatic potential.

(b) Express the charge density ρ in terms of the surface charge σ. Give the expression of the total charge q contained on the sphere.

(c) Solve the electrostatic problem with the condition $v(\|\boldsymbol{x}\| \to +\infty) = 0$, inside and outside the charged sphere: give the expression of v and $\boldsymbol{e}$, expressed in terms of q.

(d) Show that outside the sphere, the solution is identical to that for a point charge of the same value q.

Solution

(a) By substitution of $\boldsymbol{e}$ by $-\nabla v$, we get the Poisson equation

$$\Delta v = -\frac{\rho}{\varepsilon_0}. \tag{4.10.1}$$

(b) Due to the spherical symmetry of the problem, we consider a spherical coordinate system, centered at the center of the sphere. The charge is concentrated on the surface of the sphere, therefore

$$\rho(r) = \sigma\delta(r-a). \tag{4.10.2}$$

The total charge q is obtained by integrating σ, which is constant, on the surface of the sphere. This gives $q = 4\pi a^2\sigma$.

(c) The potential can be obtained by the convolution of the Green's function with the source function

$$v(\boldsymbol{x}) = \int_{\mathbb{R}^3} -\frac{\rho(\boldsymbol{x}')}{\epsilon_0} S(\boldsymbol{x}-\boldsymbol{x}')\, d\boldsymbol{x}'. \tag{4.10.3}$$

In spherical coordinates, with (r,θ,ϕ) the coordinates at which we want the value of the potential and (r',θ',ϕ') the coordinates of integration for the convolution,

$$v(r,\theta,\phi) = \int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \int_0^{+\infty} \frac{\sigma\delta(r'-a)r'^2\sin\theta'}{4\pi\varepsilon_0\|\boldsymbol{x}-\boldsymbol{x}'\|} dr'. \tag{4.10.4}$$

By symmetry, $v(r,\theta,\phi) = v(r)$. For simplicity, we can choose $\theta = 0$ (on the z-axis). Consequently, $\|\boldsymbol{x}-\boldsymbol{x}'\|$ becomes $\sqrt{(r-r'\cos\theta')^2 + r'^2\sin^2\theta'} = \sqrt{r^2 - 2r'r\cos\theta' + r'^2}$. After integration on ϕ' and using the change of variable for θ'

$$r^2 - 2r'r\cos\theta' + r'^2 = t, \tag{4.10.5}$$

$$2r'r\sin\theta' d\theta' = dt, \tag{4.10.6}$$

we get

$$v(r) = 2\pi\frac{\sigma}{4\pi\varepsilon_0}\int_0^{+\infty} dr' \int_{(r-r')^2}^{(r+r')^2} \frac{\delta(r'-a)r'}{2r}\frac{1}{\sqrt{t}}dt, \tag{4.10.7}$$

$$= \frac{\sigma}{2\varepsilon_0}\int_0^{+\infty} \frac{\delta(r'-a)r'}{2r} 2(\sqrt{(r+r')^2} - \sqrt{(r-r')^2})dr', \tag{4.10.8}$$

$$= \frac{\sigma}{2r\varepsilon_0}\int_0^{+\infty} \delta(r'-a)r'(|r+r'| - |r-r'|)dr'. \tag{4.10.9}$$

From the definition of the Dirac distribution, the integral is direct and gives

$$v(r) = \frac{\sigma a(|r+a| - |r-a|)}{2r\varepsilon_0}, \tag{4.10.10}$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{|r+a| - |r-a|}{2ra}, \tag{4.10.11}$$

$$= \begin{cases} \dfrac{q}{4\pi\varepsilon_0 r} & \text{if} \quad r \geq a, \\ \dfrac{q}{4\pi\varepsilon_0 a} & \text{if} \quad r < a. \end{cases} \tag{4.10.12}$$

The potential is constant inside the sphere, and decreases as $1/r$ to zero outside the sphere. The electric field $\boldsymbol{e}$ is given by (gradient in spherical coordinates, using the symmetry of the potential v)

$$\boldsymbol{e} = -\nabla v = -\frac{\partial v}{\partial r}\hat{\boldsymbol{r}} = \begin{cases} \dfrac{q}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}} & \text{if} \quad r \geq a, \\ \mathbf{0} & \text{if} \quad r < a. \end{cases} \tag{4.10.13}$$

(d) If the source is a point charge $\rho = q\delta(\boldsymbol{x})$, the solution is the Green's function multiplied by $-q/\varepsilon_0$, by definition (and linearity of the problem):

$$v(r) = \frac{q}{4\pi\varepsilon_0 r}, \quad r > 0 \qquad \text{and} \qquad \boldsymbol{e} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}}, \quad r > 0. \tag{4.10.14}$$

This is the same solution than for the uniformly charged surface sphere.

5 Separation of variables

5.1 Diffusion equation I 🌶🌶

Consider the one-dimensional diffusion equation on a bounded domain

$$u_t - k u_{xx} = 0 \quad \forall x \in]0,1[\tag{‡}$$

with initial condition

$$u(x,0) = \phi(x) \quad \forall x \in]0,1[$$

and with some boundary conditions at $x = 0$ and $x = 1$ (these boundary conditions will be specified later).

(a) Using separation of variables $u(x,t) = w(t)v(x)$, find all the separable solutions of Eq. (‡).

(b) Find the solution to Eq. (‡) for the homogeneous Dirichlet boundary conditions $u(0,t) = u(1,t) = 0 \, \forall t \geq 0$ and for $\phi(x) = \sin \pi x$.

(c) Find the solution to Eq. (‡) for the homogeneous Dirichlet boundary conditions $u(0,t) = u(1,t) = 0 \, \forall t \geq 0$ and for

$$\phi(x) = \begin{cases} x & \forall x \in \left]0, \dfrac{1}{2}\right], \\ 1 - x & \forall x \in \left[\dfrac{1}{2}, 1\right[. \end{cases}$$

(d) Find the solution to Eq. (‡) for the homogeneous Dirichlet boundary conditions $u(0,t) = u(1,t) = 0$ $\forall t \geq 0$ and for $\phi(x) = 1$.

(e) Find the solution to Eq. (‡) for the homogeneous Neumann boundary conditions $u_x(0,t) = u_x(1,t) = 0$ $\forall t \geq 0$ and for $\phi(x) = \cos \pi x$.

(f) Find the solution to Eq. (‡) for the non-homogeneous Dirichlet boundary conditions $u(0,t) = 0$ and $u(1,t) = 1$, $\forall t \geq 0$ and for $\phi(x) = x + \sin \pi x$.

Solution

(a) Using the ansatz $u = wv$, the diffusion equation writes as

$$w'v - kwv'' = 0 \tag{5.1.1}$$

$$\Rightarrow \quad \frac{w'}{kw} = \frac{v''}{v}. \tag{5.1.2}$$

Since the right hand side depends only on x while the left hand side depends only on t, both sides must be equal to a constant (named λ), *i.e*

$$v'' - \lambda v = 0 \quad \text{and} \quad w' - k\lambda w = 0. \tag{5.1.3}$$

Spatial dependence Depending on the sign of λ, three solutions arise, *i.e*

$$\text{if } \lambda = 0 \qquad \Rightarrow v = Ax + B, \tag{5.1.4}$$

$$\text{if } \lambda = \omega^2 > 0 \qquad \Rightarrow v = C\exp(-\omega x) + D\exp(\omega x), \tag{5.1.5}$$

$$\text{if } \lambda = -\omega^2 < 0 \qquad \Rightarrow v = E\cos(\omega x) + F\sin(\omega x). \tag{5.1.6}$$

Time dependence Whatever the sign of λ, the time dependence is given by

$$w = \exp(\lambda k t). \tag{5.1.7}$$

The constant in front the exponential is here omitted because the field of interest is $u = vw$.

(b) Among all the eigensolutions found in the previous sub-question (*i.e* for any value of λ), only those that satisfy the homogeneous boundary conditions are kept.

Stationary eigensolutions, *i.e*: $\lambda = 0$ Applying the boundary conditions to $v = Ax + B$ gives

$$\begin{cases} v(0) &= B &= 0 \\ v(1) &= A + B &= 0 \end{cases} \quad \Rightarrow A = B = 0 \tag{5.1.8}$$

such that there is no stationary eigensolution satisfying the homogeneous boundary conditions.

Time-growing eigensolutions, *i.e*: $\lambda > 0$ Applying the boundary conditions to $v = C\exp(-\omega x) + D\exp(\omega x)$ gives

$$\begin{cases} v(0) &= C + D &= 0 \\ v(1) &= C\exp(-\omega) + D\exp(\omega) &= 0 \end{cases} \quad \Rightarrow \begin{pmatrix} 1 & 1 \\ \exp(-\omega) & \exp(\omega) \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5.1.9}$$

provided that $\omega > 0$, the unique solution to Eqs(5.1.28) is

$$C = D = 0 \tag{5.1.10}$$

thus there is no time growing eigensolution satisfying the homogeneous boundary condition neither.

Time-decaying eigensolutions, *i.e*: $\lambda < 0$ Applying the boundary conditions to $v = E\cos(\omega x) + F\cos(\omega x)$ gives

$$\begin{cases} v(0) &= E &= 0 \\ v(1) &= E\cos(\omega) + F\sin(\omega) &= 0 \end{cases} \quad \Rightarrow F\sin(\omega) = 0 \quad \Rightarrow \omega \to \omega_n = n\pi, \quad n = 1, 2, 3, \dots \tag{5.1.11}$$

One should be carefull that the values $\omega_n = n\pi$ for $n < 0$ are not considered because they yield the same eigenvalue $\lambda_n = -\omega_n^2$.
The only eigensolutions compatible with the boundary conditions are therefore

$$v_n(x) = F_n \sin(\omega_n x), \quad \forall n = 1, 2, 3, \dots \tag{5.1.12}$$

and the most general solution compatible with boundary condition is then

$$u(x,t) = \sum_{n=1}^{\infty} F_n \sin(\omega_n x) \exp\left(-\omega_n^2 k t\right). \tag{5.1.13}$$

All the constants F_n must still be determined through the initial condition

$$u(x,0) = \phi(x) = \sum_{n=1}^{\infty} F_n \sin n\pi x. \tag{5.1.14}$$

Eq.(5.1.14) is actually a Fourier sine series which has the following orthogonality property

$$F_m = 2\int_0^1 \phi(x)\sin(m\pi x)\,dx. \tag{5.1.15}$$

In the particular case where $\phi(x) = \sin\pi x$, the unknwown coeficient are given by

$$F_m = 2\int_0^1 \sin(\pi x)\sin(m\pi x)\,dx = \delta_{1m} \tag{5.1.16}$$

such that the final solution is

$$u(x,t) = \sum_{n=1}^{\infty} F_n \sin(\omega_n x)\exp(-\omega_n^2 kt) = \sin(\pi x)\exp(-\pi^2 kt). \tag{5.1.17}$$

(c) The boundary conditions are the same than for the previous subquestion such that the same set of eigen-solutions must be conserved, only the coefficient F_n of the Fourier sine series change. The Fourier coefficients are given by

$$F_m = 2\int_0^1 \phi(x)\sin(m\pi x)\,dx \tag{5.1.18}$$

$$= 2\int_0^{1/2} x\sin(m\pi x)\,dx + 2\int_{1/2}^1 (1-x)\sin(m\pi x)\,dx \tag{5.1.19}$$

$$= \frac{4\sin\left(\frac{m\pi}{2}\right)}{m^2\pi^2} = \begin{cases} \dfrac{4(-1)^l}{(2l+1)^2\pi^2}, & m = 2l+1. \\ 0, & m = 2l. \end{cases} \tag{5.1.20}$$

The final solution is therfore given by

$$u(x,t) = \frac{4}{\pi^2}\sum_{l=0}^{\infty}(-1)^l \frac{\sin\Big((2l+1)\pi x\Big)}{(2l+1)^2}\exp\Big(-(2l+1)^2\pi^2 kt\Big) \tag{5.1.21}$$

(d) The boundary conditions are the same than for the previous subquestion such that the same set of eigen-solutions must be conserved, only the coefficient F_n of the Fourier sine series change. The Fourier coefficients are given by

$$F_m = 2\int_0^1 \phi(x)\sin(m\pi x)\,dx \tag{5.1.22}$$

$$= 2\int_0^1 \sin(m\pi x)\,dx \tag{5.1.23}$$

$$= 2\left[-\frac{\cos m\pi x}{m\pi}\right]_0^1 \tag{5.1.24}$$

$$= \frac{2}{m\pi}(1+(-1)^{m+1}) = \begin{cases} \dfrac{4}{(2l+1)\pi}, & m = 2l+1. \\ 0, & m = 2l. \end{cases} \tag{5.1.25}$$

The final solution is therfore given by

$$u(x,t) = \sum_{n=1}^{\infty} F_n \sin(\omega_n x) \exp(-\omega_n^2 kt) = \sum_{l=0}^{\infty} \frac{4}{(2l+1)\pi} \sin((2l+1)\pi x) \exp(-(2l+1)^2\pi^2 kt) \quad (5.1.26)$$

(e) Among all the eigensolutions found in the previous sub-question (*i.e* for any value of λ), only those that satisfy the homogeneous boundary conditions are kept.

Stationary eigensolutions, *i.e*: $\lambda = 0$ Applying the boundary conditions to $v = Ax + B$ gives

$$\begin{cases} v_x(0) &= A &= 0 \\ v_x(1) &= A &= 0 \end{cases} \Rightarrow v(x) = B. \quad (5.1.27)$$

such that only the constant functions is a stationary eigensolution satisfying the homogeneous boundary conditions.

Time-growing eigensolutions, *i.e*: $\lambda > 0$ Applying the boundary conditions to $v_x = -\omega C \exp(-\omega x) + \omega D \exp(\omega x)$ gives

$$\begin{cases} v_x(0) &= -\omega C + \omega D &= 0 \\ v(1) &= -\omega C \exp(-\omega) + \omega D \exp(\omega) &= 0 \end{cases} \Rightarrow \begin{pmatrix} -1 & 1 \\ -\exp(-\omega) & \exp(\omega) \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (5.1.28)$$

provided that $\omega > 0$, the unique solution to Eqs(5.1.28) is

$$C = D = 0 \quad (5.1.29)$$

thus there is no time growing eigensolution satisfying the homogeneous boundary condition neither.

Time-decaying eigensolutions, *i.e*: $\lambda < 0$ Applying the boundary conditions to $v_x = -\omega E \sin(\omega x) + \omega F \cos(\omega x)$ gives

$$\begin{cases} v_x(0) &= \omega F &= 0 \\ v_x(1) &= -\omega E \sin(\omega) &= 0 \end{cases} \Rightarrow E \sin(\omega) = 0 \quad \Rightarrow \omega \to \omega_n = n\pi, \quad n = 1, 2, 3, \ldots \quad (5.1.30)$$

One should be carefull that the values $\omega_n = n\pi$ for $n < 0$ are not considered because they yield the same eigenvalue $\lambda_n = -\omega_n^2$.
The only eigensolutions compatible with the boundary conditions are therefore

$$v_n(x) = E_n \cos(\omega_n x), \quad \forall n = 1, 2, 3, \ldots \quad (5.1.31)$$

and the most general solution compatible with boundary condition is then

$$u(x,t) = B + \sum_{n=1}^{\infty} E_n \cos(\omega_n x) \exp\left(-\omega_n^2 kt\right). \quad (5.1.32)$$

All the constants E_n and B must still be determined through the initial condition. By identification, it is straighforward to find that

$$B = 0 \quad \text{and} \quad E_1 = 1, E_m = 0 \forall m \neq 1 \tag{5.1.33}$$

and finally the solution is

$$u(x,t) = \cos(\pi x) \exp\left(-\pi^2 kt\right). \tag{5.1.34}$$

(f) The problem here is that the boundary conditions are not homogeneous. Therefore, if all modes satisfy the non-homogeneous boundary conditions, their sum will not satisy the boundary condition. The idea is then to first choose a group of eigenfunctions that satisfy the non-homogeneous boundary condition, then to add the eigensolutions of the problem with homogeneous boundary conditions. This is actually the same idea than when solving an initial value problem, the whole solution is the sum of a particular solution and the homogeneous solution.

First, the procedure is therefore to find a set of solutions that satisfy th non homogeneous boundary condition

$$\begin{cases} u(0,t) & = w(t)v(0) & = 0 & \forall t, \\ u(1,t) & = w(t)v(1) & = 1 & \forall t. \end{cases} \tag{5.1.35}$$

While obviously the boundary condition at $x = 0$ yields $v(0) = 0$, the boundary condition at $x = 1$ implies that $w(t)$ does not depend on t. The natural eigensolutions to consider to satisfy the non-homogeneous solutions must therefore have a temporal part that is constant, which is typically the case for $\lambda = 0$. Applying the boundary condition to the steady eigensolutions one finds

$$\begin{cases} u(0,t) & = w(t)v(0) & = \exp 0kt\,(B) & = 0 & \forall t \\ u(1,t) & = w(t)v(1) & = \exp 0kt\,(A+B) & = 1 & \forall t \end{cases} \Rightarrow \begin{cases} B & = 0 \\ A & = 1 \end{cases} \tag{5.1.36}$$

Now that a unique solution satisfying non-homogeneous boundary conditions is known, all the solutions satisfying the homogeneous boundary condition must be added. Referring to sub-questions, the solution becomes

$$u(x,t) = x \exp 0kt + \sum_{n=1}^{\infty} F_n \sin(\omega_n x) \exp(-\omega_n^2 kt), \tag{5.1.37}$$

then to initial condition yields

$$u(x,0) = x + \sum_{n=1}^{\infty} F_n \sin(\omega_n x) = \phi(x) = x + \sin(\pi x), \tag{5.1.38}$$

and, as in sub-question, the solution is

$$u(x,t) = x + \sin(\pi x) \exp(-\pi^2 kt). \tag{5.1.39}$$

5.2 Wave equation I

Consider the one-dimensional wave equation on a bounded domain

$$u_{tt} - c^2 u_{xx} = 0 \quad \forall x \in]0,1[\tag{$\diamond$}$$

with initial condition

$$u(x,0) = \phi(x) \quad \text{and} \quad u_t(x,0) = \psi(x) \quad \forall x \in]0,1[$$

and with some boundary conditions at $x = 0$ and $x = 1$ (these conditions will be specified later).

(a) Using separation of variable $u(x,t) = w(t)v(x)$, find all the separable solution of Eq. ($\diamond$).

(b) Find the solution to Eq. ($\diamond$) for the homogeneous Dirichlet boundary conditions $u(0,t) = u(1,t) = 0\,\forall t \geq 0$, for $\phi(x) = \sin \pi x$ and for $\psi(x) = 0$.

(c) Find the solution to Eq. ($\diamond$) for the homogeneous Dirichlet boundary conditions $u(0,t) = u(1,t) = 0\,\forall t \geq 0$, for

$$\phi(x) = \begin{cases} x & \forall x \in]0, \frac{1}{2}] \\ 1-x & \forall x \in [\frac{1}{2}, 1[, \end{cases}$$

and for $\psi(x) = 0$.

(d) Find the solution to Eq. ($\diamond$) for the homogeneous Dirichlet boundary conditions $u(0,t) = u(1,t) = 0\,\forall t \geq 0$ for $\phi(x) = \sin \pi x$ and for $\psi(x) = c \sin \pi x$.

