A-ϕ formulation of a mathematical model for the induction hardening process with a nonlinear law for the magnetic field

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Abstract

We derive and analyse a mathematical model for induction hardening. We assume a nonlinear relation between the magnetic field and the magnetic induction field. For the electromagnetic part, we use the vector-scalar potential formulation.

The coupling between the electromagnetic and the thermal part is provided through the temperature-dependent electric conductivity and the joule heating term, the most crucial element, considering the mathematical analysis of the model. It acts as a source of heat in the thermal part and leads to the increase in temperature. Therefore, in order to be able to control it, we apply a truncation function.

Using Rothe's method, we prove the existence of a global solution to the whole system. The nonlinearity in the electromagnetic part is handled by the theory of monotone operators.

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1. Introduction

There are many papers dealing with mathematical models of the induction hardening process. Some of them provide various numerical schemes e.g. [1, 2, 3, 4, 5, 6]. But they omit mathematical or numerical analysis of their models and numerical schemes. Other papers deal with the well-posedness of the problem and provide theoretical results e.g. [7, 8, 9, 10, 11]. The topic of induction hardening has been broadly covered in papers [12, 13] and [14]. However, all manuscripts tackling the theoretical side of the induction hardening phenomena present mathematical models with linear dependency between magnetic and magnetic induction field. The paper [15] studied a mathematical model with a nonlinear relation between those two vectorial fields (which better reflects reality), but the study was restricted just to a conductor. The authors proved solvability for a formulation with magnetic induction field as an unknown. We present the vector-scalar potential formulation for a nonlinear setting including conducting and non-conducting parts. This means that material coefficients may have jumps across the interfaces. To our best knowledge nothing similar has been done before.

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1.1. Derivation of a mathematical model

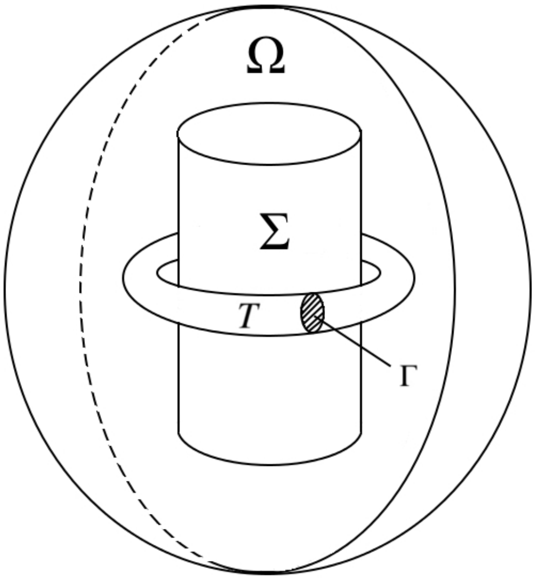


Figure 1: Illustration of the domain

We work only with a simplified model of induction hardening process (see Figure 1). The time frame is denoted by $[0, \mathcal{T}]$. Let Ω be a bounded sphere in $\mathbb{R}^3$. The workpiece and the coil are represented by Σ and T, respectively. Both Σ and T are closed subsets of Ω and the following holds

$$\Sigma \cap T = \varnothing, \quad \text{and} \quad \partial\Sigma, \partial T, \partial\Omega \quad \text{are of class } C^{1,1}. \tag{1}$$

Conductors are affected by temperature, hence we separate them from the rest of the domain Ω by denoting $\boldsymbol{\pi} = \Sigma \cup T$. Current in the coil is modeled via an interface condition on Γ. By $\boldsymbol{\nu}$ we denote the standard outer normal unit vector associated with surfaces of materials under consideration.

We start deriving our mathematical model with introducing the classical Maxwell equations (for reference, see [16])

$$\nabla \cdot \boldsymbol{D} = \rho, \tag{2}$$
$$\nabla \cdot \boldsymbol{B} = 0, \tag{3}$$
$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}, \tag{4}$$
$$\nabla \times \boldsymbol{H} = \partial_t \boldsymbol{D} + \boldsymbol{J}. \tag{5}$$

Here, $\boldsymbol{D}$ stands for displacement current and ρ is the density of electrical charge. The magnetic induction field, the electrical field and the magnetic field are denoted with $\boldsymbol{B}$, $\boldsymbol{E}$ and $\boldsymbol{H}$, respectively. At last, $\boldsymbol{J}$ indicates the source current. For the clarity, we note that equations above are true in the whole domain Ω.

In models dealing with eddy currents, the time variation of displacement current is insignificant, therefore we can neglect it. We present the nonlinear relation between $\boldsymbol{H}$ and $\boldsymbol{B}$ in the following form:

$$\boldsymbol{H} := \frac{1}{\mu^*}\boldsymbol{M}(\boldsymbol{B}) = \frac{1}{\mu^*} m(|\boldsymbol{B}|)\boldsymbol{B} = \mu \boldsymbol{M}(\boldsymbol{B}). \tag{6}$$

Magnetic permeability $\mu = \frac{1}{\mu^*}$ might behave differently in the workpiece and in the air, therefore, we specify it as a split function

$$\mu(\boldsymbol{x}) = \begin{cases} \mu_\pi(\boldsymbol{x}), & \text{if } \boldsymbol{x} \in \bar{\pi}, \\ \mu_A(\boldsymbol{x}), & \text{if } \boldsymbol{x} \in \Omega \backslash \bar{\pi}. \end{cases} \tag{7}$$

Both μ_π and μ_A are strictly positive and bounded. There is no jump in the tangential component of $\boldsymbol{H}$ along the boundaries between different materials, i.e.

$$[\mu \boldsymbol{M}(\nabla \times \boldsymbol{A}) \times \boldsymbol{\nu}]_{\partial\pi} = \boldsymbol{0}.$$

The vectorial field $\boldsymbol{M}$ is supposed to be potential i.e. *grad* $\Phi_{\boldsymbol{M}} = \boldsymbol{M}$, cf. [17] . Its potential is denoted by $\Phi_{\boldsymbol{M}}$. Moreover, we assume that $\boldsymbol{M}$ is strictly monotone and Lipschitz continuous. Furthermore, we introduce Ohm's law

$$\boldsymbol{J} = \sigma \boldsymbol{E}. \tag{8}$$

Function σ represents the electric conductivity and it is defined as follows

$$\sigma(u(\boldsymbol{x}, t)) = \begin{cases} \sigma_\pi(u(\boldsymbol{x}, t)), & \text{if } \boldsymbol{x} \in \overline{\boldsymbol{\pi}}, t \in [0, \mathcal{T}], \\ 0, & \text{if } \boldsymbol{x} \in \Omega \backslash \overline{\boldsymbol{\pi}}, t \in [0, \mathcal{T}], \end{cases} \tag{9}$$

where $u(\boldsymbol{x}, t)$ is a function of temperature in the workpiece and the coil. We consider σ to be continuous, bounded and strictly positive in $\overline{\boldsymbol{\pi}}$. Since Ω is a simply-connected domain and (3) is true in the whole Ω, we can use ([18, Theorem 3.6]) to obtain exactly one magnetic vector potential $\boldsymbol{A} \in \mathbf{H}(\mathbf{curl}\,; \Omega)$ with the following properties:

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \qquad \nabla \cdot \boldsymbol{A} = 0, \qquad \boldsymbol{A} \times \boldsymbol{\nu} = \boldsymbol{0} \ \text{ on } \ \partial\Omega. \tag{10}$$

Substituting (10) into (4) we get

$$\nabla \times (\boldsymbol{E} + \partial_t \boldsymbol{A}) = \boldsymbol{0} \ \text{ in } \Omega. \tag{11}$$

Using (11), we can apply ([18, Theorem 2.9]) to acquire a unique scalar potential $\phi \in H^1(\Omega)/\mathbb{R}$ such that:

$$\boldsymbol{E} + \partial_t \boldsymbol{A} = -\nabla\phi. \tag{12}$$

Combining (12),(10),(8),(6) and (5), we arrive at the following boundary value problem for vector potential $\boldsymbol{A}$:

$$\begin{aligned} &\sigma \partial_t \boldsymbol{A} + \nabla \times \mu \boldsymbol{M}(\nabla \times \boldsymbol{A}) + \sigma \chi_T \nabla\phi = 0 && \text{for a.e. } (\boldsymbol{x}, t) \in \Omega \times (0, \mathcal{T}) := Q_{\mathcal{T}}, \\ &\boldsymbol{A} \times \boldsymbol{\nu} = \boldsymbol{0} && \text{for a.e. } (\boldsymbol{x}, t) \in \partial\Omega \times (0, \mathcal{T}), \\ &\boldsymbol{A}(0) = \boldsymbol{A}_0 && \text{for } \boldsymbol{x} \in \Omega, \ t = 0. \end{aligned} \tag{13}$$

Characteristic function χ_T has value 1, if $\boldsymbol{x} \in T$ and 0 otherwise. We use it, because the external source of the current, which is defined by the gradient of the scalar potential, is present only in the coil (T, see Figure 1).

Scalar potential ϕ is determined by the following elliptic equation with homogenous Neumann boundary condition on ∂T and interface condition on Γ:

$$\begin{aligned} &-\nabla \cdot (\sigma_\pi \nabla\phi) = 0 && \text{for a.e. } (\boldsymbol{x}, t) \in T \times (0, \mathcal{T}), \\ &-\sigma_\pi \frac{\partial\phi}{\partial\boldsymbol{\nu}} = 0 && \text{for a.e. } (\boldsymbol{x}, t) \in \partial T \times (0, \mathcal{T}), \\ &\left[-\sigma_\pi \frac{\partial\phi}{\partial\boldsymbol{\nu}}\right]_\Gamma = j && \text{for a.e. } (\boldsymbol{x}, t) \in \Gamma \times (0, \mathcal{T}). \end{aligned} \tag{14}$$

External source current density is represented by function $j(\boldsymbol{x}, t)$, which is assumed to be Lipschitz continuous in time. Jump across interface Γ is indicated by $[\cdot]_\Gamma$.

Eddy currents generated in the workpiece raise temperature by a significant amount. This phenomenon is called

Joule heat and it is expressed as

$$\boldsymbol{J} \cdot \boldsymbol{E} \overset{(8)}{=} \sigma_\pi \left|\boldsymbol{E}\right|^2 \overset{(12)}{=} \sigma_\pi \left|\partial_t \boldsymbol{A} + \chi_T \nabla\phi\right|^2 . \tag{15}$$

This term is crucial and causes numerous troubles during mathematical treatment (un-boundedness), therefore, we introduce a cut-off function and work with truncated Joule-heating term.

$$\mathcal{R}_r(x) := \begin{cases} r > 0 & \text{if } x > r, \\ x & \text{if } |x| \leqslant r, \\ -r & \text{if } x < -r. \end{cases} \tag{16}$$

Evolution of temperature in the workpiece and the coil (π, see Figure 1) is characterized by the following parabolic nonlinear equation with the homogenous Neumann boundary condition:

$$\begin{aligned} &\partial_t\beta(u) - \nabla\cdot(\lambda\nabla u) = \mathcal{R}_r\left(\sigma_\pi \left|\partial_t \boldsymbol{A} + \chi_T \nabla\phi\right|^2\right) && \text{for a.e. } (\boldsymbol{x}, t) \in \boldsymbol{\pi} \times (0, \mathcal{T}), \\ &-\lambda \tfrac{\partial u}{\partial \boldsymbol{\nu}} = 0 && \text{for a.e. } (\boldsymbol{x}, t) \in \partial\boldsymbol{\pi} \times (0, \mathcal{T}), \\ &u(0) = u_0 && \text{for } \boldsymbol{x} \in \boldsymbol{\pi},\ t = 0. \end{aligned} \tag{17}$$

Continuous function $\lambda(\boldsymbol{x}, t)$ is supposed to be strictly positive and bounded. The nonlinear function β is of a linear growth and its derivative is bounded from below by a positive constant.

Equations (13),(14) and (17) model the process of induction hardening in our simplified domain Ω. They are tied together through terms $\nabla\phi$, σ and $\partial_t \boldsymbol{A}$. One could ask, whether the artificial intervention in the form of cut-off function was correct. In real applications of induction hardening, there is always a switch-off button, which is used to prevent the workpiece from thermal deformations. When the temperature reaches a certain degree, this button is turned-off, the stream of electric current is stopped and the workpiece is cooled down. Therefore, applying the cut-off function on Joule-heating term in (17), is actually a simulation of this switch-off button and indeed, necessary to be done.

2. Functional setting

2.1. Variational formulation

Let us start with some basic notations. Through the whole paper we adopt notation $(\cdot,\cdot)_\Omega$ for the standard inner product in $\mathrm{L}^2(\Omega)$ or $\mathbf{L}^2(\Omega)$. Norm induced by this inner product is indicated as $\|\cdot\|_{\mathrm{L}^2(\Omega)}$. Set of abstract functions $k : [0, \mathcal{T}] \to Y$ equipped with the norm $\max\limits_{t\in[0,\mathcal{T}]} \|\cdot\|_Y$ is denoted as $C([0,\mathcal{T}]; Y)$. In a case when $p > 1$, norm in $L^p((0,\mathcal{T}); Y)$ is defined as $\left(\int_0^{\mathcal{T}} \|\cdot\|_Y^p \,\mathrm{d}t\right)^{\frac{1}{p}}$. Set of all $\phi + c$, where $\phi \in H^1(T)$ and c is a constant is marked as ϕ_c.

Considering the vector potential $\boldsymbol{A}$, we introduce the Hilbert space

$$\boldsymbol{X}_{N,0} = \{\boldsymbol{\varphi} \in \mathbf{H}(\mathbf{curl}\,;\Omega);\ \nabla\cdot\boldsymbol{\varphi} = 0, \text{ and } \boldsymbol{\varphi}\times\boldsymbol{\nu} = 0 \text{ on } \partial\Omega\},$$

where $\mathbf{H}(\mathbf{curl}\,;\Omega) = \{\boldsymbol{\varphi} \in \mathbf{L}^2(\Omega) : \nabla\times\boldsymbol{\varphi} \in \mathbf{L}^2(\Omega)\}$. Using Friedrichs' inequality for vectorial fields (cf. [18, Lemma 3.4] or [19, Cor. 3.51]) we see that we may furnish $\boldsymbol{X}_{N,0}$ with norm $\|\boldsymbol{\varphi}\|_{\boldsymbol{X}_{N,0}} := \|\nabla\times\boldsymbol{\varphi}\|_{\mathbf{L}^2(\Omega)}$. Taking into account (1), we can use [18, Theorem 3.7] or [20, Theorem 2.12] to conclude that $\boldsymbol{X}_{N,0}$ is a closed subspace of $\mathbf{H}^1(\Omega)$.[1] Multiplying (13) by a test function $\boldsymbol{\varphi} \in \boldsymbol{X}_{N,0}$, integrating over Ω and using Green's theorem, we obtain the variational formulation for vector potential $\boldsymbol{A}$:

$$(\sigma_\pi \partial_t \boldsymbol{A}, \boldsymbol{\varphi})_\pi + (\mu \boldsymbol{M}(\nabla\times\boldsymbol{A}), \nabla\times\boldsymbol{\varphi})_\Omega + (\sigma_\pi \nabla\phi, \boldsymbol{\varphi})_T = 0 \quad \forall \boldsymbol{\varphi} \in \boldsymbol{X}_{N,0}. \tag{18}$$

[1] The relation $\boldsymbol{X}_{N,0} \subset \mathbf{H}^1(\Omega)$ is crucial for our mathematical approach. We would like to point out that the same inclusion is valid also for convex domains (with non smooth boundary). In such a case one can rely on the [20, Theorem 2.17]. All presented results hold true also for convex domains.

For equation (17) we follow identical steps as above, using $\psi \in H^1(\pi)$ as a test function, which brings us to the variational formulation for function u:

$$(\partial_t \beta(u), \psi)_\pi + (\lambda \nabla u, \nabla \psi)_\pi = \left(\mathcal{R}_r \left(\sigma_\pi \left|\partial_t \boldsymbol{A} + \chi_T \nabla \phi\right|^2\right), \psi\right)_\pi \qquad \forall \psi \in H^1(\boldsymbol{\pi}). \tag{19}$$

Norm in $H^1(\pi)$ is defined as $\|\psi\|^2_{H^1(\pi)} := \|\psi\|^2_{L^2(\pi)} + \|\nabla\psi\|^2_{\mathbf{L}^2(\pi)}$.

To obtain the variational formulation for (14), we split T in two separate parts T_1 and T_2. Flux of the scalar potential on the new interface Γ_* is supposed to be continuous. Moreover, $\Gamma_* \cap \Gamma = \emptyset$ and $T_1 \cap T_2 = \Gamma_* \cup \Gamma$ (see Figure 2). Now, we can multiply (14) by a test function $\xi \in H^1(T)/\mathbb{R}$ and integrate in T_1 and T_2. Using Green's

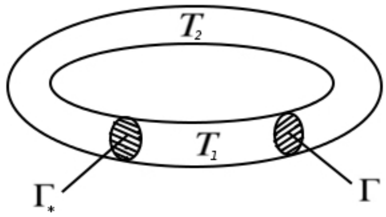


Figure 2: Dissection of T

theorem, boundary condition (14) and continuous condition on Γ_*, we arrive to the following variational formulation for scalar potential ϕ:

$$(\sigma_\pi \nabla\phi, \nabla\xi)_T + (j, \xi)_\Gamma = 0 \qquad \forall \xi \in H^1(T)/\mathbb{R}. \tag{20}$$

The choice of the test space $H^1(T)/\mathbb{R}$ is just to obtain a unique solvability.

Lemma 1. *There are positive constants c_1 and c_2 such that:*

$$c_1 \|\phi_c\|^2_{H^1(T)/\mathbb{R}} \leqslant \|\nabla\phi\|^2_{\mathbf{L}^2(T)} \leqslant c_2 \|\phi_c\|^2_{H^1(T)/\mathbb{R}}.$$

Proof. Norm in $H^1(T)/\mathbb{R}$ is defined as $\|\phi_c\|_{H^1(T)/\mathbb{R}} := \inf_{\phi \in \phi_c} \|\phi\|_{H^1(T)}$. This norm is minimal for $c = -\frac{1}{|T|}\int_T \phi \, \mathrm{d}x$, indeed, let us take a closer look.

$$0 = \frac{d}{dc}\int_T (\phi + c)^2 + |\nabla\phi|^2 \, \mathrm{d}x = 2\int_T \phi \, \mathrm{d}x + 2\int_T c \, \mathrm{d}x \Longrightarrow c = -\frac{1}{|T|}\int_T \phi \, \mathrm{d}x.$$

Now, we can write $\|\phi_c\|_{H^1(T)/\mathbb{R}} = \left\|\phi - \frac{1}{|T|}\int_T \phi \, \mathrm{d}x\right\|_{H^1(T)}$. Using Poincare-Wirtinger inequality ([21]) we conclude the following:

$$\|\phi_c\|_{H^1(T)/\mathbb{R}} = \left\|\phi - \frac{1}{|T|}\int_T \phi \, \mathrm{d}x\right\|_{L^2(T)} + \|\nabla\phi\|_{\mathbf{L}^2(T)} \leqslant c_{PW} \|\nabla\phi\|_{\mathbf{L}^2(T)} + \|\nabla\phi\|_{\mathbf{L}^2(T)} = (c_{PW} + 1) \|\nabla\phi\|_{\mathbf{L}^2(T)},$$

where c_{PW} is a positive constant. Taking $c_2 = 1$ and $c_1 = \frac{1}{1+c_{PW}}$, the proof is completed. □

2.2. Assumptions

To achieve better clarity and readability of our paper, we list all assumptions altogether:

$$
\begin{array}{lll}
(a1) & 0 < \mu_{\pi *} \leqslant \mu_\pi(\boldsymbol{x}) \leqslant {\mu_\pi}^* < \infty & \forall \boldsymbol{x} \in \overline{\Sigma}, \\
(a2) & 0 < \mu_{A*} \leqslant \mu_A(\boldsymbol{x}) \leqslant \mu_A^* < \infty & \forall \boldsymbol{x} \in \Omega \backslash \overline{\Sigma}, \\
(b) & \mu_* = \min\{\mu_{\pi *}, \mu_{A*}\},\ \mu^* = \max\{{\mu_\pi}^*, \mu_A^*\} & \\
(c1) & \mu \in H^1(\boldsymbol{\pi}) & \\
(c2) & \mu \in H^1(\Omega \backslash \boldsymbol{\pi}) & \\
(d) & 0 < \sigma_* \leqslant \sigma(u(\boldsymbol{x},t)) \leqslant \sigma^* < \infty & \forall (\boldsymbol{x},t) \in \boldsymbol{\pi} \times (0,\mathcal{T}), \\
(e1) & 0 < \lambda_* \leqslant \lambda(\boldsymbol{x},t) \leqslant \lambda^* < \infty & \forall (\boldsymbol{x},t) \in \boldsymbol{\pi} \times (0,\mathcal{T}), \\
(e2) & |\lambda(\boldsymbol{x},t_2) - \lambda(\boldsymbol{x},t_1)| \leqslant C_\lambda |t_2 - t_1| & C_\lambda > 0, \forall \boldsymbol{x} \in \boldsymbol{\pi}, \forall t_2, t_1 \in [0,\mathcal{T}] \\
(f) & |j(\boldsymbol{x},t_2) - j(\boldsymbol{x},t_1)| \leqslant C_j |t_2 - t_1| & C_j > 0, \forall \boldsymbol{x} \in \Gamma, \forall t_2, t_1 \in [0,\mathcal{T}], \\
(g) & j \in L^2((0,\mathcal{T}); H^{-1/2}(\Gamma)), \quad \int_\Gamma j \,\mathrm{d}\Gamma = 0, & \\
(h) & u_0 \in H_0^1(\boldsymbol{\pi}), & \\
(i) & \boldsymbol{A}_0 \in \boldsymbol{X}_{N,0}, & \\
(j) & \beta \text{ is continuous}, \beta(0) = 0, & \\
 & |\beta(x)| \leqslant C_\beta(1 + |x|), \quad 0 < \beta_* \leqslant \beta'(x) & C_\beta > 0, \forall x \in \mathbb{R}, \\
(k_1) & (\boldsymbol{M}(\boldsymbol{x}) - \boldsymbol{M}(\boldsymbol{y})) \cdot (\boldsymbol{x} - \boldsymbol{y}) \geqslant c_M |\boldsymbol{x} - \boldsymbol{y}|^2 & c_M > 0, \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3, \\
(k_2) & |\boldsymbol{M}(\boldsymbol{x}) - \boldsymbol{M}(\boldsymbol{y})| \leqslant C_M |\boldsymbol{x} - \boldsymbol{y}| & C_M > 0, \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3, \\
(k_3) & \boldsymbol{M}(\boldsymbol{0}) = \boldsymbol{0}. &
\end{array}
\tag{21}
$$