Solution

(a) Using the ansatz $u = wv$, the wave equation writes as

$$w''v - c^2 wv'' = 0 \tag{5.2.1}$$
$$\Rightarrow \frac{w''}{c^2 w} = \frac{v''}{v}. \tag{5.2.2}$$

Since the right hand side depends only on x while the left hand side depends only on t, both sides must be equal to a constant (named λ), *i.e*

$$v'' - \lambda v = 0 \quad \text{and} \quad w'' - c^2 \lambda w = 0. \tag{5.2.3}$$

Spatial dependence Depending on the sign of λ, three solutions arise, *i.e*

$$\text{if } \lambda = 0 \quad \Rightarrow v = Ax + B, \tag{5.2.4}$$
$$\text{if } \lambda = \omega^2 > 0 \quad \Rightarrow v = C\exp(-\omega x) + D\exp(\omega x), \tag{5.2.5}$$
$$\text{if } \lambda = -\omega^2 < 0 \quad \Rightarrow v = E\cos(\omega x) + F\sin(\omega x). \tag{5.2.6}$$

Time dependence Depending on the sign of λ, three solutions arise, *i.e*

$$\text{if } \lambda = 0 \quad \Rightarrow v = Gt + H, \tag{5.2.7}$$
$$\text{if } \lambda = \omega^2 > 0 \quad \Rightarrow v = I\exp(-c\omega t) + J\exp(c\omega t), \tag{5.2.8}$$
$$\text{if } \lambda = -\omega^2 < 0 \quad \Rightarrow v = K\cos(c\omega t) + L\sin(c\omega t). \tag{5.2.9}$$

(b) The eigenvalues are given by $\lambda_n = -\omega_n^2 = -n^2\pi^2$ with $n = 1, 2, 3, ...$ such that the only compatible eigensolutions are

$$v_n(x) = F_n \sin(w_n x), \quad \forall n = 1, 2, 3, ... \tag{5.2.10}$$

The most general way to write the solution is

$$u(x,t) = \sum_{n=1}^{\infty} [A_n \cos(nc\pi t) + B_n \sin(nc\pi t)] \sin(w_n x). \tag{5.2.11}$$

It remains to determine the constants A_n et B_n using initial conditions. Since the time dependence of the wave equation governed by a second order differential equation there are two initial conditions. One will help finding the A_n and the other will help determining the B_n.
At $t = 0$ one has

$$u(x,0) = \sin(\pi x) = \sum_{n=1}^{+\infty} A_n \sin(n\pi x) \tag{5.2.12}$$

$$u_t(x,0) = 0 = \sum_{n=1}^{+\infty} B_n cn\pi \sin(n\pi x) \tag{5.2.13}$$

The B_n are immediately identified as 0. It can be shown that $A_n = \delta_{1n}$ using the orthogonality of Fourier sine series. The final solution is then

$$u(x,t) = \cos(c\pi t)\sin(\pi x). \tag{5.2.14}$$

(c) Similarly to the preceding sub-questions, starting from Eq.(5.2.11), the constants A_n and B_n are determined with the initial conditions. First,

$$u_t(x,0) = 0 = \sum_{n+1}^{+\infty} nc\pi B_n \sin(w_n x). \tag{5.2.15}$$

As the functions $\sin(w_n x)$ form a basis of $]0,1[$, the only possible combination of B_n satisfying the equality is $B_n = 0$. The second initial condition is

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(w_n x) = \phi(x). \tag{5.2.16}$$

Using then the orthogonality property of the $\sin$ functions,

$$A_n = 2\int_0^1 \phi(x)\sin(n\pi x)dx \tag{5.2.17}$$

$$= 2\int_0^{1/2} x\sin(n\pi x)dx + 2\int_{1/2}^1 (1-x)\sin(n\pi x)dx. \tag{5.2.18}$$

Integrating by parts gives

$$A_n = \frac{4\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2} = \begin{cases} \dfrac{4(-1)^k}{(2k+1)^2\pi^2}, & n = 2k+1, \\ 0, & n = 2k. \end{cases} \tag{5.2.19}$$

Therefore the solution is

$$u(x,t) = \frac{4}{\pi^2}\sum_{k=0}^{\infty}(-1)^k \frac{\cos\Big((2k+1)c\pi t\Big)\sin\Big((2k+1)\pi x\Big)}{(2k+1)^2}. \tag{5.2.20}$$

(d) Similar to preceeding sub-questions, the A_n are given by

$$A_n = \delta_{1n} \tag{5.2.21}$$

while the B_n are derived starting from

$$u_t(x,0) = c\sin(\pi x) = \sum_{n=1}^{+\infty} B_n cn\pi \sin(n\pi x). \tag{5.2.22}$$

Using the same procedure as for the determination of the A_n one can show that

$$B_n = \frac{\delta_{1n}}{n\pi}. \tag{5.2.23}$$

$$u(x,t) = \left[\cos(c\pi t) + \frac{\sin(c\pi t)}{\pi}\right]\sin(\pi x). \tag{5.2.24}$$

5.3 Laplace equation I

Consider the two-dimensional Laplace equation on a square

$$u_{xx} + u_{yy} = 0 \quad \{x, y\} \in]0,1[\times]0,1[.$$

(a) Using separation of variables $u(x,y) = v(x)w(y)$, find all the separable solutions of this equation.

(b) Find the set of separable solutions that verify the following homogeneous boundary conditions

$$\begin{cases} u(0,y) = 0 & \forall y \in [0,1], \\ u(1,y) = 0 & \forall y \in [0,1], \\ u(x,1) = 0 & \forall x \in [0,1]. \end{cases}$$

(c) Show that the Fourier sine series expansion of

$$\phi(x) = \begin{cases} x & \forall x \in]0, \frac{1}{2}], \\ 1-x & \forall x \in [\frac{1}{2}, 1[\end{cases}$$

can be expressed as

$$\phi(x) = \frac{4}{\pi^2}\sum_{j=0}^{\infty}(-1)^j \frac{\sin\left((2j+1)\pi x\right)}{(2j+1)^2}.$$

(d) With the boundary conditions given in (b) and

$$u(x,0) = \phi(x), \quad \forall x \in [0,1]$$

for the last edge, show that the solution to the boundary value problem is then

$$u(x,y) = \frac{4}{\pi^2}\sum_{j=0}^{\infty}(-1)^j \frac{\sin\left((2j+1)\pi x\right)\sinh\left((2j+1)\pi(1-y)\right)}{(2j+1)^2\sinh\left((2j+1)\pi\right)}.$$

Solution

(a) Using separation of variables, Laplace equation can be written

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda. \tag{5.3.1}$$

Therefore, both spatial part of the solution depends on the sign of the same constant λ. The following table presents all the separable solutions as a function of λ.

	Part $X(x)$	Part $Y(y)$
$\lambda = 0$	$X(x) = Ax + B$	$Y(y) = Cy + D$
$\lambda = \omega^2 > 0$	$X(x) = E\cosh(\omega x) + F\sinh(\omega x)$	$Y(y) = G\cos(\omega y) + H\sin(\omega y)$
$\lambda = -\omega^2 < 0$	$X(x) = I\cos(\omega x) + J\sin(\omega x)$	$Y(y) = K\cosh(\omega y) + L\sinh(\omega y)$

(b) Applying the boundary condition, depending on the sign of λ yields

(1) If $\lambda = 0$

$$\begin{cases} B = 0, \\ A + B = 0, \\ C + D = 0, \end{cases} \tag{5.3.2}$$

which admits the only trivial solution $A = B = 0$.

(2) If $\lambda = \omega^2 > 0$

$$\begin{cases} E = 0, \\ E\cosh(\omega) + F\sinh(\omega) = 0, \\ G\cos(\omega) + H\sin(\omega) = 0, \end{cases} \tag{5.3.3}$$

which admits the only trivial solution $E = F = 0$.

(3) If $\lambda = -\omega^2 < 0$

$$\begin{cases} I = 0, \\ I\cos(\omega) + J\sin(\omega) = 0, \\ K\cosh(\omega) + L\sinh(\omega) = 0, \end{cases} \tag{5.3.4}$$

which admits non trivial solution for $\omega_n = n\pi$ and $L = -\dfrac{K}{\tanh\omega}$.

Therefore, the solution writes

$$u(x,y) = \sum_{n=1}^{\infty} K_n \sin(n\pi x)\left[\cosh(n\pi y) - \frac{\sinh(n\pi y)}{\tanh(n\pi)}\right]. \tag{5.3.5}$$

Alternatively, to simplify the notation, one could also write the mode $Y(y)$ when $\lambda = -\omega^2 < 0$ under the form

$$Y(y) = K\cosh(\omega(1-y)) + L\sinh(\omega(1-y)), \tag{5.3.6}$$

such that the application of the boundary conditions simply gives

$$K = 0. \tag{5.3.7}$$

With these notations, the solution writes

$$u(x,y) = \sum_{n=1}^{\infty} L_n \sin(n\pi x)\sinh(n\pi(1-y)). \tag{5.3.8}$$

It is this expression that will be used in the following because of its simplicity.

(c) The coefficient of the Fourrier expansion of the function $\phi(x)$ can be obtained by calculating the integrals

$$2\int_0^1 \sin(n\pi x)\phi(x)\mathrm{d}x, \tag{5.3.9}$$

which can be integrated by part.

(d) Applying the last boundary condition gives

$$\begin{aligned} & u(x,0) = \phi(x), \\ \Rightarrow & \sum_{n=1}^{\infty} L_n \sin(n\pi x)\sinh(n\pi) = \frac{4}{\pi^2}\sum_{j=0}^{\infty}(-1)^j \frac{\sin\left((2j+1)\pi x\right)}{(2j+1)^2}, \end{aligned} \tag{5.3.10}$$

Identifying the terms in those series, one can find the expression for L_n:

$$L_n = \begin{cases} 0 & \text{if } n \text{ pair}, \\ \dfrac{-4}{\pi^2 n^2 \sinh(n\pi)} & \text{if } n \text{ odd and } n \mod 4 = 1, \\ \dfrac{4}{\pi^2 n^2 \sinh(n\pi)} & \text{if } n \text{ odd and } n \mod 4 = 3. \end{cases} \tag{5.3.11}$$

Replacing this in Eq. (5.3.8) and redefining the summation index, finally gives

$$u(x,y) = \frac{4}{\pi^2}\sum_{j=0}^{\infty}(-1)^j \frac{\sin\left((2j+1)\pi x\right)\sinh\left((2j+1)\pi(1-y)\right)}{(2j+1)^2\sinh\left((2j+1)\pi\right)}. \tag{5.3.12}$$

5.4 Laplace equation II

Consider the following equation on a square

$$u_{xx} + \frac{1}{4}u_{yy} = 0 \quad \{x, y\} \in]0,1[\times]0,1[$$

with the boundary conditions

$$\begin{cases} u_x(0,y) = 0 & \forall y \in [0,1], \\ u_x(1,y) = 0 & \forall y \in [0,1], \\ u(x,1) = 0 & \forall x \in [0,1], \\ u(x,0) = 2\cos 2\pi x - 1 & \forall x \in [0,1]. \end{cases}$$

Using separation of variables $u(x, y) = v(x)w(y)$, find the solution to this boundary value problem.

Solution

Using separation of variables, the equation can be written:

$$\frac{X''}{X} = -\frac{Y''}{4\,Y} = \lambda. \tag{5.4.1}$$

Therefore, both spatial part of the solution depends on the sign of the same constant λ.
The following table presents all the separable solutions as a function of λ.

	Part $X(x)$	Part $Y(y)$
$\lambda = 0$	$X(x) = Ax + B$	$Y(y) = Cy + D$
$\lambda = \omega^2 > 0$	$X(x) = E\cosh(\omega x) + F\sinh(\omega x)$	$Y(y) = G\cos(2\omega y) + H\sin(2\omega y)$
$\lambda = -\omega^2 < 0$	$X(x) = I\cos(\omega x) + J\sin(\omega x)$	$Y(y) = K\cosh(2\omega y) + L\sinh(2\omega y)$

Applying the homogeneous boundary conditions on each mode, depending on the sign of λ, one has:

(1) If $\lambda = 0$

$$\begin{cases} A = 0, \\ A = 0, \\ C + D = 0, \end{cases} \tag{5.4.2}$$

which admits non trivial solutions for $A = 0$ and $D = -C$. Redefining a new constant, the associated mode can be written

$$X_0(x)Y_0(y) = C_0(y-1). \tag{5.4.3}$$

(2) If $\lambda = \omega^2 > 0$

$$\begin{cases} \omega F = 0, \\ \omega\left[E\sinh(\omega) + F\cosh(\omega)\right] = 0, \\ G\cos(2\omega) + H\sin(2\omega) = 0, \end{cases} \tag{5.4.4}$$

which admits the only trivial solution $E = F = 0$.

(3) If $\lambda = -\omega^2 < 0$

$$\begin{cases} J = 0, \\ \omega\left[I\sin(\omega) + J\cos(\omega)\right] = 0, \\ K\cosh(2\omega) + L\sinh(2\omega) = 0, \end{cases} \tag{5.4.5}$$

which admits non trivial solution for $\omega_n = n\pi$ and $L = -\dfrac{K}{\tanh(2\omega)}$.

Therefore, the solution writes

$$u(x,y) = C_0(y-1) + \sum_{n=1}^{\infty} K_n \cos(n\pi x)\left[\cosh(2n\pi y) - \frac{\sinh(2n\pi y)}{\tanh(2n\pi)}\right]. \tag{5.4.6}$$

Finally, applying the remaining boundary condition

$$\begin{aligned} u(x,0) &= 2\cos 2\pi x - 1, \\ \Rightarrow -C_0 &+ \sum_{n=1}^{\infty} K_n\cos(n\pi x) = 2\cos(2\pi x) - 1. \end{aligned} \tag{5.4.7}$$

Identifying the coefficient, the final solution is given by

$$u(x,y) = y - 1 + 2\cos(2\pi x)\left[\cosh(4\pi y) - \frac{\sinh(4\pi y)}{\tanh(4\pi)}\right]. \tag{5.4.8}$$

5.5 Laplace-like equation

Consider the following equation on a square

$$u_{xx} + 2u_y + u_{yy} = 0 \quad \{x,y\} \in]0,1[\times]0,1[$$

with the boundary conditions, for a given function f,

$$\begin{cases} u(0,y) = 0 & \forall y \in [0,1], \\ u(1,y) = 0 & \forall y \in [0,1], \\ u(x,1) = f(x) & \forall x \in [0,1], \\ u(x,0) = 0 & \forall x \in [0,1]. \end{cases}$$

Using separation of variables $u(x,y) = v(x)w(y)$, find the solution to this boundary value problem. As the function f is not specified, some constants remain in the final solution. Explain and show how to compute these constants when f is known.

Solution

Using separation of variables, the equation can be written

$$-\frac{X''}{X} = \frac{2Y'}{Y} + \frac{Y''}{Y} = \lambda. \tag{5.5.1}$$

Therefore, both spatial part of the solution depend on the sign of the same constant λ.
First focus on the equation for $X(x)$

$$\begin{cases} X'' + \lambda X = 0, \\ X(0) = 0, \\ X(1) = 0. \end{cases} \tag{5.5.2}$$

This problem admits some not trivial solutions only for $\lambda = (n\pi)^2 > 0$, where n is a positive integer. Those solution write

$$X_n(x) = \tilde{A}_n \sin(n\pi x). \tag{5.5.3}$$

After this, the equation for Y writes

$$Y'' + 2Y' - (n\pi)^2 Y = 0. \tag{5.5.4}$$

This is an ODE with constant coefficients which can thus simply be solved using the exponential polynomial method. The general solution is given by

$$Y_n(y) = \exp(-y)\left[C_n \sinh\left(\sqrt{(n\pi)^2 + 1}y\right) + D_n \cosh\left(\sqrt{(n\pi)^2 + 1}y\right)\right]. \tag{5.5.5}$$

Applying the last homogeneous boundary condition directly leads to $D_n = 0\ \forall n$. Therefore, defining a new constants A_n, the solution writes

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)\exp(-y)\sinh\left(\sqrt{(n\pi)^2 + 1}y\right). \tag{5.5.6}$$

The determination of the coefficients A_n is obtained by applying the non homogeneous boundary condition to Eq.(5.5.6). It can easily be shown that the application of this boundary condition will lead to the decomposition of $f(x)$ in a sine Fourier series.

5.6 Helmholtz equation

Consider the two-dimensional Helmholtz equation on a square

$$u_{xx} + u_{yy} - u = 0 \quad \{x,y\} \in]0,1[\times]0,1[$$

with the boundary conditions, for a given function f,

$$\begin{cases} u(0,y) = 0 & \forall y \in [0,1], \\ u(1,y) = 0 & \forall y \in [0,1], \\ u(x,1) = f(x) & \forall x \in [0,1], \\ u(x,0) = 0 & \forall x \in [0,1]. \end{cases}$$

Using separation of variables $u(x,y) = v(x)w(y)$, find the solution to this boundary value problem. As the function f is not specified, some constants remain in the final solution. Explain and show how to compute these constants when f is known.
Hint: introduce two eigenvalues k_x and k_y such that $v''/v = \lambda_x$ and $w''/w = \lambda_y$.

Solution

Using the separation of variables $u(x, y) = X(x)Y(y)$, the Helmoltz equation can be written as

$$\frac{X''}{X} + \frac{Y''}{Y} - 1 = 0. \tag{5.6.1}$$

Introducing $\lambda_x = \frac{X''}{X}$ and $\lambda_y = \frac{Y''}{Y}$, Eq.(5.6.1) becomes

$$\begin{cases} X'' - \lambda_x X = 0, \\ Y'' - \lambda_y Y = 0, \\ \lambda_x + \lambda_y = 1, \end{cases} \tag{5.6.2}$$

i.e.

$$\begin{cases} X'' - \lambda_x X = 0, \\ Y'' - (1 - \lambda_x)Y = 0. \end{cases} \tag{5.6.3}$$

Each possible values for λ_x should be investigated

(1) $\lambda_x = 0$

(2) $\lambda_x = -\omega_x^2 < 0$

(3) $0 < \lambda_x < 1$

(4) $\lambda_x = 1$

(5) $\lambda_x > 1$

However, due to the homogeneous Dirichlet boundary condition $u(0, y) = u(1, y) = 0$, $X(x) = 0 \quad \forall \lambda_x \geq 0$. Only the value of λ such that $\lambda_x = -\omega^2 < 0$ will lead to non trivial solutions. When $\lambda_x = -\omega^2 < 0$, the solutions for X and Y read

$$\begin{cases} X(x) = A\cos(\omega x) + B\sin(\omega x), \\ Y(y) = C\cosh\left(\sqrt{1+\omega^2}y\right) + D\sinh\left(\sqrt{1+\omega^2}y\right). \end{cases} \tag{5.6.4}$$

Using the initial conditions gives $A = C = 0$ and

$$\omega_n = n\pi, \qquad n = 1, 2, \ldots \tag{5.6.5}$$

Finally, the solution to the problem can be written as

$$u(x, y) = \sum_{n=1}^{+\infty} A_n \sin(n\pi x)\sinh(\sqrt{1+n^2\pi^2}y). \tag{5.6.6}$$

The constant A_n can be inferred by using the remaining initial condition

$$u(x, 1) = \sum_{n=1}^{+\infty} A_n \sin(n\pi x)\sinh(\sqrt{1+n^2\pi^2}) = f(x),$$

which is a sine Fourier expansion of $f(x)$.

5.7 Diffusion equation II

Consider the one-dimensional diffusion equation on a bounded domain

$$u_t - k u_{xx} = 0 \quad \forall x \in]0,1[. \tag{$\star$}$$

(a) Using separation of variables $u(x,t) = w(t)v(x)$, find all the separable solutions of Eq. ($\star$).

(b) Find the solution to Eq. ($\star$) for the homogeneous Robin boundary conditions at one end of the domain,

$$u_x(1,t) + \beta u(1,t) = 0,$$

with a constant $\beta \in \mathbb{R}$, the homogeneous Dirichlet boundary condition at the other end,

$$u(0,t) = 0.$$

and an initial condition, for a given function ϕ,

$$u(x,0) = \phi(x) \quad \forall x \in]0,1[.$$

In particular, show that the solution of this problem requires (among other things) to solve the transcendental equation

$$\omega = -\beta \tan \omega.$$

Draw a schematics and point out (qualitatively) the solutions of this equation. As the function ϕ is not specified, some constants remain in the final solution. Explain why finding these constants is not immediate in this case.

(c) The domain in now extended to the interval $]-1,1[$. Find the solution to Eq. ($\star$) for the periodic boundary conditions

$$u(-1,t) = u(1,t) \quad \text{and} \quad u_x(-1,t) = u_x(1,t),$$

and an initial condition, for a given function ϕ,

$$u(x,0) = \phi(x) \quad \forall x \in]0,1[.$$

As the function ϕ is not specified, some constants remain in the final solution. Explain and show how to compute these constants when ϕ is known.

Solution

(a) Using the ansatz $u = wv$, the diffusion equation writes as

$$w'v - kwv'' = 0 \tag{5.7.1}$$

$$\Rightarrow \quad \frac{w'}{kw} = \frac{v''}{v}. \tag{5.7.2}$$

Since the right hand side depends only on x while the left hand side depend only on t, both sides must equal a constant (named λ), *i.e*

$$v'' - \lambda v = 0 \quad \text{and} \quad w' - k\lambda w = 0. \tag{5.7.3}$$

Spatial dependency Depending on the sign of λ, three solutions arise, *i.e*

$$\begin{aligned}
&\text{if } \lambda = 0 && \Rightarrow v = Ax + B, && (5.7.4)\\
&\text{if } \lambda = \omega^2 > 0 && \Rightarrow v = C \exp -\omega x + D \exp \omega x, && (5.7.5)\\
&\text{if } \lambda = -\omega^2 < 0 && \Rightarrow v = E \cos \omega x + F \sin \omega x. && (5.7.6)
\end{aligned}$$

Time dependency Whatever is the sign of λ, the time dependency is given by

$$w = \exp \lambda k t. \tag{5.7.7}$$

The constant in front the the exponential is here omitted because the field of interest is $u = vw$.

(b) Among all the eigensolutions found in the previous sub-question (*i.e* for any value of λ), only those that satisfy the homogeneous boundary conditions are kept.

Steady eigenvalues, $\lambda = 0$ The boundary conditions yields

$$\begin{cases} v(0) & = B = 0 \qquad (5.7.8)\\ v_x(1) + \beta v(1) = 0 & = A(1+\beta) \qquad (5.7.9)\end{cases}$$

which implies

$$A = 0 \quad \text{if } \beta \neq -1 \tag{5.7.10}$$

but if $\beta = -1$ then $v(x) = Ax$ satisfy both boundary condition. This means that the steady mode is non-trivial only if $\beta = -1$.

Time growing eigenvalues, $\lambda = \omega^2 > 0$ The boundary conditions yields

$$\begin{cases} v(0) & = 0 = C + D \qquad (5.7.11)\\ v_x(1) + \beta v(1) & = 0 = -\omega C \exp -\omega + \omega D \exp \omega x + \beta \left(C \exp -\omega + D \exp \omega\right) \qquad (5.7.12)\end{cases}$$

which implies that

$$C = D = 0 \quad \text{if } (\beta - \omega) \exp -\omega - (\beta + \omega) \exp \omega \neq 0 \tag{5.7.13}$$

but if

$$(\beta - \omega) \exp -\omega - (\beta + \omega) \exp \omega = 0 \quad \Rightarrow \omega = -\beta \tanh \omega \tag{5.7.14}$$

then $v(x) = C\left(\exp -\omega x - \exp \omega x\right)$ is a solution. It can be shown that there exists a single ω that verifies that equation only if $\beta < -1$. Indeed, consider the function $f(\omega) = \omega + \beta \tanh \omega$ which is differentiable and is such that $f(0) = 0$, and $\lim_{\omega \to +\infty} f(\omega) = +\infty$. We have $f'(0) = 1 + \beta$. If $\beta < -1$ then $f'(0) < 0$ and there exists at least one ω where f is negative. By the theorem of intermediate values, there has to be a root after that point. At some point, the function becomes increasing again and goes monotonically to $+\infty$. Therefore, there is only one root.

Time decaying eigenvalues, $\lambda = -\omega^2 < 0$ The boundary conditions once again yields

$$v(0) = E = 0 \tag{5.7.15}$$
$$v_x(1) + \beta v(1) = 0 \quad \Rightarrow F\left(\omega \cos \omega + \beta \sin \omega\right) = 0 \tag{5.7.16}$$

Once again, there is a non trivial solution only if

$$(\omega \cos \omega + \beta \sin \omega) = 0 \Rightarrow \omega = -\beta \tan \omega. \tag{5.7.17}$$

It can be shown that these solutions are infinitely many. There are denoted by ω_n.
The solution compatible with boundary conditions is then, for $\beta > -1$,

$$u(x,t) = \sum_{n=1}^{\infty} F_n \sin \omega_n x \exp -\omega_n^2 kt. \tag{5.7.18}$$

If $\beta = -1$, then the steady mode must be added to the solution, *i.e*

$$u(x,t) = A_0 x + \sum_{n=1}^{\infty} F_n \sin \omega_n x \exp -\omega_n^2 kt. \tag{5.7.19}$$

If $\beta < -1$, then the growing mode must be added to the solution, *i.e*

$$u(x,t) = B\left(\exp -\omega t - \exp \omega t\right) \exp \omega^2 kt + \sum_{n=1}^{\infty} F_n \sin \omega_n x \exp -\omega_n^2 kt. \tag{5.7.20}$$

Finally the initial condition must be imposed, *i.e* (in the case $\beta > -1$)

$$\phi(x) = \sum_{n=1}^{\infty} F_n \sin \omega_n x \tag{5.7.21}$$

which is not a Fourier series (**as the frequencies are not harmonics**) so the identification is tricky and not necessarily possible.

5.8 Wave equation II 🌶🌶🌶

Consider the two-dimensional wave equation inside an open subset D of $\mathbb{R}^2$ (in this exercise, D will either be a square or a disk),

$$u_{tt} - c^2 \Delta u = 0 \quad \forall \{x, y\} \in D,$$

with Dirichlet condition on the boundary ∂D of D

$$u(x, y, t) = 0 \quad \forall \{x, y\} \in \partial D$$

and with initial conditions

$$u(x, y, 0) = \phi(x, y) \quad \text{and} \quad u_t(x, y, 0) = \psi(x, y).$$

The Laplace operator is denoted by Δ. In Cartesian coordinate, $\Delta u = u_{xx} + u_{yy}$, while in polar coordinates $\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$.

(a) Using separation of variables $u(x, y, t) = w(t)v(x)q(y)$, find the solution to this boundary value problem if $D =]0, 1[\times]0, 1[$. As the functions ϕ and ψ are not specified, some constants remain in the final solution. Explain how to compute these constants when ϕ and ψ are known. What is the physical interpretation of any term in the final sum taken individually? How do these terms evolve with time?

(b) Using separation of variables $u(x, y, t) = w(t)g(r)h(\theta)$, find the solution to this boundary value problem if $D = \{x, y \,|\, x^2 + y^2 < 1\}$, *i.e*, if D is a disk of radius 1. As the functions ϕ and ψ are not specified, some constants remain in the final solution. Explain how to compute these constants when ϕ and ψ are known. Make the parallel with (a) as far as eigenfunctions and time evolution is concerned.

Solution

(a) Using the ansatz $u = wvq$, the wave equation writes as

$$\frac{w''}{c^2 w} = \frac{v''}{v} + \frac{q''}{q}. \tag{5.8.1}$$

Since the right hand side depends only on x and y while the left hand side depends only on t, each of the three terms in Eq.(5.8.1) must be a constant. These constant are respectively called λ, λ_x and λ_y. The system to solve is then

$$\begin{cases} v'' - \lambda_x v &= 0, \quad (5.8.2)\\ q'' - \lambda_y q &= 0, \quad (5.8.3)\\ w'' - c^2\lambda w &= 0 \quad (5.8.4)\end{cases}$$

with

$$\lambda_x + \lambda_y = \lambda. \tag{5.8.5}$$

Time dependency (t) Depending on the sign of λ, three solutions arise, *i.e*

$$\begin{aligned} &\text{if } \lambda = 0 && \Rightarrow w = A_t t + B_t, && (5.8.6)\\ &\text{if } c^2\lambda = \omega^2 > 0 && \Rightarrow w = C_t \exp -\omega t + D_t \exp \omega t, && (5.8.7)\\ &\text{if } c^2\lambda = -\omega^2 < 0 && \Rightarrow v = E_t \cos \omega t + F_t \sin \omega t. && (5.8.8)\end{aligned}$$

Spatial dependency (x) Depending on the sign of λ_x, three solutions arise, *i.e*

$$\begin{aligned} &\text{if } \lambda_x = 0 && \Rightarrow v = A_x x + B_x, && (5.8.9)\\ &\text{if } \lambda_x = k_x^2 > 0 && \Rightarrow v = C_x \exp -k_x x + D_x \exp k_x x, && (5.8.10)\\ &\text{if } \lambda_x = -k_x^2 < 0 && \Rightarrow v = E_x \cos k_x x + F_x \sin k_x x. && (5.8.11)\end{aligned}$$

The same solutions appears for the spatial dependency in y.
Among all the eigensolutions found previously, only those satisfying the homogeneous boundary condition are kept. Using the ansatz, the boundary conditions become

$$v(0) = v(1) = 0 \tag{5.8.12}$$
$$q(0) = q(1) = 0. \tag{5.8.13}$$

Stationary eigensolutions, *i.e* $\lambda_x = 0$ Applying the boundary conditions gives $A_x = B_x = 0$ and similarly $A_y = B_y = 0$.

Evanescent eigensolutions, *i.e* $\lambda_x > 0$ Applying the boundary conditions gives $C_x = D_x = 0$ and similarly $C_y = D_y = 0$.

Propagating eigensolutions, *i.e* $\lambda_x < 0$ Applying the boundary condition gives

$$v_n(x) = F_x^n \sin k_x^n x \quad \text{with} \quad k_x^n = n\pi \quad n = 1, 2, 3, \dots \tag{5.8.14}$$

and

$$q_m(x) = F_y^m \sin k_y^m y \quad \text{with} \quad k_x^m = m\pi \quad m = 1, 2, 3, \dots \tag{5.8.15}$$

Because

$$\lambda = \lambda_x + \lambda_y, \tag{5.8.16}$$

the pulsation are given by

$$\omega_{m,n} = c\sqrt{m^2 + n^2}\pi. \tag{5.8.17}$$

The most general solution compatible with boundary condition is then

$$u(x, y, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} F^n F^m \sin n\pi x \sin n\pi y \left(A_{m,n} \cos\left(c\sqrt{m^2 + n^2}\pi t\right) + B_{m,n} \sin\left(c\sqrt{m^2 + n^2}\pi^2 t\right)\right) \tag{5.8.18}$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A'_{m,n} \sin n\pi x \sin m\pi y \cos\left(c\sqrt{m^2 + n^2}\pi t\right)$$

$$+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B'_{m,n} \sin n\pi x \sin m\pi y \sin\left(c\sqrt{m^2 + n^2}\pi t\right). \tag{5.8.19}$$

Finally using the initial condition, one finds

$$u(x, y, 0) = \phi(x, y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A'_{m,n} \sin n\pi x \sin m\pi y \tag{5.8.20}$$

and

$$u_t(x, y, 0) = \psi(x, y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B'_{m,n} c\sqrt{m^2 + n^2}\pi \sin n\pi x \sin m\pi y \tag{5.8.21}$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B''_{m,n} \sin n\pi x \sin m\pi y. \tag{5.8.22}$$

Then using the result from 2D Fourier sine series, the coefficients are given by

$$A'_{m,n} = 4\int_0^1 \int_0^1 \phi(x, y) \sin n\pi x \sin m\pi y \, dxdy \tag{5.8.23}$$

and

$$B''_{m,n} = 4\int_0^1\int_0^1 \psi(x,y)\sin n\pi x \sin m\pi y \, dxdy. \tag{5.8.24}$$

From Eq.(5.8.19), it can be seen that the solution is a superposition of modes of increasing spatial frequencies, $k_x^n = n\pi$ and $k_y^m = m\pi$ and of increasing time pulsation $\omega_{m,n} = c\sqrt{m^2+n^2}\pi$. At the opposite of the heat equation, none of the modes are damped. Consequently, the discontinuity and the high frequency variations of the initial data are conserved and not smoothed as it is the case for the heat equation.

(b) If D is a cylinder, the problem written in cylindrical coordinates is

$$u_{tt} - u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad \forall\{\theta, r\} \in [0, 2\pi] \times [0, 1[\tag{5.8.25}$$

with the boundary condition

$$u(1, \theta, t) = 0 \quad \forall\theta \in [0, 2\pi]. \tag{5.8.26}$$

Moreover, to ensure that the solution is single valued,

$$u(r, \theta + 2\pi, t) = u(r, \theta, t) \tag{5.8.27}$$

must be verified.
As previously, the initial conditions remain

$$u(r, \theta, 0) = \phi(r, \theta) \quad \text{and} \quad u_t(r, \theta, 0) = \psi(r, \theta) \quad \forall\{r, \theta\} \in D. \tag{5.8.28}$$

Using the ansatz $u = wgh$, the wave equation writes as

$$\frac{w''}{c^2 w} = \frac{1}{gh}\left(hg'' + \frac{1}{r}hg' + \frac{1}{r^2}gh''\right). \tag{5.8.29}$$

The left hand side only depends on t while the right hand side only depends on r and θ such that both sides must equal the same constant called λ. The wave equation then becomes

$$\begin{cases} r^2hg'' + rhg' + gh'' &= r^2\lambda gh, \quad (5.8.30)\\ r^2\dfrac{g''}{g} + r\dfrac{g'}{g} - r^2\lambda + \dfrac{h''}{h} = 0. & \quad (5.8.31) \end{cases}$$

Once again each term should equal a constant, denoted by $-\mu$ because they only depend on r or θ. The system to solve is then

$$\begin{cases} w'' - c^2\lambda w &= 0, \quad (5.8.32)\\ h'' + \mu h &= 0, \quad (5.8.33)\\ r^2g'' + rg' + (-r^2\lambda - \mu)g &= 0. \quad (5.8.34) \end{cases}$$

Time dependency Depending on λ, this equations yields different time behaviours. However here, only propagating modes are considered, *i.e* $\lambda = -\omega^2 < 0$. To formally reject $\lambda \geq 0$, the same procedure as in the previous sub-question should be done, *i.e* show that the spatial part associated with steady and evanescent modes does not satisfy the boundary conditions.
For propagating modes, the time dependency is

$$w(t) = A_t \cos c\omega t + B_t \sin c\omega t. \tag{5.8.35}$$

Azimutal dependency The only solution that ensures the unicity is $\mu = m^2$ where m is an integer, the solution is then

$$h(\theta) = A_\theta \cos m\theta + B_\theta \sin m\theta. \tag{5.8.36}$$

Radial dependency The radial equation is now

$$r^2 g'' + rg' + \left(r^2\omega^2 - m^2\right) g = 0. \tag{5.8.37}$$

Now consider the change of variable $x = \omega r$, the equation becomes

$$x^2 g'' + xg' + (x^2 - m^2)g = 0. \tag{5.8.38}$$

This equation is known as the *Bessel's equation*. Solving this equations is hard and outside the scope of these exercises.
The solution of this equation can be expressed as the superposition of the Bessel function of the first kind of order m, $J_m(x)$, and of the Bessel function of the second kind of order m, $Y_m(x)$, *i.e*

$$g(x) = A_x J_m(x) + B_x Y_m(x) \tag{5.8.39}$$

The boundary condition are $g(1) = 0$ and $|g(0)| < \infty$. Thus $B_x = 0$ because $Y_m(x)$ is unbounded at the origin. Then the Dirichlet boundary condition yields

$$g(r = 1) = 0 = J_m(x(1)) = J_m(\omega) = 0. \tag{5.8.40}$$

The nth eigenvalues that satisfy $J_m(\omega) = 0$ is denoted by $\omega_{m,n}$. It can be shown that there are an infinite numbers of such values, the root $\omega_{m,n} = 0$ (*i.e* for $n = 0$) should however be withdrawn because $\omega > 0$. The solution then writes

$$u(r, \theta, t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} [A_t^{m,n} \cos c\omega_{m,n}t + B_t^{m,n} \sin c\omega_{m,n}t] [A_\theta^{m,n} \cos m\theta + B_\theta^{m,n} \sin m\theta] J_m(\omega_{m,n}r) \tag{5.8.41}$$

$$= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \left[A_\theta'^{m,n} \cos m\theta + B_\theta'^{m,n} \sin m\theta\right] J_m(\omega_{m,n}r) \cos c\omega_{m,n}t \tag{5.8.42}$$

$$+ \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \left[A_\theta''^{m,n} \cos m\theta + B_\theta''^{m,n} \sin m\theta\right] J_m(\omega_{m,n}r) \sin c\omega_{m,n}t. \tag{5.8.43}$$

It is interesting to point out that the values $m = 0$ yields radially symmetric eigenfunctions for all values of n.
As previously, the constant can be withdrawn from the initial conditions

$$u(r, \theta, 0) = \phi(r, \theta) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \left[A_\theta'^{m,n} \cos m\theta + B_\theta'^{m,n} \sin m\theta\right] J_m(\omega_{m,n}r) \tag{5.8.44}$$

$$u_t(r, \theta, 0) = \psi(r, \theta) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \omega_{m,n} \left[A_\theta''^{m,n} \cos m\theta + B_\theta''^{m,n} \sin m\theta\right] J_m(\omega_{m,n}r) \tag{5.8.45}$$

Previously, the initial data are decomposed into a Fourier sine series. In this case, the basis used is slightly more complcated and one has to decomposed the initial data in a so called Fourrier-Bessel basis.

As in the case of Fourier sine or cosine series, here the coefficient of the Fourier-Bessel series are given by

$$A_\theta'^{m,n} = \frac{2}{\pi\left(J_{m+1}(\omega_{m,n})\right)^2}\int_0^1\int_{-\pi}^{\pi}\phi(r,\theta)J_m(\omega_{m,n}r)r\cos m\theta\, d\theta dr, \tag{5.8.46}$$

$$B_\theta'^{m,n} = \frac{2}{\pi\left(J_{m+1}(\omega_{m,n})\right)^2}\int_0^1\int_{-\pi}^{\pi}\phi(r,\theta)J_m(\omega_{m,n}r)r\sin m\theta\, d\theta dr \tag{5.8.47}$$

and

$$\omega_{m,n}A_\theta''^{m,n} = \frac{2}{\pi\left(J_{m+1}(\omega_{m,n}r)\right)^2}\int_0^1\int_{-\pi}^{\pi}\psi(r,\theta)J_m(\omega_{m,n}r)r\cos m\theta\, d\theta dr, \tag{5.8.48}$$

$$\omega_{m,n}B_\theta''^{m,n} = \frac{2}{\pi\left(J_{m+1}(\omega_{m,n}r)\right)^2}\int_0^1\int_{-\pi}^{\pi}\psi(r,\theta)J_m(\omega_{m,n}r)r\sin m\theta\, d\theta dr. \tag{5.8.49}$$

5.9 Starting flow in circular pipe

Consider an infinite circular pipe of radius a, filled with an incompressible and Newtonian fluid of density ρ and dynamic viscosity μ. Consider a cylindrical coordinate system, centered in the pipe, the z-direction is parallel to the pipe. The only non-zero velocity component of the fluid u is in the z-direction. With the azimuthal symmetry and invariance along the z-direction, Navier-Stokes momentum equation for the velocity $u = u(r,t)$ inside the pipe writes

$$\frac{\partial u}{\partial t} - \nu\,\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = -\frac{p_z}{\rho}, \quad \text{with} \quad (r,t)\in\,]0,a[\times]0,+\infty[,$$

with $\nu = \mu/\rho$ the kinematic viscosity and p_z a constant pressure gradient in the z-direction. The boundary conditions are $u(0,t) < \infty$ and $u(a,t) = 0$, $\forall t \in]0,+\infty[$. The initial condition is $u(r,0) = 0$, $\forall r \in]0,a[$.

(a) Show that the steady-state velocity $u_s = u_s(r)$ is given by

$$u_s(r) = \frac{p_z}{4\mu}(r^2 - a^2).$$

(b) Rewrite the initial boundary value problem in terms of $v(r,t) = u(r,t) - u_s(r)$.

(c) Using the separation of variables $v(r,t) = R(r)T(t)$, give the transient solution to the problem.

Help: The equation $x^2y'' + xy' + (x^2 - m^2)y = 0$ is the Bessel's equation of order $m \in \mathbb{C}$. For a given m, this equation has two linearly independent solutions: J_m the Bessel function of the first kind, which has a finite value at $x = 0$, and Y_m the Bessel function of the second kind, which diverges at $x = 0$. Both functions have an infinite number of roots. See Fig. 5. Some useful properties of the J_m are, with α_k the k^{th} non-zero root of J_m,

$$J_{m-1}(x) = \frac{2m}{x}J_m(x) - J_{m+1}(x), \qquad x^m J_{m-1}(x) = \frac{d}{dx}\big(x^m J_m(x)\big), \qquad J_m(0) = 0, \forall m \in \mathbb{N}_0,$$

$$\int_0^1 xJ_m(\alpha_k x)J_m(\alpha_l x)\,dx = \begin{cases} 0, & \text{if } k \neq l,\\ \dfrac{J_{m+1}^2(\alpha_k)}{2}, & \text{if } k = l.\end{cases}$$

The equation $x^2y'' + xy' - (x^2 + n^2)y = 0$ is the modified Bessel's equation of order $n \in \mathbb{C}$. For a given n, it has two linearly independent solutions: I_n the modified Bessel function of the first kind, which has a finite value at $x = 0$ and no non zero root, and K_n the modified Bessel function of the second kind, which diverges at $x = 0$ and has no roots. Unlike the ordinary Bessel functions J_m and Y_m, the modified Bessel functions are exponentially growing or decaying, they exhibit no oscillations.

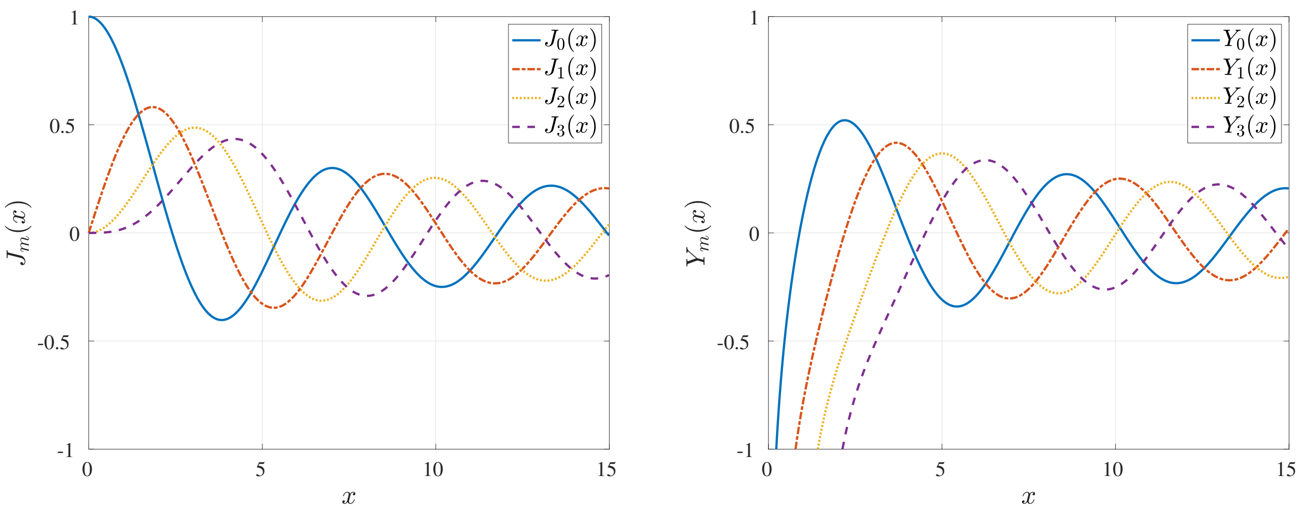

Figure 5: First four Bessel functions for integer orders m.