Following [17, Theorem 5.1], we can see that potential $\Phi_{\boldsymbol{M}}$ of vectorial field $\boldsymbol{M}$ with properties $(k_1) - (k_3)$, is strictly convex. Applying [17, Theorem 8.4], we get

$$
\boldsymbol{M}(\boldsymbol{x}) \cdot (\boldsymbol{x} - \boldsymbol{y}) \geqslant \Phi_{\boldsymbol{M}}(\boldsymbol{x}) - \Phi_{\boldsymbol{M}}(\boldsymbol{y}) \qquad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3. \tag{22}
$$

Thanks to (k_1) and (k_2), we can bound $\Phi_{\boldsymbol{M}}$ from below

$$
\Phi_{\boldsymbol{M}}(\boldsymbol{x}) = \int_0^1 \boldsymbol{M}(\boldsymbol{x}p) \cdot \boldsymbol{x} \,\mathrm{d}p = \int_0^1 \boldsymbol{M}(\boldsymbol{x}p) \cdot (\boldsymbol{x}p) p^{-1} \,\mathrm{d}p \geqslant \int_0^1 c_M |\boldsymbol{x}p|^2 p^{-1} \,\mathrm{d}p = \frac{c_M}{2} |\boldsymbol{x}|^2 \tag{23}
$$

and from above

$$
\Phi_{\boldsymbol{M}}(\boldsymbol{x}) = \int_0^1 \boldsymbol{M}(\boldsymbol{x}p) \cdot \boldsymbol{x} \,\mathrm{d}p \leqslant \int_0^1 |\boldsymbol{M}(\boldsymbol{x}p)| \, |\boldsymbol{x}| \,\mathrm{d}p \leqslant C_M \int_0^1 |\boldsymbol{x}p| \, |\boldsymbol{x}| \,\mathrm{d}p = \frac{C_M}{2} |\boldsymbol{x}|^2 . \tag{24}
$$

3. Existence of a solution

3.1. Time discretization scheme and a priori estimates

In this section we discretize the time interval $[0,\mathcal{T}]$ and solve a system of steady-state differential equations on each time step. Afterwards, we construct piece-wise constant and piece-wise linear in time functions and show convergence of sub-sequences of these functions in appropriate functional spaces to the weak solution. This approach is called Rothe's method ([22, 23]). Consider a time step τ. We split the time interval in n equidistant parts i.e. $n\tau = \mathcal{T}$, where $n \in \mathbb{N}$. Denoting $t_i = i\tau$ we can write the following for any function f:

$$
f_i = f(t_i), \qquad \delta f_i = \frac{f_i - f_{i-1}}{\tau}.
$$

Applying this method to the system (18), (19), (20), we are able to approximate it on every time step t_i, for $i = 1 \dots n$

$$(\delta\beta(u_i), \psi)_\pi + (\lambda_i \nabla u_i, \psi)_\pi = \left(\mathcal{R}_r\left(\sigma_\pi(u_{i-1})\left|\delta \boldsymbol{A}_i + \chi_T \nabla\phi_i\right|^2\right), \psi\right)_\pi \qquad \text{for any } \psi \in H^1(\boldsymbol{\pi}), \tag{25}$$

$$(\sigma_\pi(u_{i-1})\delta \boldsymbol{A}_i, \boldsymbol{\varphi})_\pi + (\mu \boldsymbol{M}(\nabla \times \boldsymbol{A}_i), \nabla \times \boldsymbol{\varphi})_\Omega + (\sigma_\pi(u_{i-1})\nabla\phi_i, \boldsymbol{\varphi})_T = 0 \qquad \text{for any } \boldsymbol{\varphi} \in \boldsymbol{X}_{N,0}, \tag{26}$$

$$(\sigma_\pi(u_{i-1})\nabla\phi_i, \nabla\xi)_T + (j_i, \xi)_\Gamma = 0 \qquad \text{for any } \xi \in H^1(T)/\mathbb{R}. \tag{27}$$

To prove the solvability on each time step, we use the theory of monotone operators (for more details, see [17, 24]).

Lemma 2. *Assume that* (21) *holds. Then, for any $i = 1 \dots n$, there exists a uniquely determined triplet $\phi_{c_i} \in H^1(T)/\mathbb{R}$, $\boldsymbol{A}_i \in \boldsymbol{X}_{N,0}$ and $u_i \in H^1(\boldsymbol{\pi})$ solving system* (25)-(27).

Proof. Let us define operators: $\mathcal{F}_\sigma : \boldsymbol{X}_{N,0} \to (\boldsymbol{X}_{N,0})^*$ and $\mathcal{G}_\lambda : H^1(\boldsymbol{\pi}) \to (H^1(\boldsymbol{\pi}))^*$

$$\langle \mathcal{F}_\sigma(\boldsymbol{A}), \boldsymbol{\varphi}\rangle := \left(\sigma\frac{\boldsymbol{A}}{\tau}, \boldsymbol{\varphi}\right)_\pi + (\mu \boldsymbol{M}(\nabla \times \boldsymbol{A}), \nabla \times \boldsymbol{\varphi})_\Omega,$$

$$\langle \mathcal{G}_\lambda(u), \psi\rangle := \left(\frac{\beta(u)}{\tau}, \psi\right)_\pi + (\lambda\nabla u, \nabla\psi)_\pi.$$

Assuming that τ is small enough, i.e. $0 < \tau < 1$, these operators are strictly monotone, coercive and demicontinuous. Rest of the proof serves as a guideline for obtaining a solution-triplet on every time step $t = t_i$, for $i = 1 \dots n$. Applying Lax-Milgram lemma (see [19, Lemma 2.21]) to (27), we obtain a unique solution $\phi_{c_i} \in H^1(\boldsymbol{\pi})/\mathbb{R}$ on a time step $t = t_i$ (u_{i-1} is known on this time step).

To obtain a unique solution $\boldsymbol{A}_i$ at a time step t_i, we have to solve the following identity:

$$\left\langle \mathcal{F}_{\sigma_\pi(u_{i-1})}(\boldsymbol{A}_i), \boldsymbol{\varphi}\right\rangle = \left(\sigma_\pi(u_{i-1})\frac{\boldsymbol{A}_{i-1}}{\tau}, \boldsymbol{\varphi}\right)_\pi - (\sigma_\pi(u_{i-1})\nabla\phi_i, \boldsymbol{\varphi})_T.$$

Since the right-hand side (RHS) is known, we can use [17, Theorem 18.2] to provide the solution. Now, we can involve the same theorem again to acquire a unique solution $u_i \in H^1(\boldsymbol{\pi})$ of the setting below (taking into account that the RHS is known)

$$\langle \mathcal{G}_{\lambda_i}(u_i), \psi\rangle = \left(\frac{\beta(u_{i-1})}{\tau}, \psi\right)_\pi + \left(\mathcal{R}_r\left(\sigma_\pi(u_{i-1})\left|\delta \boldsymbol{A}_i + \chi_T\nabla\phi_i\right|^2\right), \psi\right)_\pi$$

This provides us with the solution-triplet $\{\phi_{c_i}, \boldsymbol{A}_i, u_i\}$ on a time step $t = t_i$, for $i = 1 \dots n$. □

To wrap everything together we state a pseudo-scheme for obtaining the solution-triplet $\{\phi_{c_i}, \boldsymbol{A}_i, u_i\}$ for every time step $t = t_i$:

1. Let i be given and assume that u_{i-1}, j_i and λ_i are known
2. Solve: $(\sigma_\pi(u_{i-1})\nabla\phi_i, \nabla\xi)_T + (j_i, \xi)_\Gamma = 0$
3. Solve: $\left(\sigma_\pi(u_{i-1})\frac{\boldsymbol{A}_i}{\tau}, \boldsymbol{\varphi}\right)_\pi + (\mu \boldsymbol{M}(\nabla \times \boldsymbol{A}_i), \nabla \times \boldsymbol{\varphi})_\Omega = \left(\sigma_\pi(u_{i-1})\frac{\boldsymbol{A}_{i-1}}{\tau}, \boldsymbol{\varphi}\right)_\pi - (\sigma_\pi(u_{i-1})\nabla\phi_i, \boldsymbol{\varphi})_T$ (28)
4. Solve: $\left(\frac{\beta(u_i)}{\tau}, \psi\right)_\pi + (\lambda_i\nabla u_i, \nabla\psi)_\pi = \left(\frac{\beta(u_{i-1})}{\tau}, \psi\right)_\pi + \left(\mathcal{R}_r\left(\sigma_\pi(u_{i-1})\left|\delta \boldsymbol{A}_i + \chi_T\nabla\phi_i\right|^2\right), \psi\right)_\pi$
5. Set $i = i + 1$ and repeat the process.

At this point it is necessary to make a small remark. In system (25)-(27), we use u_{i-1} as an argument for function σ. The reason to take this action is, to be able to decouple the whole system. As we will see in the sequel, this small adjustment does not affect convergence results. Before we proceed to the main theorem, we have to derive some basic energy estimates for ϕ_{c_i}, $\boldsymbol{A}_i$ and u_i. They are covered by the following lemmas.

Lemma 3. *Suppose* (21). *Then there exists a positive constant C such that*

$$\sum_{i=1}^{n} \|\nabla\phi_i\|^2_{\mathbf{L}^2(T)}\, \tau \leqslant C.$$

Proof. Take $\xi = \phi_{c_i}\tau$ in (27) and sum it up for $i = 1, \ldots, l \leqslant n$ to get

$$\sum_{i=1}^{l} (\sigma_\pi(u_{i-1})\nabla\phi_i, \nabla\phi_i)_T\, \tau = -\sum_{i=1}^{l} (j_i, \phi_{c_i})_\Gamma\, \tau.$$

We can bound the left-hand side (LHS) from below

$$\sigma_* \sum_{i=1}^{l} \|\nabla\phi_i\|^2_{\mathbf{L}^2(T)}\, \tau \leqslant \sum_{i=1}^{l} (\sigma_\pi(u_{i-1})\nabla\phi_i, \nabla\phi_i)_T\, \tau.$$

Using Cauchy-Schwarz's and Young's inequalities, we can bound the RHS

$$\sum_{i=1}^{l} (j_i, \phi_{c_i})_\Gamma\, \tau \leqslant \frac{1}{2\varepsilon}\sum_{i=1}^{l} \|j_i\|^2_{H^{-1/2}(\Gamma)}\, \tau + \frac{\varepsilon}{2}\sum_{i=1}^{l} \|\phi_{c_i}\|^2_{H^{1/2}(\Gamma)}\, \tau \leqslant C_\varepsilon + \varepsilon \sum_{i=1}^{l} \|\phi_{c_i}\|^2_{H^{1/2}(\Gamma)}\, \tau,$$

where $\varepsilon > 0$. Since $H^1(T)/\mathbb{R} \subset H^{1/2}(\Gamma)$ we can use Lemma 1 to write

$$\sum_{i=1}^{l} \|\phi_{c_i}\|^2_{H^{1/2}(\Gamma)}\, \tau \leqslant C \sum_{i=1}^{l} \|\nabla\phi_i\|^2_{\mathbf{L}^2(T)}\, \tau.$$