Solution (brief solution only)

(a) Impose $\partial_t = 0$ and solve the resulting ODE.

(b) The diffusion equation writes $v_t - \nu(r^{-1}v_r + v_{rr}) = 0$. The boundary conditions write $v(0,t) < \infty$ and $v(a,t) = 0$, $\forall t \in]0, +\infty[$. And the initial condition is $v(r, 0) = -u_s(r)$, $\forall r \in]0, a[$.

(c) The separation of variables yields one ODE for R and one ODE for T, with a separation constant $\lambda^2 \in \mathbb{R}$. Three cases are discussed for the sign of λ^2.

Stationary solution: $\lambda = 0$. We get $R(r) = B \ln r + C$, for $B, C \in \mathbb{R}$, which diverges at $r = 0$ so the identification with the initial condition will give $B = 0$. The boundary condition at $r = a$ yields $C = 0$.

Time-growing solutions: $\lambda^2 < 0$. We get the modified Bessel's equation of order 0 after a change of variables. Because the solutions I_0 and K_0 have no non zero roots, we cannot satisfy the boundary conditions with them and the associated constants are zero. Time-growing solutions thus do not contribute to the final solution (which makes sense physically).

Time-decaying solutions: $\lambda^2 > 0$. We get the Bessel's equation of order 0, whose solutions are J_0 and Y_0. The boundary condition at $r = 0$ discards the solution Y_0. The no-slip boundary condition $v(a,t) = 0$, $\forall t \in]0, +\infty[$, forces λ to have discrete values, related to the roots of J_0.

Before applying the initial condition, the most general solution expresses

$$v(r,t) = \sum_{k=1}^{\infty} A_k J_0\left(\frac{\lambda_k r}{\sqrt{\nu}}\right) \exp\left(-\lambda_k^2 t\right), \tag{5.9.1}$$

with $\lambda_k = \alpha_k\sqrt{\nu}/a$, α_k the k^{th} root of J_0, and the A_k constants to be determined with the initial condition:

$$\sum_{k=1}^{\infty} A_k J_0\left(\frac{\lambda_k r}{\sqrt{\nu}}\right) = -\frac{p_z}{4\mu}(r^2 - a^2). \tag{5.9.2}$$

Multiplying this condition by $rJ_0(\alpha_l r/a)$, $l \in \mathbb{N}_0$, integrating over $]0, a[$, and using the orthogonality property yields

$$\int_0^a rJ_0(\alpha_l r/a)A_l J_0(\alpha_l r/a)dr = -\int_0^a rJ_0(\alpha_l r/a)\frac{p_z}{4\mu}(r^2 - a^2)dr. \tag{5.9.3}$$

The change of variable $x = r/a$ and the norm property leads to the following integral for computing each A_l:

$$A_l = -\frac{a^2 p_z}{2\mu}\frac{1}{J_1^2(\alpha_l)}\int_0^1 x(x^2 - 1)J_0(\alpha_l x)dx. \tag{5.9.4}$$

Using the given recurrence relation, we have $x^3 J_0(\alpha_l x) = 2x^2 J_1(\alpha_l x)/\alpha_l - x^3 J_2(\alpha_l x)$. The given derivative property and the fact that $J_m(0) = 0$, $\forall m \in \mathbb{N}_0$, allows to compute directly the integral. We get

$$A_l = -\frac{a^2 p_z}{2\mu}\frac{1}{J_1^2(\alpha_l)}\left(\frac{2J_2(\alpha_l)}{\alpha_l^2} - \frac{J_3(\alpha_l)}{\alpha_l} - \frac{J_1(\alpha_l)}{\alpha_l}\right), \tag{5.9.5}$$

and with $J_1(x) + J_3(x) = 4J_2(x)/x$, then with $J_2(\alpha_l) = 2J_1(\alpha_l)/\alpha_l$ (because $J_0(\alpha_l) = 0$, by definition of α_l being a root of J_0) we finally get

$$A_l = \frac{a^2 p_z}{\mu}\frac{J_2(\alpha_l)}{\alpha_l^2 J_1^2(\alpha_l)} = \frac{2a^2 p_z}{\mu\alpha_l^3 J_1(\alpha_l)}, \tag{5.9.6}$$

such that the transient solution writes (see Fig. 6 for an illustration of the solution)

$$u(r,t) = \frac{p_z}{4\mu}(r^2 - a^2) + \sum_{k=1}^{\infty}\frac{2a^2 p_z}{\mu\alpha_k^3 J_1(\alpha_k)}J_0\left(\alpha_k\frac{r}{a}\right)\exp\left(-\frac{\alpha_k^2\nu t}{a^2}\right), \text{ with } \alpha_k \text{ the } k^{\text{th}} \text{ root of } J_0(x). \tag{5.9.7}$$

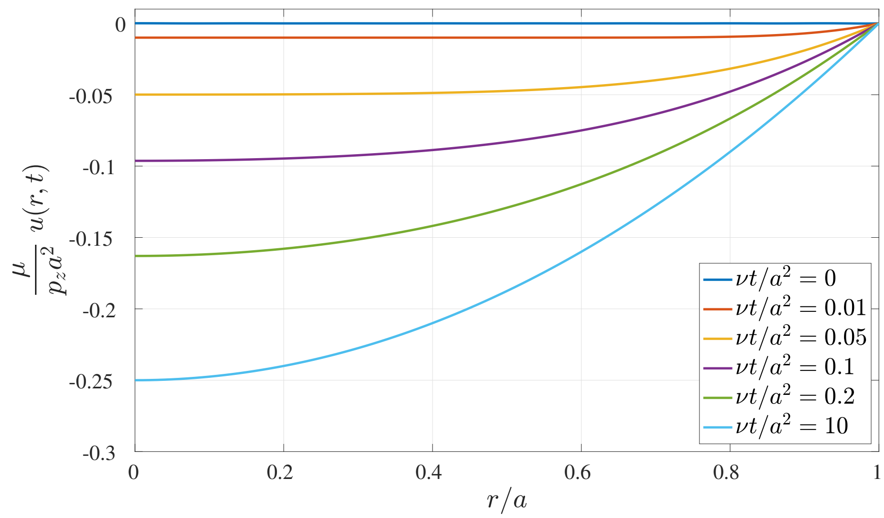

Figure 6: Velocity profile at different time instants.

5.10 Spherical harmonics [Olver, Sec. 12.2] 🌶🌶🌶

Consider the three-dimensional Laplace equation inside the unit ball $B = \{r \leq 1\}$, in spherical coordinates $(r, \theta, \phi) \in [0, 1[\times [0, \pi] \times] - \pi, \pi]$,

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cos\theta}{r^2 \sin\theta}\frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0,$$

with arbitrary boundary conditions of the sphere $r = 1$.

Warning: In this course (as in other courses), θ is the zenith angle (or latitude angle) and ϕ is the azimuthal angle (or longitude angle). The associated volume Jacobian is $r^2 \sin\theta$. This convention is usually used in physics. In mathematics, the opposite convention is often used (as in the Olver reference). Be very careful when comparing different formulae.

In order to solve this problem, follow successively the steps below. Explain your developments. (You are not asked to show that the various sets of functions form bases of the associated vector spaces.)

(a) Using the separation of variables $u(r, \theta, \phi) = v(r)p(\theta)q(\phi)$, show that we obtain the following three ordinary differential equations for v, p and q,

$$\begin{cases} r^2 v'' + 2rv' - \mu v = 0, \\ \sin^2\theta\, p'' + \sin\theta\cos\theta\, p' + \left(\mu \sin^2\theta - m^2\right) p = 0, \\ q'' + m^2 q = 0. \end{cases}$$

(b) Solve the azimuthal equation for $q(\phi)$. Remember to impose the 2π-periodicity condition, explain why it is necessary.

(c) Solve the zenith equation for $p(\theta)$, with the following steps:

- Use the change of variable $t = \cos\theta$ to write the equation for p in a more convenient form, with $p(\theta) = P(\cos\theta) = P(t)$. The obtained equation is the *Legendre equation of order* m. Combined with the boundary conditions $|P(-1)| < \infty$ and $|P(1)| < \infty$, this yields the order m Legendre boundary value problem.

- As a lemma for the next questions, show by induction that the function $Q_n(t) = (1-t^2)^n$, $n = 0, 1, 2\ldots$, verifies the induction equation, with the notation $Q_n^{(k)}(t) = \frac{d^k Q_n(t)}{dt^k}$,

$$(1-t^2)Q_n^{(k+2)} = -2(n-k-1)\,t\,Q_n^{(k+1)} - (k+1)(2n-k)Q_n^{(k)}, \quad \forall k = 0, 1, 2\ldots.$$

- Show that, in the particular case $m = 0$, $Q_n^{(n)}(t)$ solve the Legendre boundary value problem for the particular value $\mu = n(n+1)$. Consequently, the functions

$$P_n(t) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dt^n}(1-t^2)^n, \quad \forall n \in \mathbb{N},$$

are also solutions order 0 Legendre boundary value problem. These functions are the *Legendre polynomials*. The factor is a common convention.

- In the general case $m \geq 0$, the solutions are the *associated Legendre functions*, or *Ferrers functions*, defined by

$$P_n^m(t) = (1-t^2)^{m/2}\frac{d^m}{dt^m}P_n(t) = (-1)^n\frac{(1-t^2)^{m/2}}{2^n n!}\frac{d^{n+m}}{dt^{n+m}}(1-t^2)^n, \quad n = m, m+1\ldots.$$

Show that these functions are indeed solutions to the order m Legendre boundary value problem, by writing the induction equation with $k = m+n$ and the substitution $Q_n^{(m+n)}(t) = (1-t^2)^{-m/2}S_n^m(t)$.

One can show that the Ferrers functions provide a complete list of solutions to the order m Legendre boundary value problem. It has the eigenvalues $\mu_n = n(n+1)$ for $n \in \mathbb{N}$ and the associated eigenfunctions $P_n^m(t)$, with $m = 0, \ldots, n$ (notice that m is at most n). The Ferrers eigenfunctions form an orthogonal basis relative to the L^2 inner product in $[-1, 1]$.

- Express (directly) the Ferrers functions in terms of the original variable θ of the problem to get the zenith function $p_n^m(\theta)$.

(d) Combine the azimuthal and zenith solution to get the *spherical harmonics*:

$$Y_n^m(\theta, \phi) = p_n^m(\theta)\cos m\phi,$$
$$\tilde{Y}_n^m(\theta, \phi) = p_n^m(\theta)\sin m\phi,$$

with $n = 0, 1, 2, \ldots$ and $m = 0, 1, \ldots, n$. Following the steps below, show that these functions are orthogonal with respect to the classical inner product $\langle\cdot,\cdot\rangle = \iint \cdot \cdot \sin\theta d\theta d\phi$ on the sphere:

- Show that the *spherical Laplacian* Δ_S, defined by,

$$\Delta_S(w) = \frac{\partial^2 w}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial w}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 w}{\partial\phi^2},$$

for a function $w = w(\theta, \phi)$, is a self-adjoint operator with respect to the inner product $\langle\cdot,\cdot\rangle$.
Reminder: an operator L is self-adjoint if $\langle u, L(v)\rangle = \langle L(u), v\rangle$ for any vector u and v.

- Show that the spherical harmonics are the eigenvectors of the eigenvalue problem

$$\Delta_S(w) = -\mu_n w.$$

- In general, show that eigenfunctions associated with different eigenvalues of a self-adjoint operator are mutually orthogonal.
- Using the results above and a direct calculation for the remaining cases, show that all spherical harmonics are orthogonal.

The norms of these functions are given by (if interested, see the nice proof in Olver, exercise 12.2.16):

$$\|Y_n^0\|^2 = \frac{4\pi}{2n+1}, \qquad \|Y_n^m\|^2 = \|\tilde{Y}_n^m\|^2 = \frac{2\pi(n+m)!}{(2n+1)(n-m)!}, \quad m = 0, 1, \ldots, n.$$

Finally, one can also show that these functions form a complete orthogonal system of functions on the unit sphere (this is not easy).

(e) Solve the radial equation for $v(r)$. Discard unbounded solutions at $r \to 0$ and combine them with the spherical harmonics to get the *harmonic polynomials*. Show that they are expressed as

$$H_n^m = r^n Y_n^m(\theta, \phi),$$
$$\tilde{H}_n^m = r^n \tilde{Y}_n^m(\theta, \phi).$$

Show that, in particular, H_0^0, H_1^0, H_1^1, $\tilde{H}_1^1$ and H_2^0 are indeed polynomials when expressed in Cartesian coordinates. This is actually the case for all of them (try some other ones). It is also possible to show that these functions form a complete system in the unit ball.

(f) Because the harmonic polynomials form a complete system in the unit ball, any (sufficiently smooth) harmonic function $u(r, \theta, \phi)$ can be expressed as ($1/2$ factors for convenience)

$$u(r, \theta, \phi) = \frac{c_{0,0}}{2} + \sum_{n=1}^{\infty} \left(\frac{c_{0,n}}{2} \, r^n \, Y_n^0(\theta, \phi) + \sum_{m=1}^{n} \left(c_{m,n} \, r^n \, Y_n^m(\theta, \phi) + \tilde{c}_{m,n} \, r^n \, \tilde{Y}_n^m(\theta, \phi) \right) \right),$$

with coefficients $c_{m,n}$ and $\tilde{c}_{m,n}$ to be fixed by the boundary conditions. Explain how to compute them for a Dirichlet boundary condition $u(1, \theta, \phi) = h(\theta, \phi)$ on the unit sphere.

(g) How is the final solution modified if the mathematical domain is *outside* the unit ball?

Solution (brief solution only)

(a) Algebraic manipulations only.

(b) The periodicity condition is necessary to ensure the matching of the solution at $\phi = -\pi$ and $\phi = \pi$. The solution is indeed asked to be continuous. If $m^2 > 0$, the solutions are $q(\phi) = \cos m\phi$ and $\sin m\phi$, with m an integer, that can be chosen non-negative without loss of generality. If $m^2 < 0$, the solutions are real exponentials and no non-zero solution is periodic. Therefore, eigenfunctions exist only for the eigenvalues $m \in \mathbb{N}$.

(c)
- The Legendre equation of order m writes, for $p(\theta) = P(\cos\theta) = P(t)$,

$$(1-t^2)^2 P''(t) - 2t(1-t^2)P'(t) + \left(\mu(1-t^2) - m^2\right) P(t) = 0, \tag{5.10.1}$$

with $t \in [-1, 1]$ (because $\theta \in [0, \pi]$).

- Show this is valid for $k = 0$; then show that if it is valid for a given k, then this is also valid for $k+1$; conclude (do not forget to conclude!).
- Immediate.
- Follow the prescribed steps.
- Immediate.

(d)
- This follows by integration by parts. Using

$$v w_{\theta\theta} = (v w_\theta)_\theta - v_\theta w_\theta = (v w_\theta - v_\theta w)_\theta + v_{\theta\theta} w, \tag{5.10.2}$$

$$\sin\theta\,(v w_\theta - v_\theta w)_\theta = (\sin\theta\,(v w_\theta - v_\theta w))_\theta - \cos\theta\,(v w_\theta - v_\theta w) \tag{5.10.3}$$

and

$$v w_{\phi\phi} = v_{\phi\phi} w + (v w_\phi - v_\phi w)_\phi, \tag{5.10.4}$$

as well as the fact that $[\sin\theta\,(v w_\theta - v_\theta w)]_0^\pi = 0$ (trivial, because v and w are bounded) and $[(v w_\phi - v_\phi w)]_{-\pi}^{\pi} = 0$ (by periodicity), we have

$$\begin{aligned} \langle v, \Delta_S(w)\rangle &= \iint \left(\sin\theta\, v\, w_{\theta\theta} + \cos\theta\, v\, w_\theta + \frac{1}{\sin\theta} v\, w_{\phi\phi}\right) d\theta d\phi, && (5.10.5)\\ &= \iint \left(\sin\theta\, v_{\theta\theta}\, w + \cos\theta\, v_\theta w + \frac{1}{\sin\theta} v_{\phi\phi}\, w\right) d\theta d\phi && (5.10.6)\\ &= \langle \Delta_S(v), w\rangle. && (5.10.7) \end{aligned}$$

- Direct from the separation of variables $u(r,\theta,\phi) = v(r)w(\theta,\phi)$, the resulting angular equation is the eigenvalue problem under consideration whose solutions have been found above, they are the spherical harmonics.
- For a self-adjoint operator L and eigenfunctions v_i and v_j associated with eigenvalues λ_i and λ_j, respectively, we have

$$\begin{aligned} & \langle v_j, L(v_i)\rangle = \langle L(v_j), v_i\rangle, && (5.10.8)\\ \Leftrightarrow\quad & \langle \lambda_j v_j, v_i\rangle = \langle v_j, \lambda_i v_i\rangle, && (5.10.9)\\ \Leftrightarrow\quad & (\lambda_i - \lambda_j)\langle v_j, v_i\rangle = 0, && (5.10.10) \end{aligned}$$

and thus if $\lambda_i \neq \lambda_j$, the eigenvectors are orthogonal.

- The orthogonality relations are $\iint Y_n^m\, Y_l^k dS = \delta_{nl}\delta_{mk}$, $\iint \tilde{Y}_n^m\, \tilde{Y}_l^k dS = \delta_{nl}\delta_{mk}$ and $\iint Y_n^m\, \tilde{Y}_l^k dS = 0$, $\forall n, m, l, k$. If $n \neq l$, the functions are eigenfunctions associated with different eigenvalues (μ_n and μ_l) of a self-adjoint operator Δ_S. Therefore, they are orthogonal (with respect to the same inner product, of course). If $n = l$ but $m \neq k$, the azimuthal integration gives zero for the first two orthogonality relations; and if $n = l$, the azimuthal integration gives zero for all couples (m, k) for the third orthogonality relation.

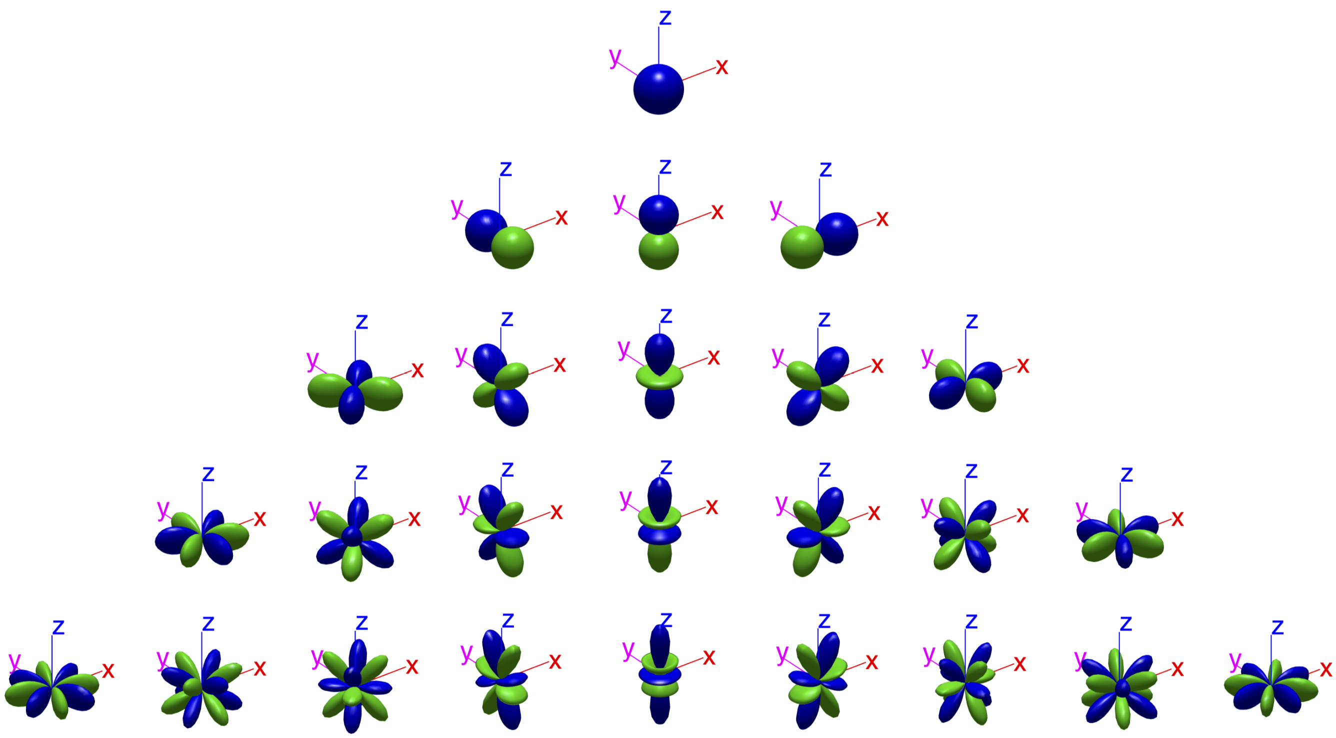

Figure 7: First few spherical harmonics, for information. What is represented is the absolute value of $Y_n^m(\theta,\phi)$ in the angular direction (θ,ϕ). Blue indicates positive regions and green indicates negative regions. Try to find the (n,m) couple for each graph.

(e) Radial solutions: r^n and r^{-n-1}. Must be bounded for $r\to 0$ so that we only keep r^n. We have, by simple change of coordinates and algebraic manipulations, $H_0^0=1$, $H_1^0=z$, $H_1^1=x$, $\tilde{H}_1^1=y$ and $H_2^0=z^2-0.5x^2-0.5y^2$.

(f) The boundary condition expresses

$$u(1,\theta,\phi)=\frac{c_{0,0}}{2}+\sum_{n=1}^{\infty}\left(\frac{c_{0,n}}{2}\,Y_n^0(\theta,\phi)+\sum_{m=1}^{n}\left(c_{m,n}\,Y_n^m(\theta,\phi)+\tilde{c}_{m,n}\,\tilde{Y}_n^m(\theta,\phi)\right)\right)=h(\theta,\phi), \tag{5.10.11}$$

and the coefficients $c_{m,n}$ or $\tilde{c}_{m,n}$ are the projection of h on the spherical harmonics Y_n^m and $\tilde{Y}_n^m$, respectively, taking care of the norms, as follows:

$$c_{0,n}=\frac{2\langle h,Y_n^0\rangle}{\|Y_n^0\|^2},\qquad c_{m,n}=\frac{\langle h,Y_n^m\rangle}{\|Y_n^m\|^2},\qquad \tilde{c}_{m,n}=\frac{\langle h,\tilde{Y}_n^m\rangle}{\|\tilde{Y}_n^m\|^2}, \tag{5.10.12}$$

with $0\le n$ and $1\le m\le n$ (notice the squared norm).

(g) We would have kept r^{-n-1} instead of r^n for the radial solution, to ensure that the solution is bounded at infinity.

6 Nonlinear transport

The simplest nonlinear partial differential equation to solve is certainly the homogenous transport equation whose most general form is expressed as

$$u_t + a(u)u_x = 0, \tag{$\diamond$}$$

or equivalently as

$$\frac{d}{dt}[u(x(t), t)] = 0,$$

or even

$$u_t + [A(u)]_x = 0,$$

with $A'(u) = a(u)$.

Such equations can be solved with the characteristic lines technique. In certain cases, the characteristic lines can display an intersection point meaning that two different solutions are possible. One technique to circumvent such problem would be to allow a jump discontinuity in the solution, also called a *shock*.
When a *shock* is present, it is very important to notice that the PDE ($\diamond$) does not hold anymore, as the derivatives u_t and u_x stop existing at a discontinuity. Only the integral form of the equation on a compact $\mathcal{C}$ still holds:

$$\frac{d}{dt}\int_{\mathcal{C}} u dx = 0. \tag{$\star$}$$

Therefore, a solution in the sense of distributions is necessary to solve the problem but its uniqueness would not be guaranteed anymore. It implies the admissibility of the solution should always be discussed.

A *shock wave* is a function, which must satisfy along its curves of discontinuity the *RankineHugoniot formula* and the *entropy criterion*.
The *RankineHugoniot formula* is given by

$$\frac{A(u^+) - A(u^-)}{u^+ - u^-} = s(t),$$

with $s(t)$ the speed of the shock wave.

To determine whether a solution is rejected or not, the *entropy criterion* must be verified. For a shock wave, the *entropy criterion* is defined as

$$a(u^-) > s(t) > a(u^+),$$

It requires that $a(u^-)$ the wave speed behind the shock is greater than $a(u^+)$ the wave speeds ahead of it.

6.1 Nonlinear Transport 1 [Strauss 14.1, Ex.3] 🌶

Solve the nonlinear equation $u_t + uu_x = 0$ with the auxiliary condition $u(x, 0) = x$. Sketch some of the characteristic lines. Give a physical interpretation of the solution.

Solution

From the definition of the total derivative

$$\frac{d}{dt}[u(x(t),t)] = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x}, \tag{6.1.1}$$

the PDE can be written equivalently as

$$\frac{d}{dt}[u(x(t),t)] = 0, \tag{6.1.2}$$

provided u_x and u_t are defined on the line $x = x(t)$ and provided $\frac{dx}{dt} = u(x(t),t)$.
Then along any such line the solution is constant, *i.e.*

$$u(x(t),t) = C_1, \quad C_1 \in \mathbb{R}. \tag{6.1.3}$$

The characteristic curves are given by

$$\frac{dx}{dt} = u(x(t),t), \tag{6.1.4}$$

$$\Rightarrow \qquad \frac{dx}{dt} = C_1, \tag{6.1.5}$$

$$\Rightarrow \qquad x(t) = C_1 t + C_2, \quad C_1, C_2 \in \mathbb{R}. \tag{6.1.6}$$

Then using the initial condition gives ($x_0 \triangleq x(0)$)

$$u(x(t),t) = C_1 = u(x(0),0) = \phi(x_0). \tag{6.1.7}$$

Note that the above equation ceases to hold if u_x and/or u_t are not defined at some point on the line between $(x_0, 0)$ and $(x(t), t)$.
This result can be replaced in the characteristic line equation, *i.e.*

$$x(t) = \phi(x_0)t + C_2. \tag{6.1.8}$$

At $t = 0$, one has

$$x(0) = x_0 = C_2. \tag{6.1.9}$$

Hence

$$x(t) = \phi(x_0)t + x_0. \tag{6.1.10}$$

Now, the given initial condition $\phi(x_0) = x_0$ is replaced into the characteristic line equation

$$x(t) = x_0 t + x_0, \tag{6.1.11}$$

$$\Rightarrow \qquad x(t) = x_0(1+t), \tag{6.1.12}$$

$$\Rightarrow \qquad x_0 = \frac{x}{1+t}. \tag{6.1.13}$$

Therefore

$$u(x,t) = \phi(x_0) = \frac{x}{1+t}. \tag{6.1.14}$$

To draw the characteristic lines, the Eq.(6.1.12) can be used for different x_0. It is possible to observe on the graph that all the lines are never crossing each other as long as $t \geq 0$. It means that in a physical situation, no shock will occur with such initial condition. This is a perfect example of a *rarefaction wave* causing the solution to spread out as time progress.

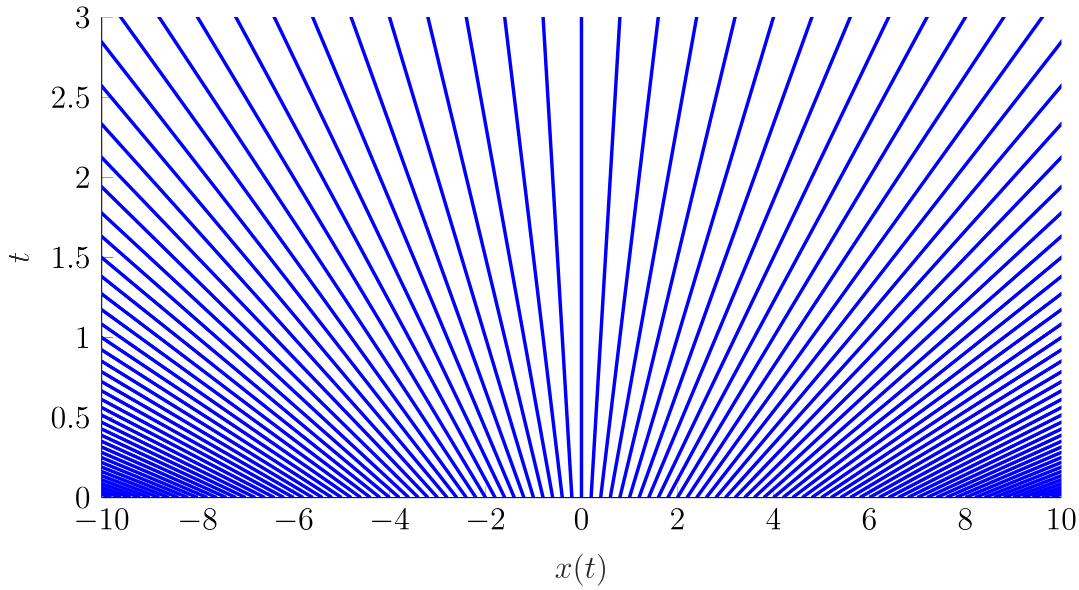

Figure 8: The characteristic lines are represented in blue.

6.2 Nonlinear Transport 2 [Strauss 14.1, Ex.5] 🌶️🌶️

Solve $u_t + u^2 u_x = 0$ with $u(x,0) = 2 + x$. Sketch some of the characteristic lines.

Solution

From the definition of the total derivative

$$\frac{d}{dt}[u(x(t),t)] = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x}, \tag{6.2.1}$$

the PDE can be written equivalently as

$$\frac{d}{dt}[u(x(t),t)] = 0, \tag{6.2.2}$$

provided u_x and u_t are defined on the line $x = x(t)$ and provided $\frac{dx}{dt} = u(x(t),t)^2$. Then along any such line the solution is constant, *i.e.*

$$u(x(t),t) = C_1, \quad C_1 \in \mathbb{R}. \tag{6.2.3}$$

The characteristic curves are given by

$$\frac{dx}{dt} = u(x(t),t)^2, \tag{6.2.4}$$

$$\Rightarrow \qquad \frac{dx}{dt} = C_1^2, \tag{6.2.5}$$

$$\Rightarrow \qquad x(t) = C_1^2 t + C_2, \quad C_2 \in \mathbb{R}. \tag{6.2.6}$$

Using the initial condition and Eq(6.2.3), one has

$$u(x(t),t) = C_1 = u(x(0),0) = \phi(x_0). \tag{6.2.7}$$

Note that the above equation ceases to hold if u_x and/or u_t are not defined at some point on the line between $(x_0, 0)$ and $(x(t), t)$.
Inserting this result in the equation of the characteristic lines gives

$$x(t) = \phi^2(x_0)t + x_0, \tag{6.2.8}$$
$$= (2 + x_0)^2 t + x_0. \tag{6.2.9}$$

To determine which curves (*i.e.* which x_0) pass through the point (x, t), the following polynomial equation has to be solved for x_0

$$tx_0^2 + (4t + 1)x_0 + (4t - x) = 0. \tag{6.2.10}$$

The two roots are

$$x_{0_{\frac{1}{2}}} = \frac{-(4t+1) \pm \sqrt{1 + 4t(2 + x)}}{2t} \tag{6.2.11}$$

which means that for a given point (x, t), there are two characteristic lines passing through that point. These lines are plotted in Figure 9. No characteristic line penetrates the region

$$1 + 4t(2 + x) < 0 \tag{6.2.12}$$

because Eq.(6.2.10) admits no solution for such (x, t).

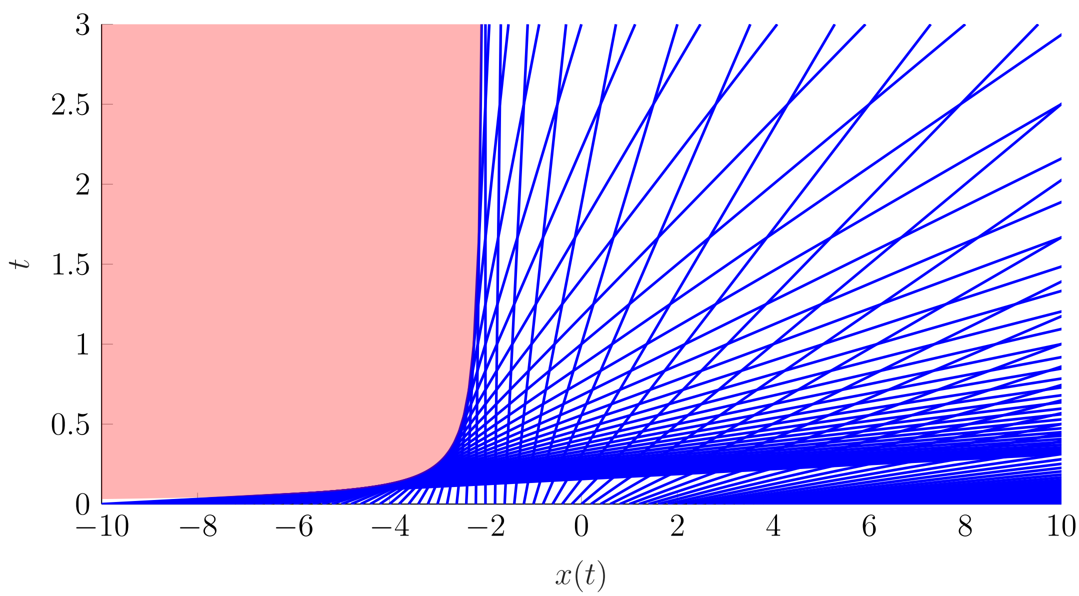

Figure 9: Characteristic lines are depicted in blue while the red region represents the values impossible to take.

Therefore, the two possible solutions are

$$u_{\frac{1}{2}}(x, t) = 2 + x_{0_{\frac{1}{2}}}, \tag{6.2.13}$$
$$= \frac{-1 \pm \sqrt{1 + 4t(2 + x)}}{2t}. \tag{6.2.14}$$

The solution procedure used here is valid provided u_x and u_t are defined at all points on the line between $(x_0, 0)$ and $(x(t), t)$. Using the above possible solutions, the derivative w.r.t x is given by

$$\partial_x u_{\frac{1}{2}} = \pm \frac{1}{\sqrt{1+4t(2+x)}} \tag{6.2.15}$$

which means that $\partial_x u_{\frac{1}{2}}$ does not exist when $1+4t(2+x) = 0$. As can be seen from Figure 9, the characteristic line corresponding to x_{0^2} is actually tangent to the hyperbola $1+4t(2+x) = 0$ before it arrives at the point (x, t). This characteristic line is thus not valid beyond the tangent point. Consequently, there is only one valid characteristic curve that passes through (x, t) given by

$$x_0 = \frac{-(4t+1) + \sqrt{1+4t(2+x)}}{2t} \tag{6.2.16}$$

and thus

$$u(x,t) = \frac{-1 + \sqrt{1+4t(2+x)}}{2t}. \tag{6.2.17}$$

If one is not convinced that the solution u_1 should not be kept, one can check if the initial conditions are verified.

$$\lim_{t\to 0} u_1(x,t) = \lim_{t\to 0} \frac{-1 + \sqrt{1+4t(2+x)}}{2t}. \tag{6.2.18}$$

L'Hôpital's rule can thus be applied to this expression

$$\lim_{t\to 0} u_1(x,t) \overset{\text{H}}{=} \lim_{t\to 0} \frac{2+x}{\sqrt{1+4t(2+x)}} = 2+x. \tag{6.2.19}$$

The initial condition is recovered as expected, and the solution $u_1(x,t)$ is thus valid.
If the same procedure is now applied to the second solution

$$\lim_{t\to 0} u_2(x,t) = \lim_{t\to 0} \frac{-1 - \sqrt{1+4t(2+x)}}{2t} = -\infty \neq 2+x. \tag{6.2.20}$$

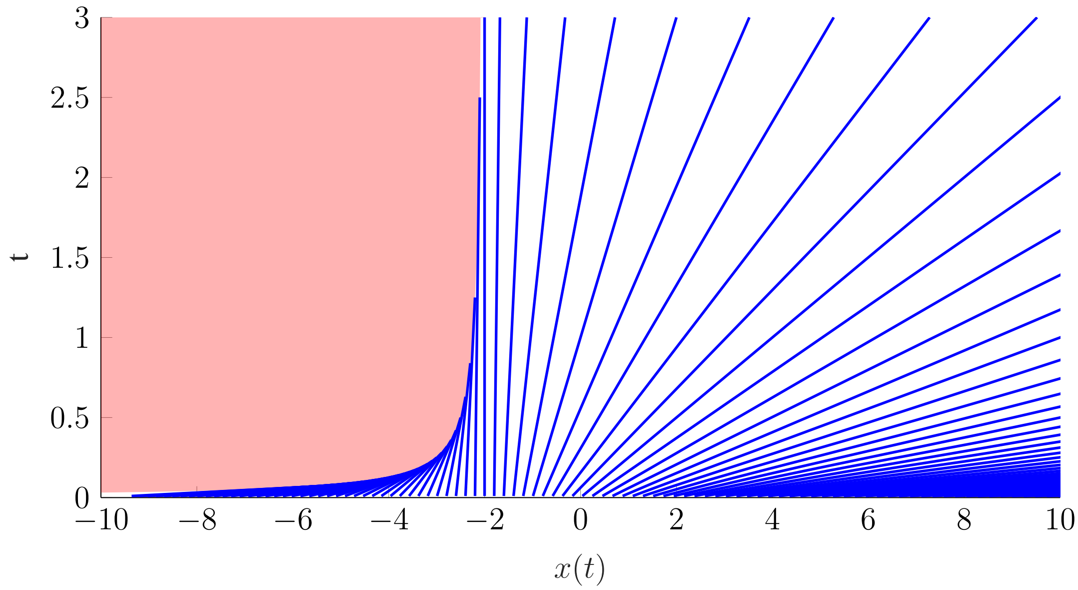

Figure 10: Characteristic lines of the valid solution are depicted in blue while the red region represents the values impossible to take.

6.3 Shocks and Entropy criterion 1 [Strauss 14.1, Examples 6 and 7] 🌶

Solve $u_t + uu_x = 0$ with the following initial conditions

(a)

$$u(x,0) = \phi(x) = \begin{cases} 0 & \forall x > 0, \\ 1 & \forall x < 0. \end{cases} \tag{6.3.1}$$

(b)

$$u(x,0) = \phi(x) = \begin{cases} 1 & \forall x > 0, \\ 0 & \forall x < 0. \end{cases} \tag{6.3.2}$$

In both cases, sketch the characteristic lines and determine the speed of the shock waves. Which one is physically correct?

Solution

(a) From the definition of the total derivative

$$\frac{d}{dt}[u(x(t),t)] = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x}, \tag{6.3.3}$$

the PDE can be written equivalently as

$$\frac{d}{dt}[u(x(t),t)] = 0, \tag{6.3.4}$$

provided u_x and u_t are defined on the line $x = x(t)$ and provided $\frac{dx}{dt} = u(x(t), t)$.
Then along any such line the solution is constant, *i.e.*

$$u(x(t), t) = C_1, \quad C_1 \in \mathbb{R} \tag{6.3.5}$$

The characteristic curves are given by

$$\frac{dx}{dt} = u(x(t), t), \tag{6.3.6}$$
$$\Rightarrow \qquad \frac{dx}{dt} = C_1, \tag{6.3.7}$$
$$\Rightarrow \qquad x(t) = C_1 t + C_2, \quad C_1, C_2 \in \mathbb{R}. \tag{6.3.8}$$

Then using the initial condition gives ($x_0 \triangleq x(0)$)

$$u(x(t), t) = C_1 = u(x(0), 0) = \phi(x_0). \tag{6.3.9}$$

Note that the above equation ceases to hold if u_x and/or u_t are not defined at some point on the line between $(x_0, 0)$ and $(x(t), t)$.
This result can be replaced in the characteristic line equation, *i.e.*

$$x(t) = \phi(x_0)t + C_2. \tag{6.3.10}$$

At $t = 0$, one has

$$x(0) = x_0 = C_2. \tag{6.3.11}$$

Hence

$$x(t) = \phi(x_0)t + x_0 \tag{6.3.12}$$

and the characteristic lines are given by

$$\begin{cases} x(t) = x_0 & \forall x_0 > 0, \\ x(t) = t + x_0 & \forall x_0 < 0. \end{cases} \tag{6.3.13}$$

The line(s) that passes through a given (x, t) is(are) therefore

$$\begin{cases} x_0 = x & 0 < t < x, \\ x_0 = x - t & x < 0, \\ x_{0_2^1} = \begin{cases} x \\ x - t \end{cases} & 0 < x < t. \end{cases} \tag{6.3.14}$$

Thus for (x, t) such that $0 < t < x$ or $x < 0$, the solution takes the value of the unique characteristic curve passing through (x, t), *i.e.*

$$u(x, t) = \begin{cases} 0 & 0 < t < x, \\ 1 & x < 0 \end{cases} \tag{6.3.15}$$

but for (x, t) such that $0 < x < t$, the solution is not defined as there is two possible values corresponding to the two possible characteristics.