Now fixing a sufficiently small ε we conclude the proof. □

Lemma 4. *Assume* (21). *Then there exists a positive constant C such that*

(i) $$\sum_{i=1}^{n} \|\delta \boldsymbol{A}_i\|^2_{\mathbf{L}^2(\pi)}\, \tau + \max_{1\leqslant l\leqslant n} \|\nabla \times \boldsymbol{A}_l\|^2_{\mathbf{L}^2(\Omega)} \leqslant C$$

(ii) $$\sum_{i=1}^{n} \|\nabla \times (\mu \boldsymbol{M}(\nabla \times \boldsymbol{A}_i))\|^2_{\mathbf{L}^2(\pi)}\, \tau \leqslant C.$$

Proof. (i) Taking $\boldsymbol{\varphi} = \delta \boldsymbol{A}_i\tau$ in (26) and summing up for $i = 1, \ldots, l \leqslant n$ yields

$$\sum_{i=1}^{l} (\sigma_\pi(u_{i-1})\delta\boldsymbol{A}_i, \delta\boldsymbol{A}_i)_\pi\, \tau + \sum_{i=1}^{l} (\mu\boldsymbol{M}(\nabla\times\boldsymbol{A}_i), \nabla\times\boldsymbol{A}_i - \nabla\times\boldsymbol{A}_{i-1})_\Omega = -\sum_{i=1}^{l} (\sigma_\pi(u_{i-1})\nabla\phi_i, \delta\boldsymbol{A}_i)_T\, \tau.$$

Using Lemma 3, Cauchy-Schwarz's and Young's inequalities, we can bound the first term on the LHS and the term on the RHS as follows

$$\begin{aligned} \sigma_* \sum_{i=1}^{l} \|\delta\boldsymbol{A}_i\|^2_{\mathbf{L}^2(\pi)}\, \tau &\leqslant \sum_{i=1}^{l} (\sigma_\pi(u_{i-1})\delta\boldsymbol{A}_i, \delta\boldsymbol{A}_i)_\pi\, \tau, \\ -\sum_{i=1}^{l} (\sigma_\pi(u_{i-1})\nabla\phi_i, \delta\boldsymbol{A}_i)_T\, \tau &\leqslant \frac{\sigma^*}{2\varepsilon}\sum_{i=1}^{l} \|\nabla\phi_i\|^2_{\mathbf{L}^2(T)}\, \tau + \frac{\varepsilon\sigma^* C_\pi}{2}\sum_{i=1}^{l} \|\delta\boldsymbol{A}_i\|^2_{\mathbf{L}^2(\pi)}\, \tau \\ &\leqslant C\frac{\sigma^*}{2\varepsilon} + \frac{\varepsilon\sigma^* C_\pi}{2}\sum_{i=1}^{l} \|\delta\boldsymbol{A}_i\|^2_{\mathbf{L}^2(\pi)}\, \tau. \end{aligned}$$

To estimate the second term on the LHS, we take into account (23) and (24)

$$\sum_{i=1}^{l}\int_{\Omega}\mu\left\{\boldsymbol{M}(\nabla\times\boldsymbol{A}_i)\cdot(\nabla\times\boldsymbol{A}_i-\nabla\times\boldsymbol{A}_{i-1})\right\}\,\mathrm{d}x \geqslant \sum_{i=1}^{l}\int_{\Omega}\mu(\Phi_{\boldsymbol{M}}(\nabla\times\boldsymbol{A}_i)-\Phi_{\boldsymbol{M}}(\nabla\times\boldsymbol{A}_{i-1}))\,\mathrm{d}x$$
$$=\int_{\Omega}\mu\Phi_{\boldsymbol{M}}(\nabla\times\boldsymbol{A}_l)\,\mathrm{d}x-\int_{\Omega}\mu\Phi_{\boldsymbol{M}}(\nabla\times\boldsymbol{A}_0)\,\mathrm{d}x \geqslant \frac{c_M\mu_*}{2}\left\|\nabla\times\boldsymbol{A}_l\right\|^2_{\mathbf{L}^2(\Omega)}-\frac{C_M\mu^*}{2}\left\|\nabla\times\boldsymbol{A}_0\right\|^2_{\mathbf{L}^2(\Omega)}.$$

We relocate the terms to get

$$\left(\sigma_*-\frac{\varepsilon}{2}\sigma^*C_\pi\right)\sum_{i=1}^{l}\left\|\delta\boldsymbol{A}_i\right\|^2_{\mathbf{L}^2(\pi)}\tau+\frac{c_M\mu_*}{2}\left\|\nabla\times\boldsymbol{A}_l\right\|^2_{\mathbf{L}^2(\Omega)} \leqslant C\frac{\sigma^*}{2\varepsilon}+\frac{C_M\mu^*}{2}\left\|\nabla\times\boldsymbol{A}_0\right\|^2_{\mathbf{L}^2(\Omega)}.$$

Fixing $\varepsilon\in\left(0,\frac{2\sigma_*}{\sigma^*C_\pi}\right)$ and assuming that $\boldsymbol{A}_0\in\boldsymbol{X}_{N,0}$, we obtain

$$\sum_{i=1}^{l}\left\|\delta\boldsymbol{A}_i\right\|^2_{\mathbf{L}^2(\pi)}\tau+\left\|\nabla\times\boldsymbol{A}_l\right\|^2_{\mathbf{L}^2(\Omega)}\leqslant C.$$

This is valid for any $1\leqslant l\leqslant n$, which concludes the proof of (i).

(ii) Take $\boldsymbol{\varphi}\in\mathbf{C}_0^\infty(\boldsymbol{\pi})$. It holds

$$(\sigma_\pi(u_{i-1})\delta\boldsymbol{A}_i,\boldsymbol{\varphi})_\pi+(\sigma_\pi(u_{i-1})\nabla\phi_i,\boldsymbol{\varphi})_T=-\left(\mu\boldsymbol{M}(\nabla\times\boldsymbol{A}_i),\nabla\times\boldsymbol{\varphi}\right)_\Omega$$
$$\stackrel{Green's\,theorem}{=}-\left(\nabla\times\left(\mu\boldsymbol{M}(\nabla\times\boldsymbol{A}_i)\right),\boldsymbol{\varphi}\right)_\Omega.$$

Based on Lemma 3 and Lemma 4 (i) we see that the LHS can be seen as a linear bounded functional in $L^2((0,\mathcal{T});\mathbf{L}^2(\boldsymbol{\pi}))$. According to the Hahn-Banach theorem the same holds true for the RHS, i.e.

$$\sum_{i=1}^{n}\left\|\nabla\times\left(\mu\boldsymbol{M}(\nabla\times\boldsymbol{A}_i)\right)\right\|^2_{\mathbf{L}^2(\pi)}\tau\leqslant C.$$

□

Lemma 5. *Let* (21) *be fulfilled. Then there exists a positive constant C_r, depending only on parameter r of truncation function $\mathcal{R}_r$, such that*

$$(i)\quad \sum_{i=1}^{n}\left\|\delta u_i\right\|^2_{L^2(\pi)}\tau+\sum_{i=1}^{n}\left\|\nabla u_i-\nabla u_{i-1}\right\|^2_{\mathbf{L}^2(\pi)}+\max_{1\leqslant i\leqslant n}\left\|\nabla u_i\right\|_{\mathbf{L}^2(\pi)}\leqslant C_r,$$
$$(ii)\quad \max_{1\leqslant i\leqslant n}\left\|u_i\right\|^2_{L^2(\pi)}\leqslant C_r,$$
$$(iii)\quad \max_{1\leqslant i\leqslant n}\left\|\delta\beta(u_i)\right\|^2_{(H^1(\pi))^*}\leqslant C_r.$$

Proof. (i) Take $\psi=\delta u_i\tau$ in (25) and sum it up for $i=1,\dots,l\leqslant n$ to have

$$\sum_{i=1}^{l}\left(\delta\beta(u_i),\delta u_i\right)_\pi\tau+\sum_{i=1}^{l}\left(\lambda_i\nabla u_i,\nabla u_i-\nabla u_{i-1}\right)_\pi=\sum_{i=1}^{l}\left(\mathcal{R}_r\left(\sigma_\pi(u_{i-1})\left|\delta\boldsymbol{A}_i+\chi_T\nabla\phi_i\right|^2\right),\delta u_i\right)_\pi\tau.$$

Utilizing the mean value theorem and (21), we can bound the first term on the LHS

$$\sum_{i=1}^{l}\left(\delta\beta(u_i),\delta u_i\right)_\pi\tau=\sum_{i=1}^{l}\left(\beta'(\eta)(u_i-u_{i-1}),\delta u_i\right)_\pi\geqslant\beta_*\sum_{i=1}^{l}\left\|\delta u_i\right\|^2_{L^2(\pi)}\tau.$$

For the term on the RHS we use Cauchy's and Young's inequalities

$$\sum_{i=1}^{l}\left(\mathcal{R}_r\left(\sigma_\pi(u_{i-1})\left|\delta\boldsymbol{A}_i+\chi_T\nabla\phi_i\right|^2\right),\delta u_i\right)_\pi\tau\leqslant\frac{C_r^2}{2\varepsilon}\left|\pi\right|\mathcal{T}+\frac{\varepsilon}{2}\sum_{i=1}^{l}\left\|\delta u_i\right\|^2_{L^2(\pi)}\tau=C_{r,\varepsilon}+\frac{\varepsilon}{2}\sum_{i=1}^{l}\left\|\delta u_i\right\|^2_{L^2(\pi)}\tau.$$

Thanks to Lipschitz continuity of λ in time, we can bound the last term as follows (cf. [25])

$$\begin{aligned}\sum_{i=1}^{l}\left(\lambda_i\nabla u_i,\nabla u_i-\nabla u_{i-1}\right)_\pi &= \frac{1}{2}\int_\pi\lambda_l\left|\nabla u_l\right|^2\,\mathrm{d}x+\frac{1}{2}\sum_{i=1}^{l}\int_\pi\lambda_i\left|\nabla u_i-\nabla u_{i-1}\right|^2\,\mathrm{d}x-\frac{1}{2}\int_\pi\lambda_1\left|\nabla u_0\right|^2\,\mathrm{d}x\\ &\quad-\frac{1}{2}\sum_{i=1}^{l}\int_\pi(\lambda_{i+1}-\lambda_i)\left|\nabla u_i\right|^2\,\mathrm{d}x\\ &\geqslant\frac{\lambda_*}{2}\left\|\nabla u_l\right\|^2_{\mathbf{L}^2(\pi)}+\frac{\lambda_*}{2}\sum_{i=1}^{l}\left\|\nabla u_i-\nabla u_{i-1}\right\|^2_{\mathbf{L}^2(\pi)}-\frac{C_\lambda}{2}\sum_{i=0}^{l-1}\left\|\nabla u_i\right\|^2_{\mathbf{L}^2(\pi)}\tau-\frac{\lambda^*}{2}\left\|\nabla u_0\right\|^2_{\mathbf{L}^2(\pi)}.\end{aligned}$$

Collecting all estimates above, taking $\varepsilon\in(0,2\beta_*)$ and using Grönwall's lemma we obtain (i).

(ii) This part follows readily from (i) and

$$u_l=u_0+\sum_{i=1}^{l}\delta u_i\tau\Longrightarrow\left\|u_l\right\|_{L^2(\pi)}\leqslant\left\|u_0\right\|_{L^2(\pi)}+\sum_{i=1}^{l}\left\|\delta u_i\right\|_{L^2(\pi)}\tau\leqslant C_r$$

for any $0\leqslant l\leqslant n$.