A possible way to determine the solution in the region $0 < x < t$ is to consider that there is a shock inside that region. The position of the shock determines which characteristic line (x_{0^1} or x_{0_2}) should be kept inside this region.
The shock wave speed $s(t)$ is given by the Rankine-Hugoniot formula

$$s(t) = \frac{A(u^+) - A(u^-)}{u^+ - u^-} \tag{6.3.16}$$

where $A(u)$ is the flux (defined such that $\partial_x A(u) = a(u)u_x$, *i.e.* $A(u) = \frac{1}{2}u^2$ here) while u^+ and u^- are the value of u just before and just after the shock ($u^+ = 0$ and $u^- = 1$ here).
The speed is then given by

$$s(t) = \frac{1}{2} \tag{6.3.17}$$

and the entropy criterion is respected as

$$a(u^-) \geq s(t) \geq a(u^+), \tag{6.3.18}$$

$$1 \geq \frac{1}{2} \geq 0. \tag{6.3.19}$$

Therefore, this solution is physically acceptable.
The shock wave characteristic line $x_s(t)$ is given by

$$x_s(t) = \frac{1}{2}t \qquad , \forall t > 0. \tag{6.3.20}$$

because the first intersection of characteristic occurs at $(x, t) = (0, 0)$.
Finally, a generalized solution is thus

$$u(x,t) = \begin{cases} 0 & 0 < t < 2x, \\ 1 & t > 2x. \end{cases} \tag{6.3.21}$$

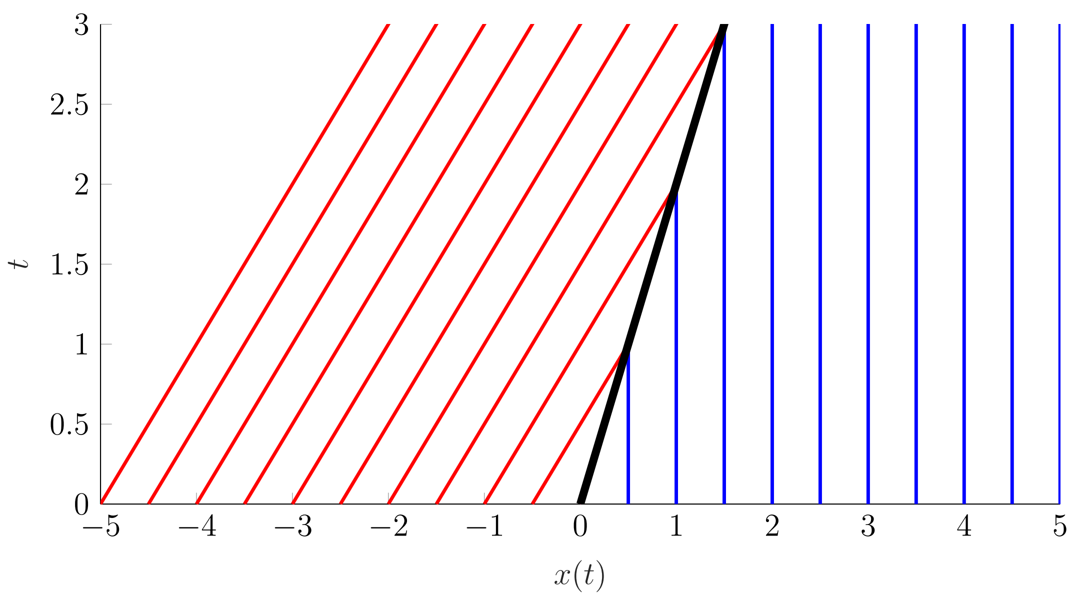

Figure 11: Characteristic lines for $x_0 < 0$ and $x_0 > 0$ are respectively represented in red and blue. The black line between both regions is the shock wave.

(b) Similarly, one finds

$$u(x,t) = \begin{cases} 1 & x > t, \\ 0 & x < 0 \end{cases} \tag{6.3.22}$$

Yet, the traditional technique does not give any direct solution in the region $0 < x < t$.

One possible solution would be to consider

$$u(x,t) = \begin{cases} 0 & 0 < t < 2x, \\ 1 & t > 2x \end{cases} \tag{6.3.23}$$

but now the entropy criterion is violated, indeed

$$a(u^-) \geq s(t) \geq a(u^+), \tag{6.3.24}$$

$$0 \geq \frac{1}{2} \geq 1. \tag{6.3.25}$$

An engineer would therefore reject this solution as it is not physically valid :(.

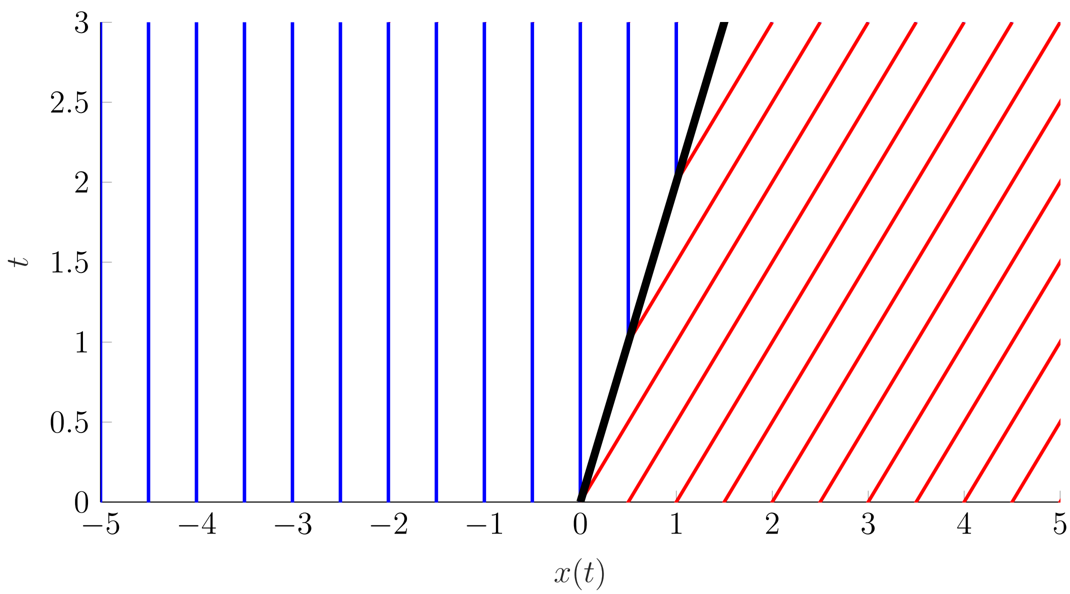

Figure 12: Representation of a non-physical solution. Characteristic lines for $x_0 < 0$ and $x_0 > 0$ are respectively depicted in blue and in red. The black line between both regions is the shock wave.

Another possibility would be to consider the physically valid classical solution

$$u(x,t) = \begin{cases} 1 & \forall\, x > t, \\ x/t & \forall\, 0 < x < t, \\ 0 & \forall\, x < 0. \end{cases} \tag{6.3.26}$$

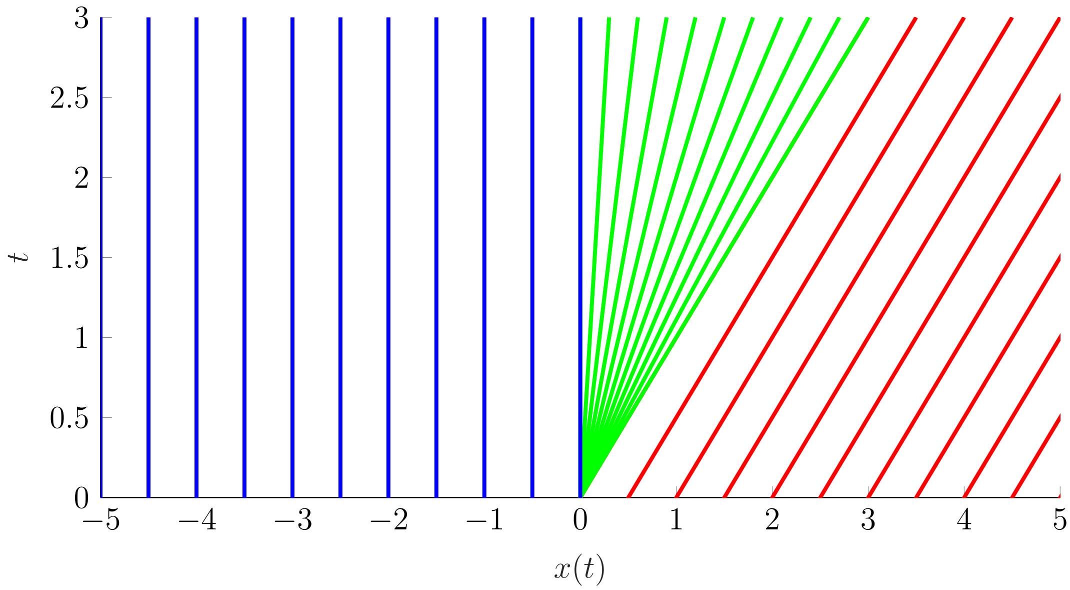

Figure 13: Representation of a physical solution. Characteristic lines for $x \leq 0$, $0 \leq x \leq t$ and $x \geq t$ are respectively depicted in blue, in green and in red.

The characteristics are not crossing each other and no shock appears; this solution is then continuous.
Caution: the most continuous solution is not always the best!

6.4 Shocks and Entropy criterion 2 [Strauss 14.1, Ex.10] 🌶🌶🌶

Solve $u_t + uu_x = 0$ with the following initial condition

$$u(x,0) = \phi(x) = \begin{cases} 1 & x \leq 0, \\ 1-x & 0 \leq x \leq 1, \\ 0 & x \geq 1. \end{cases} \tag{6.4.1}$$

Find exactly where the shock is and show that it satisfies the entropy condition. Sketch the characteristics.

Solution

From the definition of the total derivative

$$\frac{d}{dt}[u(x(t),t)] = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x}, \tag{6.4.2}$$

the PDE can be written equivalently as

$$\frac{d}{dt}[u(x(t),t)] = 0 \tag{6.4.3}$$

provided u_x and u_t are defined on the line $x = x(t)$ and provided $\frac{dx}{dt} = u(x(t),t)$.
Then along any such line the solution is constant, *i.e.*

$$u(x(t),t) = C_1, \quad C_1 \in \mathbb{R} \tag{6.4.4}$$

The characteristic curves are given by

$$
\begin{aligned}
& \frac{dx}{dt} = u(x(t),t), && (6.4.5)\\
\Rightarrow \quad & \frac{dx}{dt} = C_1, && (6.4.6)\\
\Rightarrow \quad & x(t) = C_1 t + C_2, \quad C_1, C_2 \in \mathbb{R}. && (6.4.7)
\end{aligned}
$$

Then using the initial condition gives ($x_0 \triangleq x(0)$)

$$
u(x(t),t) = C_1 = u(x(0),0) = \phi(x_0). \tag{6.4.8}
$$

Note that the above equation ceases to hold if u_x and/or u_t are not defined at some point on the line between $(x_0, 0)$ and $(x(t), t)$.
This result can be replaced in the characteristic line equation, *i.e.*

$$
x(t) = \phi(x_0)t + C_2. \tag{6.4.9}
$$

At $t = 0$, one has

$$
x(0) = x_0 = C_2. \tag{6.4.10}
$$

Hence

$$
x(t) = \phi(x_0)t + x_0, \tag{6.4.11}
$$

and the characteristic lines are given by

$$
\begin{cases}
x(t) = t + x_0 & x_0 \leq 0,\\
x(t) = (1 - x_0)t + x_0 & 0 \leq x_0 \leq 1,\\
x(t) = x_0 & x_0 \geq 1.
\end{cases} \tag{6.4.12}
$$

The line(s) that passes through a given (x,t) is(are) therefore

$$
\begin{cases}
x_0 = x - t & x \leq 1 \text{ and } x \leq t & \text{(region 1)},\\
x_0 = \dfrac{x-t}{1-t} & 0 \leq x \leq 1 \text{ and } x \geq t & \text{(region 2)},\\
x_0 = x & x \geq 1 \text{ and } x \geq t & \text{(region 3)},\\
x_{0-1,2,3} = \begin{cases} x - t \\ \dfrac{x-t}{1-t} \\ x \end{cases} & x \geq 1 \text{ and } x \leq t & \text{(region 4)}.
\end{cases} \tag{6.4.13}
$$

Thus, for (x,t) in regions 1, 2 or 3, the solution takes the value of the unique characteristic curve passing through (x,t), *i.e.*

$$
u(x,t) = \begin{cases}
1 & \text{in region } 1,\\
\dfrac{1-x}{1-t} & \text{in region } 2,\\
0 & \text{in region } 3
\end{cases} \tag{6.4.14}
$$

but for (x,t) in region 4, the solution is not defined as there are three possible values corresponding to the three possible characteristics.

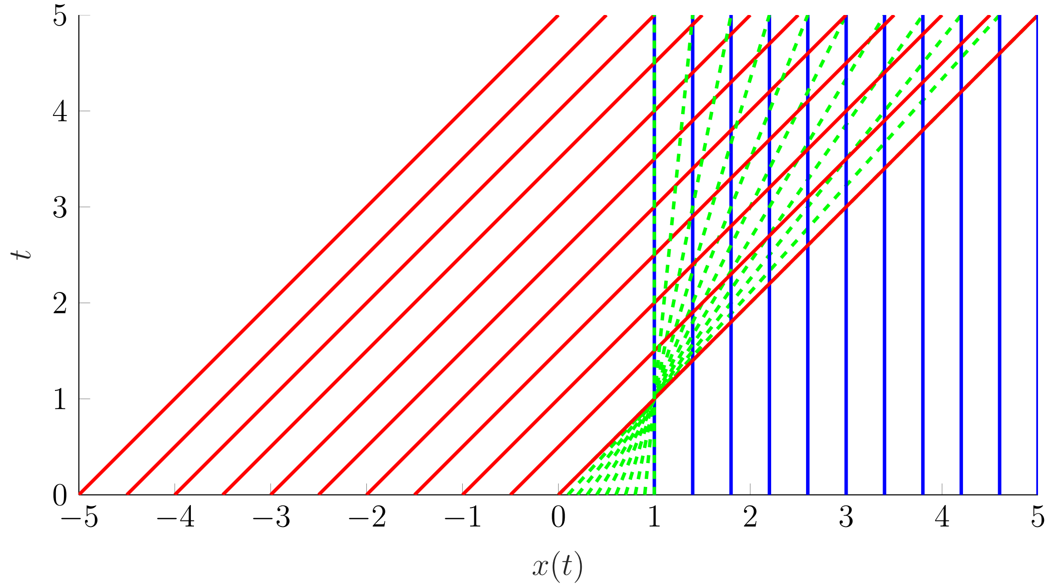

Figure 14: Characteristic lines are depicted in red for $x_0 \leq 0$, in green for $0 \leq x_0 \leq 1$ and in blue for $x_0 \geq 1$.

The solution's procedure used here is valid provided u_x and u_t are defined at all points on the line between $(x_0, 0)$ and $(x(t), t)$. Using the above incomplete solution, it appears that u_x and u_t are not defined at $(x,t) = (1,1)$ (the solution is discontinuous at that point). The characteristic lines passing through that point are therefore not valid anymore beyond that point. Consequently there are only two possibilities left in region 4.
A possible way to determine the solution in region 4 is to consider that there is a shock inside that region. The position of the shock determines which characteristic line (x_{0-1} or x_{0-3}) should be kept inside that region.
The shock wave speed $s(t)$ is given by the Rankine-Hugoniot formula

$$s(t) = \frac{A(u^+) - A(u^-)}{u^+ - u^-} \tag{6.4.15}$$

where $A(u)$ is the flux (defined such that $\partial_x A(u) = a(u)u_x$, *i.e.* $A(u) = \frac{1}{2}u^2$ here) while u^- and u^+ are the two possible values of u at the shock ($u^- = 1$ and $u^+ = 0$ here).
The speed is then given by

$$s(t) = \frac{1}{2} \tag{6.4.16}$$

and the entropy criterion is respected as

$$a(u^-) \geq s(t) \geq a(u^+), \tag{6.4.17}$$
$$1 \geq \frac{1}{2} \geq 0. \tag{6.4.18}$$

Therefore, this solution is physically acceptable.
The shock wave characteristic line $x_s(t)$ is given by

$$x_s(t) = \frac{1}{2}(t+1) \qquad , \forall t \geq 1. \tag{6.4.19}$$

because the first intersection of characteristic occurs at $(x,t) = (1,1)$.

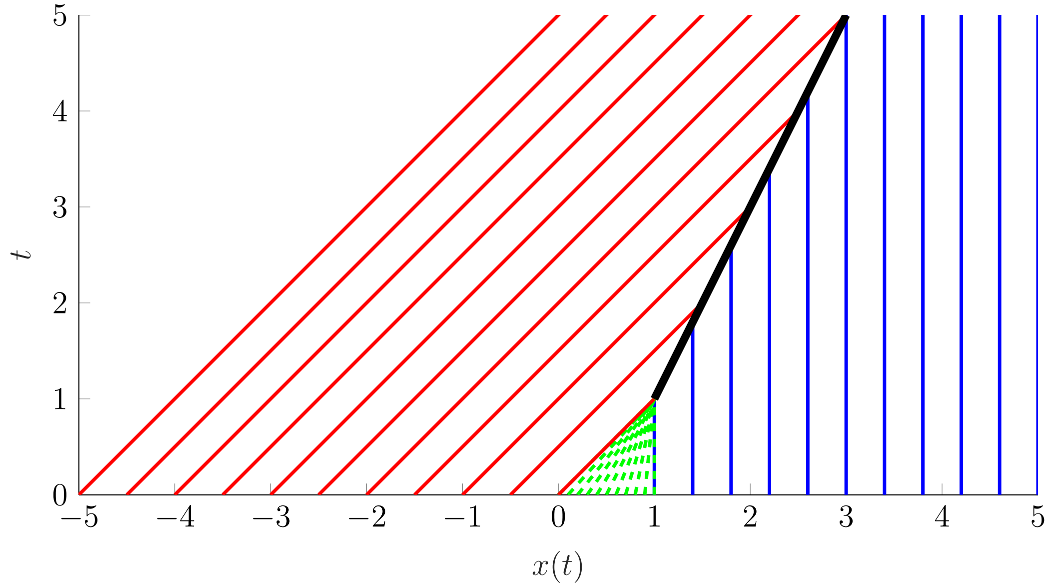

Figure 15: Characteristic lines are depicted in red for $x_0 \leq 0$, in green for $0 \geq x_0 \leq 1$ and in blue for $x_0 \geq 1$. The shock wave is represented in black.

Finally, a generalized solution is thus

$$u(x,t) = \begin{cases} 1 & 2x \leq 1+t \text{ and } x \leq t, \\ \dfrac{1-x}{1-t} & 0 \leq x \leq 1 \text{ and } x \geq t, \\ 0 & x \geq 1 \text{ and } 2x \geq 1+t. \end{cases} \tag{6.4.20}$$

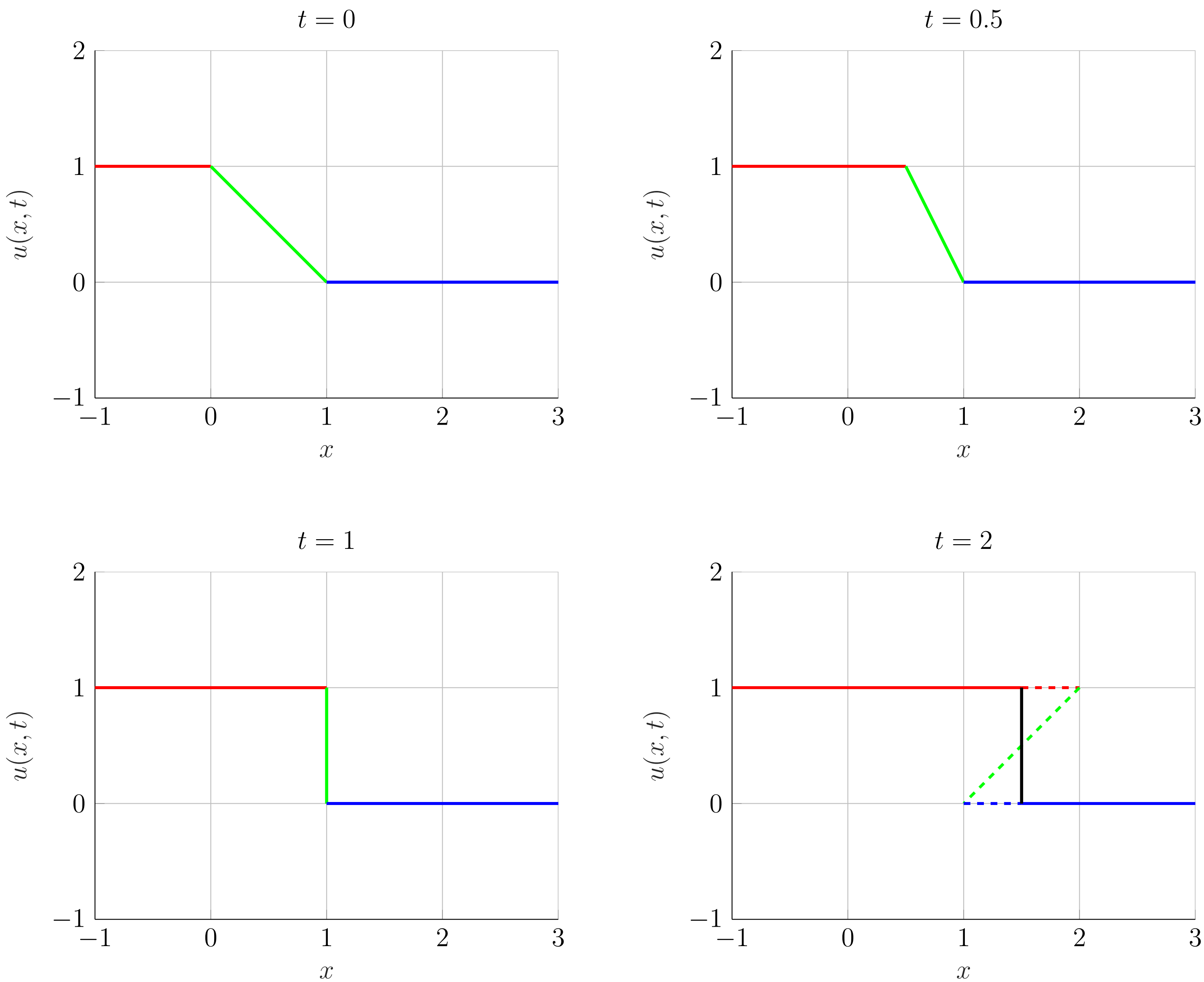

Figure 16: Representation of the solutions at different time, where the red line refers to the region 1, the green one to region 2 and the blue one to region 3. At $t = 2$, the black line represents the shock wave. The dashed lines would have been the solutions obtained along characteristics beyond the shock (the rejected solution).