(iii) Norm in $(H^1(\pi))^*$ is defined as

$$\|u\|_{(H^1(\pi))^*}:=\sup_{\psi\neq0\,\psi\in H^1(\pi)}\frac{\left|(u,\psi)_\pi\right|}{\|\psi\|_{H^1(\pi)}}.$$

Thus, deducing from (25) and using estimates above we can write

$$\begin{aligned}\left|(\delta\beta(u_i),\psi)_\pi\right| &\leqslant\left|\left(\mathcal{R}_r\left(\sigma_\pi(u_{i-1})\left|\delta\boldsymbol{A}_i+\chi_T\nabla\phi_i\right|^2\right),\psi\right)_\pi\right|+\left|(\nabla u_i,\nabla\psi)_\pi\right|\\ &\leqslant C_r\sqrt{|\pi|}\,\|\psi\|_{L^2(\pi)}+\|\nabla u_i\|_{\mathbf{L}^2(\pi)}\|\nabla\psi\|_{\mathbf{L}^2(\pi)}\\ &\leqslant\left\{C_r\sqrt{|\pi|}+\|\nabla u_i\|_{\mathbf{L}^2(\pi)}\right\}\|\psi\|_{H^1(\pi)}\\ &\leqslant C_r\|\psi\|_{H^1(\pi)},\end{aligned}$$

therefore

$$\|\delta\beta(u_i)\|_{(H^1(\pi))^*}\leqslant C_r,$$

for any $i=1,\ldots,n$. □

3.2. Convergence

We construct a piece-wise constant and piece-wise linear in time functions as follows

$$\begin{aligned}\bar{f}_n(t)&=f_i && \text{for } t\in(t_{i-1},t_i],\\ f_n(t)&=f_{i-1}+(t-t_{i-1})\delta f_i && \text{for } t\in(t_{i-1},t_i],\\ \bar{f}_n(0)&=f_n(0)=f_0.\end{aligned}$$

Using this notation, we can rewrite (25),(26) and (27) for $t \in [0,\mathcal{T}]$ as follows

$$(\partial_t\beta_n,\psi)_\pi + \left(\bar{\lambda}_n\nabla\bar{u}_n,\psi\right)_\pi = \left(\mathcal{R}_r\left(\bar{\sigma}_{\pi_n}(t-\tau)\left|\partial_t\boldsymbol{A}_n + \chi_T\nabla\bar{\phi}_n\right|^2\right),\psi\right)_\pi \qquad \text{for any } \psi \in H^1(\boldsymbol{\pi}), \tag{29}$$

$$(\bar{\sigma}_{\pi_n}(t-\tau)\partial_t\boldsymbol{A}_n,\boldsymbol{\varphi})_\pi + \left(\mu\boldsymbol{M}(\nabla\times\bar{\boldsymbol{A}}_n),\nabla\times\boldsymbol{\varphi}\right)_\Omega + (\bar{\sigma}_{\pi_n}(t-\tau)\nabla\bar{\phi}_n,\boldsymbol{\varphi})_T = 0 \qquad \text{for any } \boldsymbol{\varphi} \in \boldsymbol{X}_{N,0}, \tag{30}$$

$$(\bar{\sigma}_{\pi_n}(t-\tau)\nabla\bar{\phi}_n,\nabla\xi)_T + \left(\bar{j}_n,\xi\right)_\Gamma = 0 \qquad \text{for any } \xi \in H^1(T)/\mathbb{R}. \tag{31}$$

The proof of existence of a solution to (18)- (20) is based on the obtained stability of iterates and on the functional analysis. Technically it is very long, therefore we split it into 3 parts.

Proposition 1. *Suppose* (21). *Moreover assume that σ is globally Lipschitz continuous. Then there exist an u and a sub-sequence of u_n (denoted by the same symbol again) such that*

$$\begin{array}{lll}
(i) & u_n \to u & \text{in } C\left([0,\mathcal{T}];L^2(\pi)\right), \\
 & \bar{u}_n(t) \rightharpoonup u(t) & \text{in } H^1(\pi),\ \forall t \in [0,\mathcal{T}], \\
 & \bar{u}_n \to u & \text{in } L^2((0,\mathcal{T});L^2(\pi)), \\
(ii) & \bar{\sigma}_{\pi_n} \to \sigma_\pi(u),\ \bar{\sigma}_{\pi_n}(t-\tau) \to \sigma_\pi(u) & \text{in } L^2((0,\mathcal{T});L^2(\pi)), \\
(iii) & \bar{\beta}_n - \beta_n \to 0 & \text{in } C\left([0,\mathcal{T}];(H^1(\pi))^*\right), \\
(iv) & \bar{\beta}_n \to \beta(u) & \text{in } L^2((0,\mathcal{T});L^2(\pi)), \\
(v) & \bar{j}_n \to j & \text{in } L^2((0,\mathcal{T});H^{-1/2}(\Gamma)).
\end{array}$$

Proof. (i) Using Lemma 5, we have $\partial_t u_n \in L^2((0,\mathcal{T});L^2(\pi))$ and $\bar{u}_n \in C([0,\mathcal{T}];H^1(\pi))$. Now, since $H^1(\pi)$ is compactly embedded in $L^2(\pi)$, we can apply well-known [22, Lemma 1.3.13] to conclude the first two statements of (i). To prove the last one we only need to show that u_n and $\bar{u}_n$ have the same limit in $L^2((0,\mathcal{T});L^2(\pi))$. We may write

$$\int_0^{\mathcal{T}} \|\bar{u}_n - u_n\|^2_{L^2(\pi)}\ \mathrm{d}t = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \|u_i - u_{i-1} - (t-t_{i-1})\delta u_i\|^2_{L^2(\pi)}\ \mathrm{d}t = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \|\delta u_i(\tau - t + t_{i-1})\|^2_{L^2(\pi)}\ \mathrm{d}t$$
$$\leqslant \tau^2 \sum_{i=1}^n \|\delta u_i\|^2_{L^2(\pi)}\,\tau \leqslant C_r\tau^2 \stackrel{n\to\infty}{\longrightarrow} 0.$$

(ii) Since σ is supposed to be globally Lipschitz continuous and $\bar{u}_n$ converges strongly to u in $L^2((0,\mathcal{T});L^2(\pi))$, we conclude that $\bar{\sigma}_{\pi_n} \to \sigma(u)$ in the same space as well. The only thing left to be done is to show that $\bar{\sigma}_{\pi_n}(t-\tau)$ and $\bar{\sigma}_{\pi_n}(t)$ share the same limit in $L^2((0,\mathcal{T});L^2(\pi))$. It holds

$$\int_0^{\mathcal{T}} \|\bar{\sigma}_{\pi_n}(t) - \bar{\sigma}_{\pi_n}(t-\tau)\|^2_{L^2(\pi)}\ \mathrm{d}t = \sum_{i=1}^n \|\sigma(u_i) - \sigma(u_{i-1})\|^2_{L^2(\pi)}\,\tau \stackrel{Lipschitz}{\leqslant} C_\sigma \sum_{i=1}^n \|u_i - u_{i-1}\|^2_{L^2(\pi)}\,\tau$$
$$= C_\sigma\tau^2 \sum_{i=1}^n \|\delta u_i\|^2_{L^2(\pi)}\,\tau \leqslant C_\sigma C_r\tau^2 \stackrel{n\to\infty}{\longrightarrow} 0.$$

(iii) Results from Lemma 5 let us write

$$\left|\left(\bar{\beta}_n - \beta_n,\psi\right)\right| \leqslant \tau\,\|\partial_t\beta_n\|_{(H^1(\pi))^*}\,\|\psi\|_{H^1(\pi)} \leqslant \tau C_r\,\|\psi\|_{H^1(\pi)}$$

and therefore $\|\bar{\beta}_n - \beta_n\|_{(H^1(\pi))^*} \leqslant \tau C_r \stackrel{n\to\infty}{\longrightarrow} 0$.

(iv) Taking into account the continuity of β and the fact that u_n converges strongly towards u, allow us to use Lebesgue's dominated convergence theorem to conclude that $\bar{\beta}_n \to \beta(u)$ in $L^2((0,\mathcal{T});L^2(\pi))$.

(v) Assuming that j is Lipschitz continuous in time, we can write

$$\int_0^{\mathcal{T}} \|\bar{j}_n - j\|^2_{H^{-1/2}(\Gamma)}\ \mathrm{d}t = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \|j(t_i) - j(t)\|^2_{H^{-1/2}(\Gamma)}\ \mathrm{d}t \leqslant C\tau^2 \stackrel{n\to\infty}{\to} 0.$$

□

Proposition 2. *Suppose that all assumptions of Proposition 1 are satisfied. Then there exist an $\boldsymbol{A}$ and a sub-sequence of $\boldsymbol{A}_n$ (denoted by the same symbol again) such that*

$$
\begin{array}{llll}
(i) & \bar{\boldsymbol{A}}_n \rightharpoonup \boldsymbol{A},\ \nabla\times\bar{\boldsymbol{A}}_n \rightharpoonup \nabla\times\boldsymbol{A} & in & L^2((0,\mathcal{T});\mathbf{L}^2(\Omega)),\\
 & \mu\boldsymbol{M}(\nabla\times\bar{\boldsymbol{A}}_n) \rightharpoonup \mu\boldsymbol{M}(\nabla\times\boldsymbol{A}) & in & L^2((0,\mathcal{T});\mathbf{L}^2(\Omega\backslash\boldsymbol{\pi})),\\
 & \boldsymbol{A}_n \to \boldsymbol{A} & in & C\left([0,\mathcal{T}];\mathbf{L}^2(\boldsymbol{\pi})\right),\\
 & \boldsymbol{A}_n(t) \rightharpoonup \boldsymbol{A}(t),\ \bar{\boldsymbol{A}}_n(t) \rightharpoonup \boldsymbol{A}(t) & in & \mathbf{H}^1(\boldsymbol{\pi}),\ \forall t,\\
 & \partial_t\boldsymbol{A}_n \rightharpoonup \partial_t\boldsymbol{A} & in & L^2((0,\mathcal{T});\mathbf{L}^2(\boldsymbol{\pi})),\\
(ii) & \boldsymbol{M}(\nabla\times\bar{\boldsymbol{A}}_n) \rightharpoonup \boldsymbol{M}(\nabla\times\boldsymbol{A}) & in & L^2((0,\mathcal{T});\mathbf{L}^2(\boldsymbol{\pi})),\\
(iii) & \nabla\times\bar{\boldsymbol{A}}_n \to \nabla\times\boldsymbol{A} & in & L^2((0,\mathcal{T});\mathbf{L}^2(\boldsymbol{\pi})),\\
 & \boldsymbol{M}(\nabla\times\bar{\boldsymbol{A}}_n) \to \boldsymbol{M}(\nabla\times\boldsymbol{A}) & in & L^2((0,\mathcal{T});\mathbf{L}^2(\boldsymbol{\pi})).
\end{array}
$$

Proof. (i) Lemma 4 yields

$$\int_0^{\mathcal{T}} \|\bar{\boldsymbol{A}}_n\|^2_{\boldsymbol{X}_{N,0}}\ \mathrm{d}t \leqslant C.$$

The reflexivity of $L^2((0,\mathcal{T});\boldsymbol{X}_{N,0})$ gives for a sub-sequence that $\bar{\boldsymbol{A}}_n \rightharpoonup \boldsymbol{A}$ in that space. One can easily see that

$$\bar{\boldsymbol{A}}_n \rightharpoonup \boldsymbol{A}, \qquad \nabla\times\bar{\boldsymbol{A}}_n \rightharpoonup \nabla\times\boldsymbol{A} \qquad \text{in } L^2((0,\mathcal{T});\mathbf{L}^2(\Omega)),$$

due to the density of $\boldsymbol{C}_0^\infty(\Omega)$ in $\mathbf{L}^2(\Omega)$, see [26, Thm. 2.6.1]. Take now $\boldsymbol{\varphi} \in \mathbf{C}_0^\infty(\Omega\backslash\boldsymbol{\pi})$. Using $\mu \in H^1(\Omega\backslash\boldsymbol{\pi})$ we have

$$\int_0^{\mathcal{T}} \left(\mu\boldsymbol{M}(\nabla\times\bar{\boldsymbol{A}}_n),\boldsymbol{\varphi}\right)_\Omega\ \mathrm{d}t = \int_0^{\mathcal{T}} \left(\mu\nabla\times\bar{\boldsymbol{A}}_n,\boldsymbol{\varphi}\right)_\Omega\ \mathrm{d}t = \int_0^{\mathcal{T}} \left(\bar{\boldsymbol{A}}_n,\nabla\times(\mu\boldsymbol{\varphi})\right)_\Omega\ \mathrm{d}t.$$

Passing to the limit for $n\to\infty$ we get

$$\lim_{n\to\infty}\int_0^{\mathcal{T}} \left(\mu\nabla\times\bar{\boldsymbol{A}}_n,\boldsymbol{\varphi}\right)_\Omega\ \mathrm{d}t = \int_0^{\mathcal{T}} \left(\boldsymbol{A},\nabla\times(\mu\boldsymbol{\varphi})\right)_\Omega\ \mathrm{d}t = \int_0^{\mathcal{T}} \left(\mu\nabla\times\boldsymbol{A},\boldsymbol{\varphi}\right)_\Omega\ \mathrm{d}t.$$

Using the density argument of $\mathbf{C}_0^\infty(\Omega\backslash\boldsymbol{\pi})$ in $\mathbf{L}^2(\Omega\backslash\boldsymbol{\pi})$ we have

$$\mu\boldsymbol{M}(\nabla\times\bar{\boldsymbol{A}}_n) = \mu\nabla\times\bar{\boldsymbol{A}}_n \rightharpoonup \mu\nabla\times\boldsymbol{A} = \mu\boldsymbol{M}(\nabla\times\boldsymbol{A}) \qquad \text{in } L^2((0,\mathcal{T});\mathbf{L}^2(\Omega\backslash\boldsymbol{\pi})).$$

Lemma 4 together with $\boldsymbol{X}_{N,0} \subset \mathbf{H}^1(\Omega)$ (cf. [18, Theorem 3.7]) imply

$$\int_0^{\mathcal{T}} \|\partial_t\boldsymbol{A}_n\|^2_{L^2(\pi)}\ \mathrm{d}t \leqslant C, \qquad \|\bar{\boldsymbol{A}}_n\|_{\mathbf{H}^1(\pi)} \leqslant \|\bar{\boldsymbol{A}}_n\|_{\mathbf{H}^1(\Omega)} \leqslant C.$$

Employing [22, Lemma 1.3.13] we get for a sub-sequence that

$$
\begin{array}{lll}
\boldsymbol{A}_n \to \boldsymbol{A} & \text{in} & C\left([0,\mathcal{T}];\mathbf{L}^2(\boldsymbol{\pi})\right)\\
\boldsymbol{A}_n(t) \rightharpoonup \boldsymbol{A}(t),\ \bar{\boldsymbol{A}}_n(t) \rightharpoonup \boldsymbol{A}(t) & \text{in} & \mathbf{H}^1(\boldsymbol{\pi}),\ \forall t\\
\partial_t\boldsymbol{A}_n \rightharpoonup \partial_t\boldsymbol{A} & \text{in} & L^2((0,\mathcal{T});\mathbf{L}^2(\boldsymbol{\pi})).
\end{array}
$$

(ii) The sequence $\boldsymbol{M}(\nabla\times\bar{\boldsymbol{A}}_n)$ is bounded in $L^2((0,\mathcal{T});\mathbf{L}^2(\Omega))$. Therefore, there exists $\boldsymbol{p}$ from $L^2((0,\mathcal{T});\mathbf{L}^2(\Omega))$ such that $\boldsymbol{M}(\nabla\times\bar{\boldsymbol{A}}_n) \rightharpoonup \boldsymbol{p}$ in that space (for a sub-sequence). Now, we involve the remarkable Minty-Browder technique, cf. [27, 17]. The general idea is based on monotone character of the vectorial field $\boldsymbol{M}$. Let us investigate the following inequality

$$0 \leqslant \int_0^{\mathcal{T}} \left(\boldsymbol{M}(\nabla \times \bar{\boldsymbol{A}}_n) - \boldsymbol{M}(\boldsymbol{b}), \psi\mu\left(\nabla \times \bar{\boldsymbol{A}}_n - \boldsymbol{b}\right)\right)_\Omega \,\mathrm{d}t = I_1 + I_2 + I_3 + I_4, \tag{32}$$

where

$$I_1 = \int_0^{\mathcal{T}} \left(\boldsymbol{M}(\nabla \times \bar{\boldsymbol{A}}_n), \psi\mu\nabla \times \bar{\boldsymbol{A}}_n\right)_\Omega \,\mathrm{d}t, \qquad I_2 = \int_0^{\mathcal{T}} \left(\boldsymbol{M}(\boldsymbol{b}), \psi\mu\nabla \times \bar{\boldsymbol{A}}_n\right)_\Omega \,\mathrm{d}t,$$
$$I_3 = \int_0^{\mathcal{T}} \left(\boldsymbol{M}(\nabla \times \bar{\boldsymbol{A}}_n), \psi\mu\boldsymbol{b}\right)_\Omega \,\mathrm{d}t, \qquad I_4 = \int_0^{\mathcal{T}} \left(\boldsymbol{M}(\boldsymbol{b}), \psi\mu\boldsymbol{b}\right)_\Omega \,\mathrm{d}t.$$

This inequality holds true for any $\boldsymbol{b} \in L^2((0,\mathcal{T}); \mathbf{L}^2(\Omega))$ and any non-negative $\psi \in C_0^\infty(\boldsymbol{\pi})$. We want to pass to the limit for $n \to \infty$ in (32). We do it for each term in (32) separately.

It holds

$$\begin{aligned}
I_1 &= \int_0^{\mathcal{T}} \left(\boldsymbol{M}(\nabla \times \bar{\boldsymbol{A}}_n), \psi\mu\nabla \times \bar{\boldsymbol{A}}_n\right)_\Omega \,\mathrm{d}t \\
&= \int_0^{\mathcal{T}} \left(\boldsymbol{M}(\nabla \times \bar{\boldsymbol{A}}_n), \psi\mu\nabla \times \left(\bar{\boldsymbol{A}}_n - \boldsymbol{A}\right)\right)_\Omega \,\mathrm{d}t + \int_0^{\mathcal{T}} \left(\boldsymbol{M}(\nabla \times \bar{\boldsymbol{A}}_n), \psi\mu\nabla \times \boldsymbol{A}\right)_\Omega \,\mathrm{d}t \\
&= \int_0^{\mathcal{T}} \left(\nabla \times \left[\psi\mu\boldsymbol{M}(\nabla \times \bar{\boldsymbol{A}}_n)\right], \bar{\boldsymbol{A}}_n - \boldsymbol{A}\right)_\Omega \,\mathrm{d}t + \int_0^{\mathcal{T}} \left(\boldsymbol{M}(\nabla \times \bar{\boldsymbol{A}}_n), \psi\mu\nabla \times \boldsymbol{A}\right)_\Omega \,\mathrm{d}t \\
&= \int_0^{\mathcal{T}} \left(\psi\nabla \times \left[\mu\boldsymbol{M}(\nabla \times \bar{\boldsymbol{A}}_n)\right], \bar{\boldsymbol{A}}_n - \boldsymbol{A}\right)_\Omega \,\mathrm{d}t + \int_0^{\mathcal{T}} \left(\nabla\psi \times \left[\mu\boldsymbol{M}(\nabla \times \bar{\boldsymbol{A}}_n)\right], \bar{\boldsymbol{A}}_n - \boldsymbol{A}\right)_\Omega \,\mathrm{d}t \\
&+ \int_0^{\mathcal{T}} \left(\boldsymbol{M}(\nabla \times \bar{\boldsymbol{A}}_n), \psi\mu\nabla \times \boldsymbol{A}\right)_\Omega \,\mathrm{d}t.
\end{aligned}$$

We know that $\boldsymbol{A}_n \to \boldsymbol{A}$ in $C\left([0,\mathcal{T}]; \mathbf{L}^2(\boldsymbol{\pi})\right)$ and $\partial_t \boldsymbol{A}_n$ is bounded in $L^2((0,\mathcal{T}); \mathbf{L}^2(\boldsymbol{\pi}))$. Therefore also $\bar{\boldsymbol{A}}_n \to \boldsymbol{A}$ in $C\left([0,\mathcal{T}]; \mathbf{L}^2(\boldsymbol{\pi})\right)$. Thus, Using $\mu \in H^1(\boldsymbol{\pi})$, it is not difficult to see that

$$\lim_{n\to\infty} I_1 = \int_0^{\mathcal{T}} (\boldsymbol{p}, \psi\mu\nabla \times \boldsymbol{A})_\Omega \,\mathrm{d}t.$$

Clearly

$$\begin{aligned}
\lim_{n\to\infty} I_2 &= \int_0^{\mathcal{T}} (\boldsymbol{M}(\boldsymbol{b}), \psi\mu\nabla \times \boldsymbol{A})_\Omega \,\mathrm{d}t \\
\lim_{n\to\infty} I_3 &= \int_0^{\mathcal{T}} (\boldsymbol{p}, \psi\mu\boldsymbol{b})_\Omega \,\mathrm{d}t \\
\lim_{n\to\infty} I_4 &= \int_0^{\mathcal{T}} (\boldsymbol{M}(\boldsymbol{b}), \psi\mu\boldsymbol{b})_\Omega \,\mathrm{d}t.
\end{aligned}$$

Assembling these auxiliary results we arrive at

$$\lim_{n\to\infty} \int_0^{\mathcal{T}} \left(\boldsymbol{M}(\nabla \times \bar{\boldsymbol{A}}_n) - \boldsymbol{M}(\boldsymbol{b}), \psi\mu\left(\nabla \times \bar{\boldsymbol{A}}_n - \boldsymbol{b}\right)\right)_\Omega \,\mathrm{d}t = \int_0^{\mathcal{T}} (\boldsymbol{p} - \boldsymbol{M}(\boldsymbol{b}), \psi\mu(\nabla \times \boldsymbol{A} - \boldsymbol{b}))_\Omega \,\mathrm{d}t \geqslant 0.$$

Since $\boldsymbol{b}$ was taken as an arbitrary element of $L^2((0,\mathcal{T}); \mathbf{L}^2(\Omega))$, we can choose it as $\boldsymbol{b} = \omega\boldsymbol{q} + \nabla \times \boldsymbol{A}$, where $\boldsymbol{q} \in L^2((0,\mathcal{T}); \mathbf{L}^2(\Omega))$ and $\omega > 0$.