Note: The solution of a shock represented in Fig.16 is actually forming a step wave cutting the rejected solutions such as the removed area between the red dashed line and the green dashed line is the same as the removed area between the green dashed line and the blue dashed line. In other words, the shock obtained with the Rankine-Hugoniot formula actually verifies the Eq.($\star$).

6.5 Inhomogeneous Nonlinear Transport [Strauss 14.1, Ex.12] 🌶️🌶️

Solve $u_t + uu_x = 1$ with $u(x, 0) = x$.

Solution

From the definition of the total derivative

$$\frac{d}{dt}[u(x(t),t)] = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x}, \tag{6.5.1}$$

the PDE can be written equivalently as

$$\frac{d}{dt}[u(x(t),t)] = 1, \tag{6.5.2}$$

provided u_x and u_t are defined on the line $x = x(t)$ and provided $\frac{dx}{dt} = u(x(t),t)$.
Then along any such line the solution grows linearly, *i.e.*

$$u(x(t),t) = t + C_1, \quad C_1 \in \mathbb{R}. \tag{6.5.3}$$

The characteristic curves are given by

$$\frac{dx}{dt} = u(x(t),t), \tag{6.5.4}$$

$$\Rightarrow \qquad \frac{dx}{dt} = t + C_1, \tag{6.5.5}$$

$$\Rightarrow \qquad x(t) = \frac{1}{2}t^2 + C_1 t + C_2, \quad C_1, C_2 \in \mathbb{R}. \tag{6.5.6}$$

Then using the initial condition gives ($x_0 \triangleq x(0)$)

$$u(x(0),0) = C_1 = \phi(x_0). \tag{6.5.7}$$

Note that the above equation ceases to hold if u_x and/or u_t are not defined at some point on the line between $(x_0, 0)$ and $(x(t), t)$.
This result can be replaced in the characteristic line equation, *i.e.*

$$x(t) = \frac{1}{2}t^2 + \phi(x_0)t + C_2. \tag{6.5.8}$$

At $t = 0$, one has

$$x(0) = x_0 = C_2. \tag{6.5.9}$$

Hence

$$x(t) = \frac{1}{2}t^2 + \phi(x_0)t + x_0. \tag{6.5.10}$$

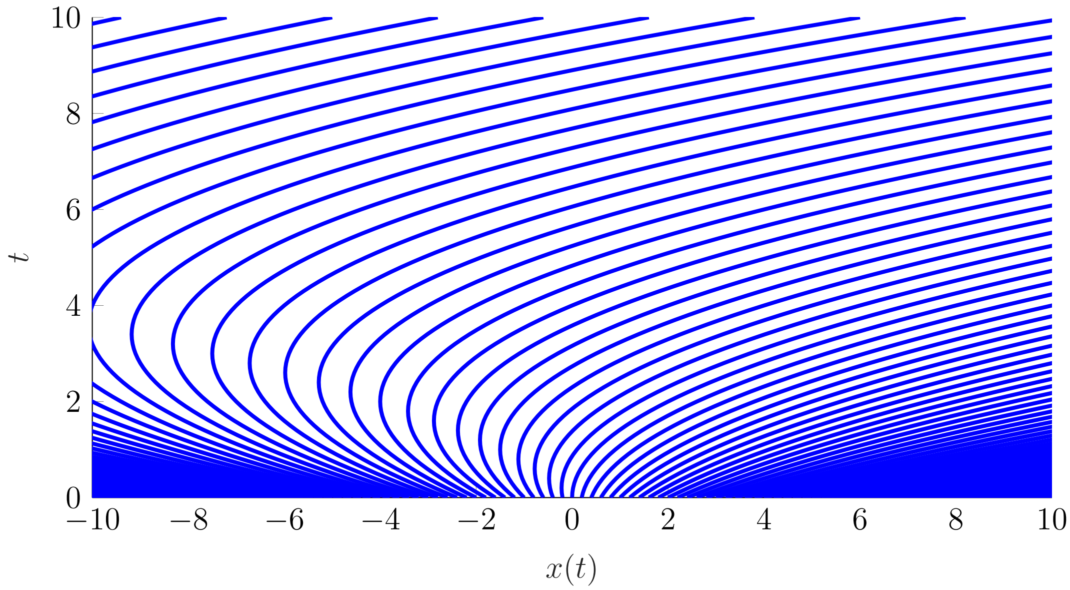

Figure 17: Characteristic lines of the problem are depicted in blue.

The line passing through a given (x, t) is therefore

$$x_0 = \frac{2x - t^2}{2(t+1)}. \tag{6.5.11}$$

and finally the solution is

$$\begin{aligned} u(x,t) &= t + x_0 && (6.5.12)\\ &= t + \frac{2x - t^2}{2(t+1)}. && (6.5.13) \end{aligned}$$

6.6 General form of Transport 🌶️🌶️

Study the partial differential equation with the following generic form

$$u_t + [x + f(u)]\, u_x = g(x,t),$$

with functions $f(u(x,t))$ and $g(x,t)$ for $(x,t) \in \mathbb{R} \times [0, +\infty[$. Consider the initial condition

$$u(x,0) = x + 2$$

and analyse the influence of the functions on the solutions and their characteristics curves. Three different scenarios will be considered:

(a) $f(u) = 1$ and $g(x,t) = 0$.

(b) $f(u) = u$ and $g(x,t) = 0$.

(c) $f(u) = u$ and $g(x,t) = 1$.

For each case, classify the equation (linearity, homogeneity, order), find the solution, determine the equation of the characteristic and sketch them.

Solution

(a) The equation to solve becomes

$$u_t + (x+1)\; u_x = 0. \tag{6.6.1}$$

From the definition of the total derivative

$$\frac{d}{dt}[u(x(t),t)] = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x}, \tag{6.6.2}$$

the PDE can be written equivalently as

$$\frac{d}{dt}[u(x(t),t)] = 0, \tag{6.6.3}$$

provided u_x and u_t are defined on the line $x = x(t)$ and provided $\frac{dx}{dt} = x+1$.
Then along any such line the solution is constant, *i.e.*

$$u(x(t),t) = C_1, \quad C_1 \in \mathbb{R}. \tag{6.6.4}$$

The characteristic curves are given by

$$\frac{dx}{dt} = x+1, \tag{6.6.5}$$

$$\Rightarrow \qquad \ln|x+1| = t + C_2, \tag{6.6.6}$$

$$\Rightarrow \qquad |x+1| = K\exp(t), \quad K \in \mathbb{R}_0^+ \tag{6.6.7}$$

$$\Rightarrow \qquad x(t) = -1 \pm K\exp(t). \tag{6.6.8}$$

Then using the initial condition gives ($x_0 \triangleq x(0)$)

$$u(x(t),t) = C_1 = u(x(0),0) = \phi(x_0). \tag{6.6.9}$$

Note that the above equation ceases to hold if u_x and/or u_t are not defined at some point on the line between $(x_0, 0)$ and $(x(t), t)$.
At $t = 0$, one has

$$x(0) = x_0 = -1 + K, \tag{6.6.10}$$

and the constant is

$$K = \pm(x_0 + 1). \tag{6.6.11}$$

Hence,

$$x(t) = -1 \pm [\pm(x_0+1)]\exp(t), \tag{6.6.12}$$

$$\Rightarrow \qquad x(t) = -1 + (x_0+1)\exp(t), \tag{6.6.13}$$

$$\Rightarrow \qquad x_0 = (x+1)\exp(-t) - 1. \tag{6.6.14}$$

Therefore

$$u(x,t) = \phi(x_0) = (x+1)\exp(-t) + 1. \tag{6.6.15}$$

To draw the characteristic lines, the Eq.(6.6.13) can be used for different x_0.

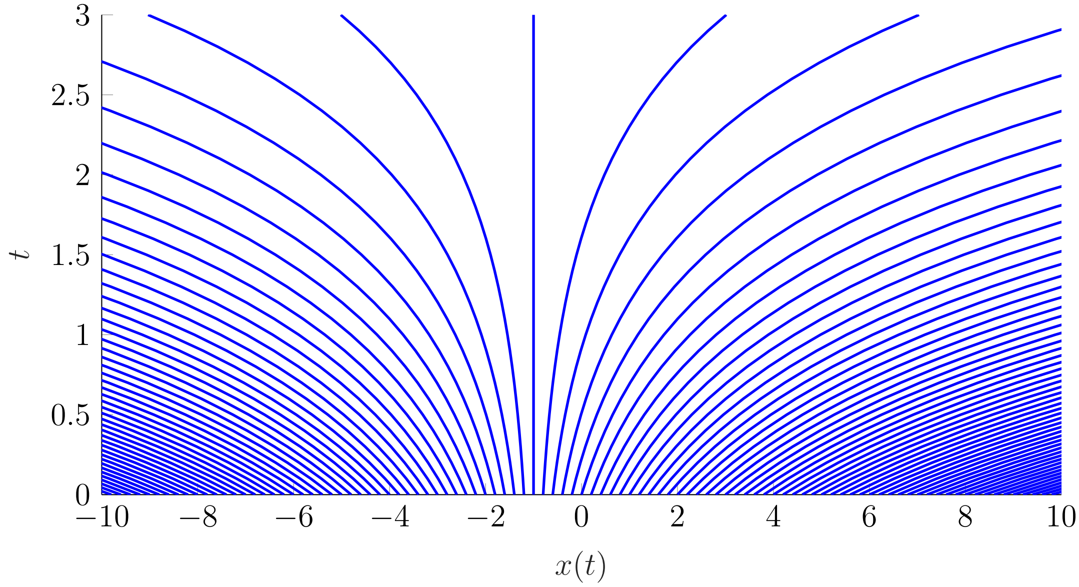

Figure 18: The characteristic lines are represented in blue.

(b) The equation to solve becomes

$$u_t + (x+u)\, u_x = 0. \tag{6.6.16}$$

From the definition of the total derivative

$$\frac{d}{dt}[u(x(t),t)] = \frac{\partial u}{\partial t} + \frac{dx}{dt}\,\frac{\partial u}{\partial x}, \tag{6.6.17}$$

the PDE can be written equivalently as

$$\frac{d}{dt}[u(x(t),t)] = 0, \tag{6.6.18}$$

provided u_x and u_t are defined on the line $x = x(t)$ and provided $\frac{dx}{dt} = x + u(x(t),t)$. Then along any such line the solution is constant, *i.e.*

$$u(x(t),t) = C_1, \quad C_1 \in \mathbb{R}. \tag{6.6.19}$$

The characteristic curves are given by

$$\frac{dx}{dt} = x + u(x(t),t), \tag{6.6.20}$$

$$\Rightarrow \qquad \ln|x + C_1| = t + C_2, \quad C_1, C_2 \in \mathbb{R}. \tag{6.6.21}$$

$$\Rightarrow \qquad x(t) = \pm K \exp(t) - C_1, \quad K \in \mathbb{R}_0^+. \tag{6.6.22}$$

If $x + C_1 = 0$, the separation cannot be done and $\frac{dx}{dt} = 0$. Then the solution is simply $x = -C_1$, equivalent to the form above with $K = 0$. The most general solution is thus :

$$x(t) = K \exp(t) - C_1, \quad K \in \mathbb{R}. \tag{6.6.23}$$

Then using the initial condition gives ($x_0 \triangleq x(0)$)

$$u(x(t),t) = C_1 = u(x(0),0) = \phi(x_0). \tag{6.6.24}$$

Note that the above equation ceases to hold if u_x and/or u_t are not defined at some point on the line between $(x_0, 0)$ and $(x(t), t)$.
This result can be replaced in the characteristic line equation, *i.e.*

$$x(t) = K\exp(t) - \phi(x_0). \tag{6.6.25}$$

At $t = 0$, one has

$$x(0) = x_0 = K - \phi(x_0), \tag{6.6.26}$$

$$\Rightarrow \qquad K = (x_0 + \phi(x_0)). \tag{6.6.27}$$

Now, the given initial condition $\phi(x_0) = x_0 + 2$ is replaced into the constant K

$$K = 2(x_0 + 1), \tag{6.6.28}$$

and the characteristic line equation becomes

$$x(t) = 2(x_0 + 1)\exp(t) - x_0 - 2, \tag{6.6.29}$$

$$\Rightarrow \qquad x_0 = \frac{x - 2(\exp(t) - 1)}{2\exp(t) - 1}. \tag{6.6.30}$$

Therefore

$$u(x,t) = \phi(x_0) = \frac{x - 2(\exp(t) - 1)}{2\exp(t) - 1} + 2, \tag{6.6.31}$$

$$= \frac{x + 1}{2\exp(t) - 1} + 1. \tag{6.6.32}$$

To draw the characteristic lines, the Eq.(6.6.29) can be used for different x_0.

(c) The equation to solve becomes

$$u_t + (x + u)\, u_x = 1 \tag{6.6.33}$$

From the definition of the total derivative

$$\frac{d}{dt}[u(x(t),t)] = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x}, \tag{6.6.34}$$

the PDE can be written equivalently as

$$\frac{d}{dt}[u(x(t),t)] = 1, \tag{6.6.35}$$

provided u_x and u_t are defined on the line $x = x(t)$ and provided $\frac{dx}{dt} = x + u(x(t),t)$.
Then along any such line the solution is constant, *i.e.*

$$u(x(t),t) = t + C_1, \quad C_1 \in \mathbb{R}. \tag{6.6.36}$$

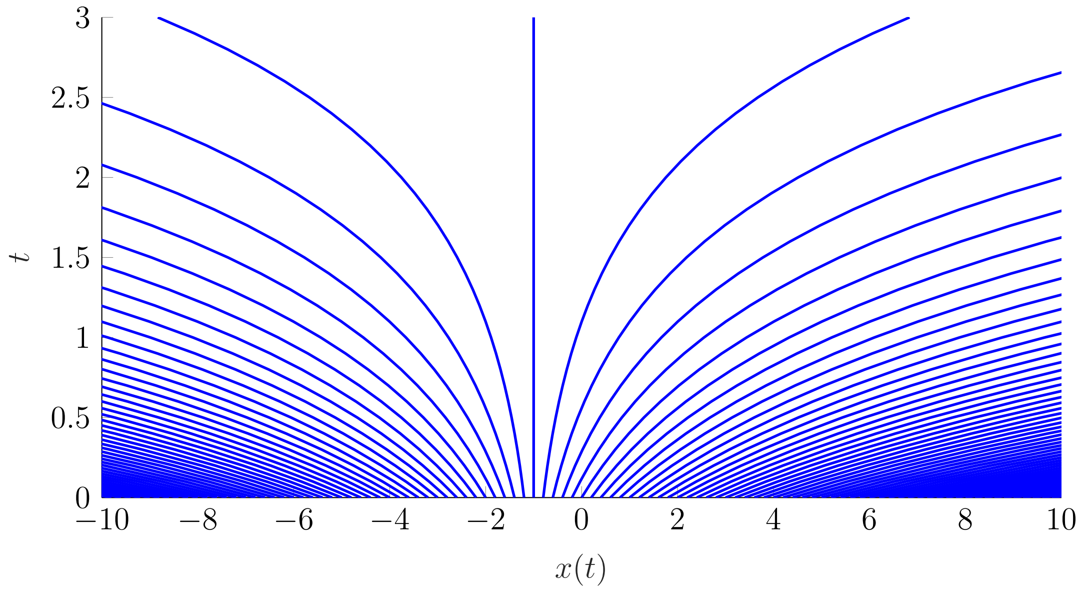

Figure 19: The characteristic lines are represented in blue.

The constant C_1 can be determined with the initial condition

$$u(x_0, 0) = C_1 = x_0 + 2. \tag{6.6.37}$$

The solution then becomes

$$u(x, t) = t + x_0 + 2. \tag{6.6.38}$$

The characteristic curves are given by

$$\frac{dx}{dt} = x + u(x(t), t), \tag{6.6.39}$$

$$\Rightarrow \quad \frac{dx}{dt} = x + t + x_0 + 2, \tag{6.6.40}$$

$$\Rightarrow \quad \frac{dx}{dt} - x(t) = t + x_0 + 2. \tag{6.6.41}$$

The general solution of the homogenous part of this differential equation is

$$x_h(t) = C_2 \exp(t), \tag{6.6.42}$$

with C_2 a constant.
The particular solution of the differential equation has the form

$$x_p(t) = At + B, \tag{6.6.43}$$

with A and B both constants that can be determined by replacing this expression in Eq.(6.6.41). Thus, $A = -1$ and $B = -(x_0 + 3)$ and the total solution is

$$x(t) = (2x_0 + 3) \exp(t) - t - (x_0 + 3). \tag{6.6.44}$$

At $x = 0$,

$$x(0) = C_2 - x_0 - 3, \tag{6.6.45}$$

$$\Rightarrow \quad C_2 = 2x_0 + 3. \tag{6.6.46}$$

Then, the characteristic equation becomes

$$x(t) = (2x_0 + 3)\exp(t) - t - (x_0 + 3), \tag{6.6.47}$$

$$\Rightarrow \quad x_0 = \frac{x - 3\exp(t) + t + 3}{2\exp(t) - 1} \tag{6.6.48}$$

Therefore,

$$u(x,t) = t + 2 + x_0 = \frac{2t\exp(t) + \exp(t) + 1 + x}{2\exp(t) - 1}. \tag{6.6.49}$$

To draw the characteristic lines, the Eq.(6.6.47) can be used for different x_0.

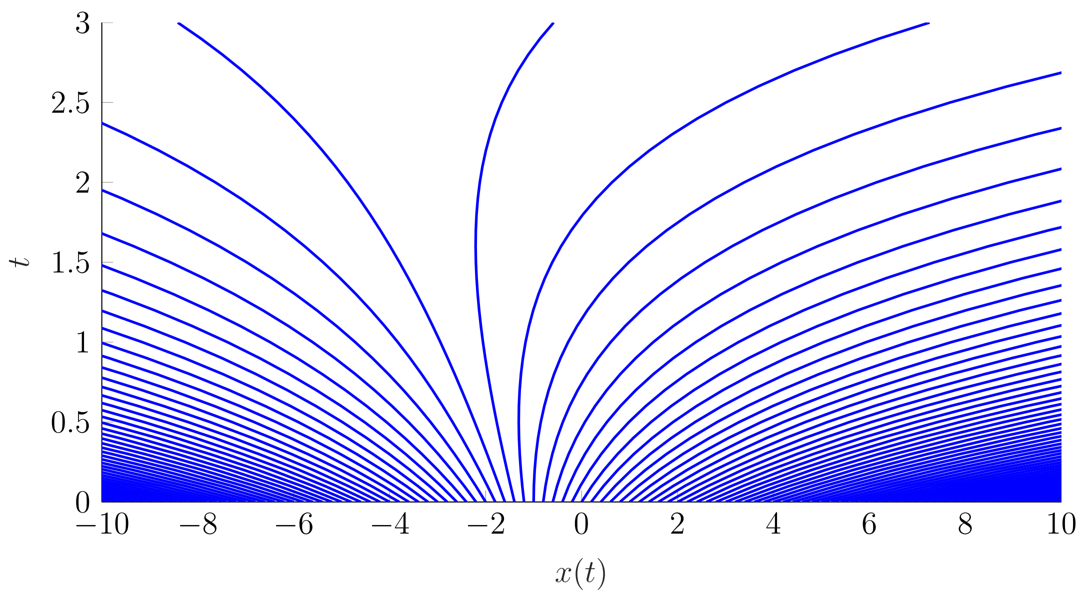

Figure 20: The characteristic lines are represented in blue.