$$\int_0^{\mathcal{T}} (\boldsymbol{p} - \boldsymbol{M}(\nabla \times \boldsymbol{A} + \omega\boldsymbol{q}), \mu\psi(-\omega\boldsymbol{q}))_\Omega \,\mathrm{d}t \geqslant 0 \quad / \cdot \frac{1}{\omega},$$

$$\int_0^{\mathcal{T}} (\boldsymbol{p} - \boldsymbol{M}(\nabla \times \boldsymbol{A} + \omega \boldsymbol{q}), \mu\psi(-\boldsymbol{q}))_\Omega \, \mathrm{d}t \geqslant 0 \qquad / \ \omega \to 0,$$

$$\int_0^{\mathcal{T}} (\boldsymbol{p} - \boldsymbol{M}(\nabla \times \boldsymbol{A}), \mu\psi(-\boldsymbol{q}))_\Omega \, \mathrm{d}t \geqslant 0 \qquad / \ \boldsymbol{q} \text{ is arbitrary, hence we can choose } \boldsymbol{q} = -\boldsymbol{q},$$

$$\int_0^{\mathcal{T}} (\boldsymbol{p} - \boldsymbol{M}(\nabla \times \boldsymbol{A}), \mu\psi(-\boldsymbol{q}))_\Omega \, \mathrm{d}t \leqslant 0.$$

The conclusion is that $\int_0^{\mathcal{T}} (\boldsymbol{p} - \boldsymbol{M}(\nabla \times \boldsymbol{A}), \mu\psi\boldsymbol{q})_\Omega \, \mathrm{d}t = 0$ for any non-negative $\psi \in C_0^\infty(\boldsymbol{\pi})$ and every $\boldsymbol{q} \in L^2((0,\mathcal{T}); \mathbf{L}^2(\Omega))$. Hence $\boldsymbol{p} = \boldsymbol{M}(\nabla \times \boldsymbol{A})$ a.e. in $(0,\mathcal{T}) \times \boldsymbol{\pi}$ and $\boldsymbol{M}(\nabla \times \bar{\boldsymbol{A}}_n) \rightharpoonup \boldsymbol{M}(\nabla \times \boldsymbol{A})$ in $L^2((0,\mathcal{T}); \mathbf{L}^2(\boldsymbol{\pi}))$.

(iii) Analogously as in (ii) using the strong monotonicity of $\boldsymbol{M}$ $(k1)$, we conclude

$$\begin{aligned} 0 &= \lim_{n\to\infty} \int_0^{\mathcal{T}} \left(\boldsymbol{M}(\nabla \times \bar{\boldsymbol{A}}_n) - \boldsymbol{M}(\nabla \times \boldsymbol{A}), \mu\psi \left(\nabla \times \bar{\boldsymbol{A}}_n - \nabla \times \boldsymbol{A} \right) \right)_\Omega \, \mathrm{d}t \\ &\geqslant \lim_{n\to\infty} c_M \int_0^{\mathcal{T}} \left(\mu\psi, \left| \nabla \times \bar{\boldsymbol{A}}_n - \nabla \times \boldsymbol{A} \right|^2 \right)_\Omega \, \mathrm{d}t \geqslant 0. \end{aligned}$$

Therefore $\lim_{n\to\infty} \int_0^{\mathcal{T}} \left(\mu\psi, \left| \nabla \times \bar{\boldsymbol{A}}_n - \nabla \times \boldsymbol{A} \right|^2 \right)_\Omega \, \mathrm{d}t = 0$ for every $0 \leqslant \psi \in C_0^\infty(\boldsymbol{\pi})$, which implies $\nabla \times \bar{\boldsymbol{A}}_n \to \nabla \times \boldsymbol{A}$ in $L^2((0,\mathcal{T}); \mathbf{L}^2(\boldsymbol{\pi}))$. Vectorial field $\boldsymbol{M}$ is also Lipschitz continuous, hence $\boldsymbol{M}(\nabla \times \bar{\boldsymbol{A}}_n) \to \boldsymbol{M}(\nabla \times \boldsymbol{A})$ in $L^2((0,\mathcal{T}); \mathbf{L}^2(\boldsymbol{\pi}))$ as well. □

Now, we are in a position to state our main result.

Theorem 1. *Suppose that all assumptions of Proposition 1 are satisfied. Then there exist a ϕ and a sub-sequence of $\bar{\phi}_n$ (denoted by the same symbol again) such that*

(i) *ϕ and u solve* (20)

(ii) *$\nabla\bar{\phi}_n \to \nabla\phi$ in $L^2((0,\mathcal{T}); \mathbf{L}^2(T))$*

(iii) *ϕ, u and $\boldsymbol{A}$ solve* (18)

(iv) *$\partial_t \boldsymbol{A}_n \to \partial_t \boldsymbol{A}$ in $L^2((0,\mathcal{T}); \mathbf{L}^2(\pi))$*

(v) *ϕ, u and $\boldsymbol{A}$ solve* (19).

Proof. (i) Existence of a potential $\phi \in H^1(T)/\mathbb{R}$ such that $\nabla\bar{\phi}_n \rightharpoonup \nabla\phi$ in $L^2((0,\mathcal{T}); \mathbf{L}^2(T))$ follows from the reflexivity of $L^2((0,\mathcal{T}); \mathbf{L}^2(T))$. The function ϕ has in fact a zero mean over T, cf. proof of Lemma 1.

Take $\xi \in H^1(T)/\mathbb{R}$ in (31) and integrate in time

$$\int_0^\zeta (\bar{\sigma}_{\pi_n}(t-\tau)\nabla\bar{\phi}_n, \xi)_T \, \mathrm{d}t + \int_0^\zeta \left(\bar{j}_n, \xi\right)_\Gamma \, \mathrm{d}t = 0.$$

Thanks to Proposition 1 (ii) and (v), we can pass to the limit for $n \to \infty$ to get

$$\int_0^\zeta (\sigma_\pi(u)\nabla\phi, \xi)_T \, \mathrm{d}t + \int_0^\zeta (j, \xi)_\Gamma \, \mathrm{d}t = 0.$$

Now, differentiating with respect to time, we can see that ϕ and u solve (20).

(ii) It holds

$$0 \leqslant \sigma_* \int_0^{\mathcal{T}} \|\nabla[\bar{\phi}_n - \phi]\|_{L^2(T)} \, \mathrm{d}t \leqslant \int_0^{\mathcal{T}} (\bar{\sigma}_{\pi_n}(t-\tau)\nabla[\bar{\phi}_n - \phi], \nabla[\bar{\phi}_n - \phi])_T \, \mathrm{d}t$$

$$
\begin{aligned}
&= \int_0^{\mathcal{T}} \left(\bar{\sigma}_{\pi_n}(t-\tau)\nabla\phi, \nabla\phi\right)_T \ \mathrm{d}t + \int_0^{\mathcal{T}} \left(\bar{\sigma}_{\pi_n}(t-\tau)\nabla\bar{\phi}_n, \nabla\bar{\phi}_n\right)_T \ \mathrm{d}t \\
&- 2\int_0^{\mathcal{T}} \left(\bar{\sigma}_{\pi_n}(t-\tau)\nabla\bar{\phi}_n, \nabla\phi\right)_T \ \mathrm{d}t \\
&\overset{(31)}{=} \int_0^{\mathcal{T}} \left(\bar{\sigma}_{\pi_n}(t-\tau)\nabla\phi, \nabla\phi\right)_T \ \mathrm{d}t - \int_0^{\mathcal{T}} \left(\bar{j}_n, \bar{\phi}_n\right)_\Gamma \ \mathrm{d}t \\
&- 2\int_0^{\mathcal{T}} \left(\bar{\sigma}_{\pi_n}(t-\tau)\nabla\bar{\phi}_n, \nabla\phi\right)_T \ \mathrm{d}t.
\end{aligned}
$$

Passing to the limit, we conclude

$$
0 \leqslant \lim_{n\to\infty} \sigma_* \int_0^{\mathcal{T}} \left\|\nabla\left[\bar{\phi}_n - \phi\right]\right\|_{L^2(T)} \ \mathrm{d}t \leqslant -\int_0^{\mathcal{T}} \left(\sigma_\pi(u)\nabla\phi, \nabla\phi\right)_T \ \mathrm{d}t - \int_0^{\mathcal{T}} (j, \phi)_\Gamma \ \mathrm{d}t \overset{(i)}{=} 0.
$$

Therefore, $\nabla\bar{\phi}_n \to \nabla\phi$ in $L^2((0,\mathcal{T}); \mathbf{L}^2(T))$.

(iii) We integrate (30) in time

$$
\int_0^{\zeta} \left(\bar{\sigma}_{\pi_n}(t-\tau)\partial_t \boldsymbol{A}_n, \boldsymbol{\varphi}\right)_\pi \ \mathrm{d}t + \int_0^{\zeta} \left(\mu \boldsymbol{M}(\nabla\times\bar{\boldsymbol{A}}_n), \nabla\times\boldsymbol{\varphi}\right)_\Omega \ \mathrm{d}t + \int_0^{\zeta} \left(\bar{\sigma}_{\pi_n}(t-\tau)\nabla\bar{\phi}_n, \boldsymbol{\varphi}\right)_T \ \mathrm{d}t = 0.
$$

Using Proposition 1 (ii), Proposition 2 and Theorem 1 (ii), we can pass to the limit for $n\to\infty$ to see

$$
\int_0^{\zeta} \left(\sigma_\pi(u)\partial_t \boldsymbol{A}, \boldsymbol{\varphi}\right)_\pi \ \mathrm{d}t + \int_0^{\zeta} \left(\mu \boldsymbol{M}(\nabla\times\boldsymbol{A}), \nabla\times\boldsymbol{\varphi}\right)_\Omega \ \mathrm{d}t + \int_0^{\zeta} \left(\sigma_\pi(u)\nabla\phi, \boldsymbol{\varphi}\right)_T \ \mathrm{d}t = 0.
$$

Thus, ϕ, u and $\boldsymbol{A}$ solve (18).