6.7 Triangular signal [partly from Olver, Example 2.12] 🌶️🌶️🌶️

Solve $u_t + uu_x = 0$ with the triangular initial data

$$u(x,0) = \begin{cases} x, & 0 \le x \le 1, \\ 2 - x, & 1 \le x \le 2, \\ 0, & \text{otherwise}, \end{cases}$$

following the successive steps below. We define $U(t) = \int_{\mathbb{R}} u(x,t)dx$.

(a) Show that u is a conserved quantity for this equation. Find the expression of the associated flux. Compute the initial total quantity $U(0)$.

(b) Determine the characteristic curves. Draw some of them to get a representative figure.

(c) At which time instant $t_\star$ do these lines start to intersect? Give the solution for $0 < t < t_\star$. Check that $U(t)$ is indeed constant in that time interval.

(d) For $t > t_\star$, the solution contains a shock. Let $\sigma(t)$ be the position of the shock with respect to time, and $\sigma_t(t) = s(t)$ be the speed of the shock, use Rankine-Hugoniot formula to derive a first order ordinary differential equation for the shock position $\sigma(t)$. Solve this equation, with the appropriate initial position $\sigma(t_\star)$ for the shock.

(e) Give the final solution. Draw it, with the shock. How does the strength of the shock $u^- - u^+$ evolve with time? Does this shock satisfy the entropy criterion?

(f) Check that $U(t)$ is still constant for $t > t_\star$.

Solution (brief solution only)

(a) /

(b) /

(c) We get $t_\star = 1$. The solution is, for $0 < t < 1$,

$$u(x,t) = \begin{cases} \dfrac{x}{1+t}, & 0 < \dfrac{x}{1+t} < 1, \\ \dfrac{2-x}{1-t}, & 1 < \dfrac{x-2t}{1-t} < 2, \\ 0, & \text{otherwise.} \end{cases} \tag{6.7.1}$$

We indeed have $U(t) = 1$ for $0 < t < 1$.

(d) The equation follows from $u^-(t) = u(\sigma(t)^-, t) = \sigma(t)/(1+t)$, and $u^+(t) = u(\sigma(t)^+, t) = 0$, and $A(u) = u^2/2$. The initial condition is $\sigma(1) = 2$. After solving the resulting equation by separation of variables, we get the shock position $\sigma(t) = \sqrt{2(t+1)}$. (The only difference with simpler exercises is that, here, $u^-(t)$ depends on the position of the shock. The shock speed is therefore not constant.)

(e) The solution is, for $t > t_\star$,

$$u(x,t) = \begin{cases} \dfrac{x}{1+t}, & 0 < x < \sigma(t), \\ 0, & \text{otherwise.} \end{cases} \tag{6.7.2}$$

The strength of the shock is $\sqrt{2/(1+t)}$. The entropy criterion is satisfied.

(f) We directly get, for $t > t_\star$,

$$U(t) = \int_{\mathbb{R}} u(x,t)dx = \int_0^{\sigma(t)} \frac{x}{1+t}\, dx = \frac{1}{1+t}\frac{\sigma(t)^2}{2} = 1. \tag{6.7.3}$$

7 Numerical Linear Algebra

Singular Value Decomposition

Let $A \in \mathbb{C}^{m\times n}$ be a rectangular matrix. A singular value decomposition of A is a matrix factorization

$$A = U\Sigma V^* \tag{$\diamond$}$$

with

$$\begin{aligned}
&U \in \mathbb{C}^{m\times m} && \text{such that } UU^* = I,\\
&V \in \mathbb{C}^{n\times n} && \text{such that } VV^* = I,\\
&\Sigma \in \mathbb{R}^{m\times n} && \text{such that } \begin{cases} \Sigma_{ij} &= 0 \qquad \text{for } i \neq j,\\ \Sigma_{ij} &= \sigma_i > 0 \quad \text{for } i = j \leq \text{rank}(A),\\ \Sigma_{ij} &= 0 \qquad \text{for } i = j > \text{rank}(A).\end{cases}
\end{aligned}$$

The columns of U and V are called the *left* and *right singular vectors* respectively. They are denoted by u_j and v_j.

Left singular vector Left multiplying definition Eq($\diamond$) by U^* gives

$$\begin{aligned}
U^*A &= U^*U\Sigma V^*\\
\Rightarrow U^*A &= \Sigma V^*\\
\Rightarrow A^*U &= V\Sigma^*\\
\Rightarrow A^*u_j &= \sigma_j v_j \quad \text{for } j \leq \text{rank}(A) \text{ (no sum on } j)
\end{aligned}$$

which means that the image by A^* of a left singular vector u_j is the corresponding right singular vector v_j weighted by the corresponding singular value σ_j.

From definition ($\diamond$) it appears that the left singular vectors are the eigenvectors of AA^* and that the corresponding eigenvalues are the singular value squared. Indeed

$$\begin{aligned}
AA^* &= U\Sigma V^*(U\Sigma V^*)^*\\
&= U\Sigma V^* V\Sigma^* U^*\\
&= U\Sigma\Sigma^* U^*\\
\Rightarrow AA^*U &= U\Sigma\Sigma^* U^* U\\
&= U\Sigma\Sigma^*
\end{aligned}$$

i.e

$$\begin{cases} AA^*u_j &= \sigma_j^2 u_j \quad \text{for } j \leq \text{rank}(A)\\ AA^*u_j &= 0 \qquad \text{for } j > \text{rank}(A).\end{cases}$$

Right singular vector Right multiplying definition Eq($\diamond$) by V gives

$$\begin{aligned}
AV &= U\Sigma V^*V\\
\Rightarrow AV &= U\Sigma\\
\Rightarrow Av_j &= \sigma_j u_j \quad \text{for } j \leq \text{rank}(A) \text{ (no sum on } j)
\end{aligned}$$

which means that the image by A of a right singular vector v_j is the corresponding left singular vector u_j weighted by the corresponding singular value σ_j.
From definition (◇) it appears that the right singular vectors are the eigenvectors of A^*A and that the corresponding eigenvalues are the singular value squared. Indeed

$$\begin{aligned} A^*A &= (U\Sigma V^*)^* U\Sigma V^* \\ &= V\Sigma^* U^* U \Sigma V^* \\ &= V\Sigma^*\Sigma V^* \\ \Rightarrow A^*AV &= V\Sigma\Sigma^* V^* V \\ &= V\Sigma\Sigma^* \end{aligned}$$

i.e

$$\begin{cases} A^*Av_j &= \sigma_j^2 v_j \quad \text{for } j \leq \text{rank}(A) \\ A^*Av_j &= 0 \quad \text{for } j > \text{rank}(A). \end{cases}$$

Computing a singular value decomposition A singular value decomposition can be obtained through the following two steps

1. Find the eigenvectors and eigenvalues of A^*A (resp. AA^*). Deduce the right (resp. left) singular vector and the singular values
2. Find the left (resp. right) singular vector through $u_j = \frac{1}{\sigma_j}Av_j$ (resp. $v_j = \frac{1}{\sigma_j}A^*u_j$).

7.1 Exercise 1 [Trefethen Ex.4.1]

Determine the singular value decomposition of the following matrices and give a geometrical interpretation to the singular values and vectors

(a)

$$\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(d)

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

(e)

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Solution

(a) A is full rank and $m = n$ such that two non-vanishing singular values are expected.

Left singular vectors

$$AA^* = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}.$$

The eigenvalues are solutions of

$$\begin{aligned} \det AA^* - \lambda I &= 0 \\ \Rightarrow \det \begin{pmatrix} 9-\lambda & 0 \\ 0 & 4-\lambda \end{pmatrix} &= (9-\lambda)(4-\lambda) = 0 \end{aligned}$$

i.e $\lambda_1 = 9$, $\lambda_2 = 4$ such that $\sigma_1 = 3 > \sigma_2 = 2$. The second eigenvector verifies

$$\begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - 9 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ -5y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thus

$$u_1 = \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}.$$

The second eigenvector verifies

$$\begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - 4 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 5x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thus

$$u_2 = \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix}$$

The $+$ sign are chosen.

Right singular vectors

$$v_1 = \frac{1}{\sigma_1} A^* u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } v_2 = \frac{1}{\sigma_2} A^* u_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \tag{7.1.1}$$

Finally

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

The geometrical interpretation of this singular value decomposition is given in Figure 21.

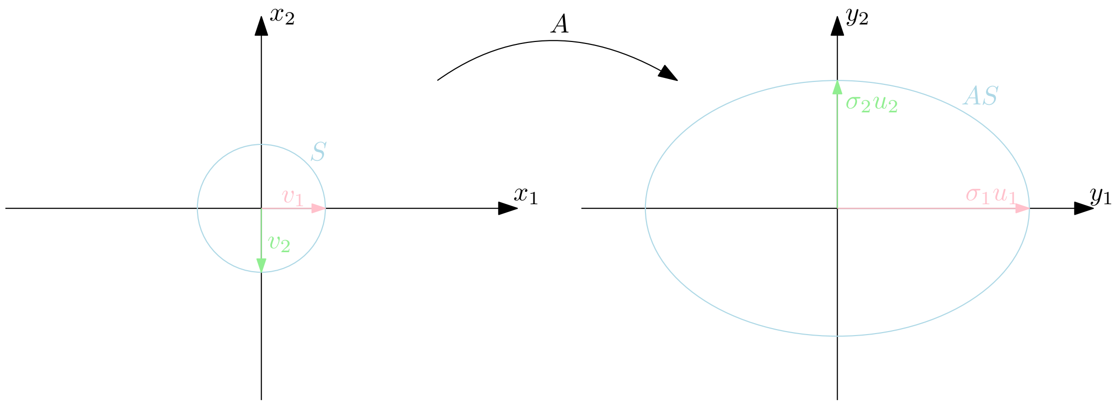

Figure 21: Geometrical interpretation of SVD.

(b) /

(c) A is of rank 1 such that only one non-vanishing singular value is expected.

Left singular eigenvectors

$$AA^* = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues are solutions of

$$\det AA^* - \lambda I = 0 \tag{7.1.2}$$

$$\Rightarrow \det \begin{pmatrix} 4-\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix} = \lambda^2(4-\lambda) = 0 \tag{7.1.3}$$

i.e $\lambda_1 = 4$, $\lambda_2 = \lambda_3 = 0$ such that $\sigma_1 = 2$.
The first eigenvector verifies

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} - 4 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ -4y \\ -4z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{7.1.4}$$

Thus

$$u_1 = \begin{pmatrix} \pm 1 \\ 0 \\ 0 \end{pmatrix}. \tag{7.1.5}$$

The $+$ sign is chosen.
To complete the basis, two orthonormal eigenvectors must be added. For simplicity, they are chosen as

$$u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{7.1.6}$$

Right singular vectors

$$v_1 = \frac{1}{\sigma_1} A^* u_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{7.1.7}$$

As previously, the basis is completed by a vector which is chosen as

$$v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{7.1.8}$$

Finally

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{7.1.9}$$

The geometrical interpretation of this singular value decomposition is given in Figure 22.

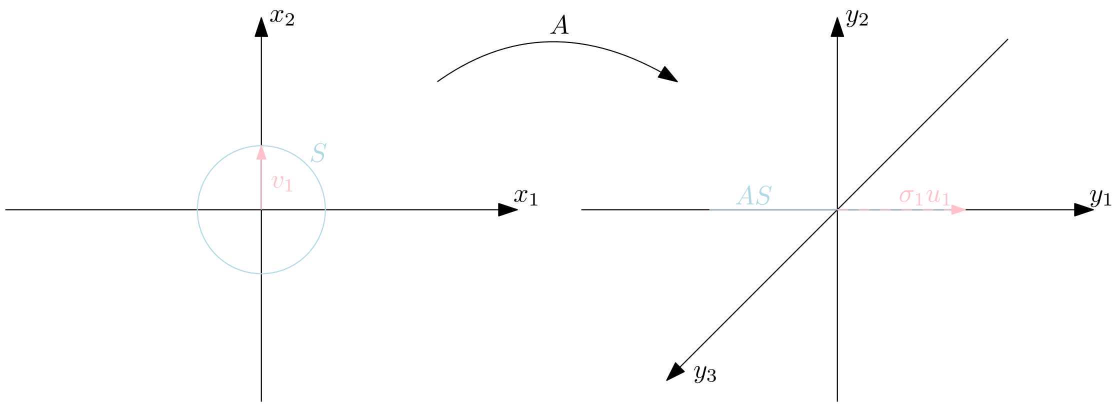

Figure 22: Geometrical interpretation of SVD.

7.2 Exercise 2: Conceptual questions on SVD 🌶️🌶️

Without any computations, think about the following questions. We strongly advise you to think about them before looking at the solutions.

(a) Consider a column vector w, i.e. a $n \times 1$ matrix. What would its SVD look like? In particular, what is/are the left and right singular vectors, as well as the singular values?

(b) Same question, but if w is a row vector.

(c) Let A be a diagonal matrix with complex entries $A_{jj} = r_j \, e^{i\theta_j}$ (with $i^2 = -1$ and $j = 1, .., n$). What does its SVD look like ? What is the best rank k approximation of A ?

Solution

(a) A column matrix has 1 right singular vector (of dimension 1) and n left singular vectors (of dimension n). The simplest choice of right eigenvector is simply $v_1 = (1)$. Then, the corresponding left singular vector is $\frac{u_1}{\|u_1\|} = \frac{wv_1}{\sigma_1}$, i.e. the vector w but normalized. The only non-zero singular value is thus $\|w\|$. The $n-1$ other left singular vectors are an arbitrary basis of the set of vectors orthogonal to w.

(b) Since A^* as the same singular values as A but with left and right singular vectors swapped, we can reuse the results above: the only non-zero singular value is $\|w\|$, there is a single left singular vector (1), the first right singular vector is w but normalized and the other ones are an orthogonal basis of the space of vectors orthogonal to w. It can be interpreted as follows: a row vector is a linear function from a vector to a scalar. The first right singular vector represents the only direction that does not get mapped to 0, and the non-zero singular value represents the (modulus of) output of that function when its argument is this unit right singular vector.
Note: Be careful to not forget complex conjugations where needed.

(c) Since the matrix is diagonal, the right singular vectors are the standard basis vectors (Vectors e_i with a single non-zero component of 1 at position i). However they have to be sorted according to the singular values: v_1 is the basis vector e_j where j maximises r_j, and so on. The singular values are naturally the moduli of the diagonal elements. Finally, the left singular vectors retain the phase information and are the same unit vectors (with a similar permutation), but multiplied by the relevant complex number.

For instance, let us consider the following matrix:

$$\begin{pmatrix} e^{i\alpha} & 0 & 0 \\ 0 & 3e^{i\beta} & 0 \\ 0 & 0 & 2e^{i\gamma}. \end{pmatrix} \tag{7.2.1}$$

Its SVD is given by:

$$U = \begin{pmatrix} 0 & 0 & e^{i\alpha} \\ e^{i\beta} & 0 & 0 \\ 0 & e^{i\gamma} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7.2.2}$$

A low-rank approximation is obtained by only keeping the k elements with the biggest moduli.

7.3 From Steepest descent to Conjugate Gradients

Let A be a real symmetric, positive definite matrix. One can show that the function

$$J(x) = \frac{1}{2}x^T Ax - b^T x \tag{7.3.1}$$

has gradient $\nabla J(x) = Ax - b$. Solving the linear system $Ax = b$ is then equivalent to finding the minimum[3] of $J(x)$. In this exercise, we will always denote by r_i the vector $b - Ax_i$. The error $e_i = x - x_i$ where x is the true solution is such that $Ae_i = r_i$.

[3]If A has positive eigenvalues, the extremum is a minimum.

(a) For a given search direction d_i and an initial guess x_i, we compute an updated guess $x_{i+1} = x_i + \alpha_i d_i$. Show that the value of α_i that minimizes $J(x_{i+1})$ is given by

$$\alpha_i = \frac{r_i{}^T d_i}{d_i{}^T A d_i}. \tag{7.3.2}$$

(b) The *Steepest Descent* method solves a linear system by repeatedly applying the method above, with direction $d_i = r_i$. For the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \tag{7.3.3}$$

and right-hand side

$$b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{7.3.4}$$

apply two iterations of the algorithm with a zero initial guess, *i.e.* compute x_1, x_2 if $x_0 = 0$. What can you say about the rate of convergence of the error ?
Hint: To compute the error, you can use the true solution $x = e_0 = (1,\ \frac{1}{3})^T$.

(c) To avoid the "zigzag" effect of the previous algorithm, we use conjugate directions, *i.e.* directions such that $d_i{}^T A d_j = 0$ when $i \neq j$. This is the key idea behind the *Conjugate Gradients* algorithm.
To make d_1 and d_0 conjugate, we use $d_1 = r_1 + \beta d_0$. (With $d_0 = r_0$ still holding). Show that the value of β_0 that gives a conjugate direction is

$$\frac{r_1{}^T r_1}{r_0{}^T r_0}. \tag{7.3.5}$$

Hint: Remember that $r_1 = r_0 - \alpha_0 A d_0$. Furthermore, the vectors r_0 and r_1 are orthogonal.

(d) Show that using this updated direction in the computation of x_2 of the problem defined in (b) yields the exact solution.

Solution

(a) To minimize $J(x_1 + \alpha_i d_i)$, we cancel the derivative with respect to α_i:

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} J(x_i + \alpha_i d_i) = \nabla J(x_i + \alpha_i d_i) \cdot d_i \tag{7.3.6}$$

$$= (A(x_i + \alpha_i d_i) - b)^T d_i \tag{7.3.7}$$

$$= (\alpha_i A d_i - r_i)^T d_i, \tag{7.3.8}$$

which is zero if α_i is defined as in 7.3.2. We can check that this is indeed a minimum, since $\frac{\mathrm{d}^2 J}{\mathrm{d}\alpha^2} = d_i{}^T A d_i > 0$ when the matrix is definite positive.

(b) The first residual is $r_0 = b = (1,\ 1)^T$. Computing α_0 with $d_0 = r_0$ yields

$$\alpha_0 = \frac{{r_0}^T r_0}{{r_0}^T A r_0} = \frac{2}{4}. \tag{7.3.9}$$

This gives an update $x_1 = (\frac{1}{2},\ \frac{1}{2})^T$, and a residual $r_1 = (\frac{1}{2},\ -\frac{1}{2})^T$. The error is $e_1 = (\frac{1}{2},\ -\frac{1}{6})^T$, since the correct solution is $x = e_0 = (1,\ \frac{1}{3})^T$.

For the second iteration, we use $d_1 = r_1$ and compute the new step length:

$$\alpha_1 = \frac{{r_1}^T r_1}{{r_1}^T A r_1} = \frac{0.5}{1} = \frac{1}{2}. \tag{7.3.10}$$

This gives $x_2 = (\frac{3}{4},\ \frac{1}{4})^T$, and a residual $r_2 = (\frac{1}{4},\ \frac{1}{4})^T$. The new error is $e_2 = (\frac{1}{4},\ \frac{1}{12})^T$.

After two iterations, the error is divided by 4, and has the same direction has before. Further iterations will yield the same corrections, but at a smaller scale. Overall, the rate of convergence is such that $\|e_n\| \sim \frac{\|e_0\|}{2^n}$. The error decays exponentially but never completely reaches zero.

(c) The value of β is chosen such that ${d_0}^T A d_1 = 0$. Plugging the definition of d_1, we get

$${d_0}^T A \left(r_1 + \beta d_0\right) = 0, \tag{7.3.11}$$

which holds if $\beta = \frac{-{d_0}^T A r_1}{{d_0}^T A d_0}$. This can be simplified to $\frac{{r_1}^T r_1}{{r_0}^T r_0}$, since $r_1 = r_0 - \alpha_0 A d_0$ allows us to eliminate Ad_0 (thus avoiding an additional multiplication) and r_1 is orthogonal to r_0.

Note that to derive the full Conjugate Gradients algorithm, additional care must be taken to ensure this formula can be used at all iterations. However, the main components (residuals orthogonal to the previous search direction, conjugate directions) were shown here. The full derivation can be achieved through induction.

(d) Using the formula derived above, we find $\beta = \frac{1}{4}$, and $d_1 = (\frac{1}{2},\ -\frac{1}{2})^T + \frac{1}{4}(1,\ 1)^T = (\frac{3}{4},\ \frac{-1}{4})^T$. The relevant step length α_1 is then $\frac{2}{3}$ (using equation 7.3.2), which yields $x_2 = (1,\ \frac{1}{3})^T$, the exact solution.