(iv) The strong convergence of $\nabla\times\bar{\boldsymbol{A}}_n \to \nabla\times\boldsymbol{A}$ in $L^2((0,\mathcal{T}); \mathbf{L}^2(\boldsymbol{\pi}))$ is guaranteed by Proposition 2 (iii). Let us take any $\zeta \in [0,\mathcal{T}]$ such that $\nabla\times\bar{\boldsymbol{A}}_n(\zeta) \to \nabla\times\boldsymbol{A}(\zeta)$ in $\mathbf{L}^2(\boldsymbol{\pi})$. This set is dense in $[0,\mathcal{T}]$. Take any non-negative $\psi \in C_0^\infty(\boldsymbol{\pi})$. We use the positiveness of σ to estimate the following

$$
\begin{aligned}
0 \leqslant \sigma_* \int_0^{\zeta}\int_\pi \psi\left|\partial_t\boldsymbol{A}_n - \partial_t\boldsymbol{A}\right|^2 \ \mathrm{d}x\,\mathrm{d}t &\leqslant \int_0^{\zeta}\int_\pi \psi\bar{\sigma}_{\pi_n}(t-\tau)\left|\partial_t\boldsymbol{A}_n - \partial_t\boldsymbol{A}\right|^2 \ \mathrm{d}x\,\mathrm{d}t \\
= -2\int_0^{\zeta} \left(\psi\bar{\sigma}_{\pi_n}(t-\tau)\partial_t\boldsymbol{A}_n, \partial_t\boldsymbol{A}\right)_\pi \ \mathrm{d}t &+ \int_0^{\zeta} \left(\psi\bar{\sigma}_{\pi_n}(t-\tau)\partial_t\boldsymbol{A}, \partial_t\boldsymbol{A}\right)_\pi \ \mathrm{d}t + \int_0^{\zeta} \left(\psi\bar{\sigma}_{\pi_n}(t-\tau)\partial_t\boldsymbol{A}_n, \partial_t\boldsymbol{A}_n\right)_\pi \ \mathrm{d}t.
\end{aligned}
$$

We use Lebesgue's dominated convergence theorem combined with Proposition 1 (ii) and Proposition 2 (i) to pass to the limit for $n\to\infty$ in the first two terms

$$
\begin{aligned}
&\lim_{n\to\infty} -2\int_0^{\zeta} \left(\psi\bar{\sigma}_{\pi_n}(t-\tau)\partial_t\boldsymbol{A}_n, \partial_t\boldsymbol{A}\right)_\pi \ \mathrm{d}t = -2\int_0^{\zeta} \left(\psi\sigma_\pi(u)\partial_t\boldsymbol{A}, \partial_t\boldsymbol{A}\right)_\pi \ \mathrm{d}t, \\
&\lim_{n\to\infty} \int_0^{\zeta} \left(\psi\bar{\sigma}_{\pi_n}(t-\tau)\partial_t\boldsymbol{A}, \partial_t\boldsymbol{A}\right)_\pi \ \mathrm{d}t = \int_0^{\zeta} \left(\psi\sigma_\pi(u)\partial_t\boldsymbol{A}, \partial_t\boldsymbol{A}\right)_\pi \ \mathrm{d}t.
\end{aligned}
$$

We can assume that $\zeta \in (t_{j-1}, t_j]$ and use variational formulation (30) to rewrite the third term as

$$
\begin{aligned}
\int_0^{\zeta} \left(\psi\bar{\sigma}_{\pi_n}(t-\tau)\partial_t\boldsymbol{A}_n, \partial_t\boldsymbol{A}_n\right)_\pi \ \mathrm{d}t &= -\int_0^{\zeta} \left(\mu\boldsymbol{M}(\nabla\times\bar{\boldsymbol{A}}_n), \nabla\times(\psi\partial_t\boldsymbol{A}_n)\right)_\Omega \ \mathrm{d}t - \int_0^{\zeta} \left(\bar{\sigma}_{\pi_n}(t-\tau)\nabla\bar{\phi}_n, \psi\partial_t\boldsymbol{A}_n\right)_T \ \mathrm{d}t \\
&= -\int_0^{\zeta} \left(\psi\mu\boldsymbol{M}(\nabla\times\bar{\boldsymbol{A}}_n), \nabla\times\partial_t\boldsymbol{A}_n\right)_\Omega \ \mathrm{d}t - \int_0^{\zeta} \left(\mu\boldsymbol{M}(\nabla\times\bar{\boldsymbol{A}}_n), \nabla\psi\times\partial_t\boldsymbol{A}_n\right)_\Omega \ \mathrm{d}t \\
&- \int_0^{\zeta} \left(\bar{\sigma}_{\pi_n}(t-\tau)\nabla\bar{\phi}_n, \psi\partial_t\boldsymbol{A}_n\right)_T \ \mathrm{d}t
\end{aligned}
$$

$$=: R_1 + R_2 + R_3.$$

Let us rewrite the first term on the RHS and examine it closely

$$\begin{aligned}
R_1 &= -\int_0^{t_j} \left(\psi\mu\boldsymbol{M}(\nabla\times\bar{\boldsymbol{A}}_n), \nabla\times\partial_t\boldsymbol{A}_n\right)_\Omega \,\mathrm{d}t + \int_\zeta^{t_j} \left(\psi\mu\boldsymbol{M}(\nabla\times\bar{\boldsymbol{A}}_n), \nabla\times\partial_t\boldsymbol{A}_n\right)_\Omega \,\mathrm{d}t \\
&= -\sum_{i=1}^{t_j} \int_\Omega \psi\mu\boldsymbol{M}(\nabla\times\boldsymbol{A}_i)\cdot(\nabla\times\boldsymbol{A}_i - \nabla\times\boldsymbol{A}_{i-1})\,\mathrm{d}x + \int_\zeta^{t_j} \left(\nabla\times\left(\psi\mu\boldsymbol{M}(\nabla\times\bar{\boldsymbol{A}}_n)\right), \partial_t\boldsymbol{A}_n\right)_\Omega \,\mathrm{d}t \\
&\overset{(22)}{\leqslant} -\sum_{i=1}^{t_j} \int_\Omega \psi\mu\left\{\Phi_{\boldsymbol{M}}(\nabla\times\boldsymbol{A}_i) - \Phi_{\boldsymbol{M}}(\nabla\times\boldsymbol{A}_{i-1})\right\}\mathrm{d}x \\
&+ \int_\zeta^{t_j} \left(\nabla\psi\times\left(\mu\boldsymbol{M}(\nabla\times\bar{\boldsymbol{A}}_n)\right), \partial_t\boldsymbol{A}_n\right)_\Omega \,\mathrm{d}t + \int_\zeta^{t_j} \left(\psi\nabla\times\left(\mu\boldsymbol{M}(\nabla\times\bar{\boldsymbol{A}}_n)\right), \partial_t\boldsymbol{A}_n\right)_\Omega \,\mathrm{d}t \\
&= -\int_\Omega \psi\mu\Phi_{\boldsymbol{M}}(\nabla\times\boldsymbol{A}_j)\,\mathrm{d}x + \int_\Omega \psi\mu\Phi_{\boldsymbol{M}}(\nabla\times\boldsymbol{A}_0)\,\mathrm{d}x \\
&+ \int_\zeta^{t_j} \left(\nabla\psi\times\left(\mu\boldsymbol{M}(\nabla\times\bar{\boldsymbol{A}}_n)\right), \partial_t\boldsymbol{A}_n\right)_\Omega \,\mathrm{d}t + \int_\zeta^{t_j} \left(\psi\nabla\times\left(\mu\boldsymbol{M}(\nabla\times\bar{\boldsymbol{A}}_n)\right), \partial_t\boldsymbol{A}_n\right)_\Omega \,\mathrm{d}t \\
&= -\int_\Omega \psi\mu\Phi_{\boldsymbol{M}}(\boldsymbol{M}(\nabla\times\bar{\boldsymbol{A}}_n(\zeta))\,\mathrm{d}x + \int_\Omega \psi\mu\Phi_{\boldsymbol{M}}(\nabla\times\boldsymbol{A}_0)\,\mathrm{d}x \\
&+ \int_\zeta^{t_j} \left(\nabla\psi\times\left(\mu\boldsymbol{M}(\nabla\times\bar{\boldsymbol{A}}_n)\right), \partial_t\boldsymbol{A}_n\right)_\Omega \,\mathrm{d}t + \int_\zeta^{t_j} \left(\psi\nabla\times\left(\mu\boldsymbol{M}(\nabla\times\bar{\boldsymbol{A}}_n)\right), \partial_t\boldsymbol{A}_n\right)_\Omega \,\mathrm{d}t.
\end{aligned}$$

Now, we are able to pass to the limit for $n \to \infty$ to find

$$\lim_{n\to\infty} R_2 = -\int_0^\zeta \left(\mu\boldsymbol{M}(\nabla\times\boldsymbol{A}), \nabla\psi\times\partial_t\boldsymbol{A}\right)_\Omega \,\mathrm{d}t,$$

$$\lim_{n\to\infty} R_3 = -\int_0^\zeta \left(\sigma_\pi(u)\nabla\phi, \psi\partial_t\boldsymbol{A}\right)_T \,\mathrm{d}t,$$

and

$$\begin{aligned}
\lim_{n\to\infty} R_1 &\leqslant -\int_\Omega \psi\mu\Phi_{\boldsymbol{M}}(\nabla\times\boldsymbol{A}(\zeta))\,\mathrm{d}x + \int_\Omega \psi\mu\Phi_{\boldsymbol{M}}(\nabla\times\boldsymbol{A}(0))\,\mathrm{d}x &&= -\int_0^\zeta\int_\Omega \psi\mu\frac{\mathrm{d}\Phi_{\boldsymbol{M}}(\nabla\times\boldsymbol{A})}{\mathrm{d}t}\,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_0^\zeta\int_\Omega \psi\mu\boldsymbol{M}(\nabla\times\boldsymbol{A})\cdot\partial_t\left(\nabla\times\boldsymbol{A}\right)\,\mathrm{d}x\,\mathrm{d}t &&= -\int_0^\zeta \left(\mu\boldsymbol{M}(\nabla\times\boldsymbol{A}), \psi\nabla\times(\partial_t\boldsymbol{A})\right)_\Omega \,\mathrm{d}t.
\end{aligned}$$

Thus

$$\begin{aligned}
\lim_{n\to\infty} R_1 + R_2 + R_3 &\leqslant -\int_0^\zeta \left(\mu\boldsymbol{M}(\nabla\times\boldsymbol{A}), \nabla\times(\psi\partial_t\boldsymbol{A})\right)_\Omega \,\mathrm{d}t - \int_0^\zeta \left(\sigma_\pi(u)\nabla\phi, \psi\partial_t\boldsymbol{A}\right)_T \,\mathrm{d}t \\
&\overset{(18)}{=} \int_0^\zeta \left(\psi\sigma_\pi(u)\partial_t\boldsymbol{A}, \partial_t\boldsymbol{A}\right)_\pi \,\mathrm{d}t.
\end{aligned}$$

Thus, collecting all estimates above, we can see that

$$0 \leqslant \lim_{n\to\infty}\int_0^\zeta\int_\pi \psi\left|\partial_t\boldsymbol{A}_n - \partial_t\boldsymbol{A}\right|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant 0.$$

Please note that this is valid for any non-negative $\psi \in C_0^\infty(\boldsymbol{\pi})$. Since the set of $\zeta \in [0, \mathcal{T}]$ for which $\nabla\times\bar{\boldsymbol{A}}_n(\zeta) \to \nabla\times\boldsymbol{A}(\zeta)$ in $\mathbf{L}^2(\Omega)$ is dense in $[0, \mathcal{T}]$, we achieve a strong convergence of $\partial_t\boldsymbol{A}_n$ in $L^2((0, \mathcal{T}); \mathbf{L}^2(\pi))$ i.e. $\partial_t\boldsymbol{A}_n \to \partial_t\boldsymbol{A}$

in $L^2((0,\mathcal{T});\mathbf{L}^2(\pi))$.

(v) Take $\psi \in H^1(\pi)$ in (29) and integrate in time

$$\left(\bar{\beta}_n(t)-\beta_n(0),\psi\right)_\pi + \left(\beta_n(t)-\bar{\beta}_n(t),\psi\right)_\pi + \int_0^t \left(\bar{\lambda}_n \nabla \bar{u}_n, \nabla\psi\right)_\pi \,\mathrm{d}s = \int_0^t \left(\mathcal{R}_r\left(\bar{\sigma}_{\pi_n}(s-\tau)\left|\partial_t \boldsymbol{A}_n + \chi_T \nabla\bar{\phi}_n\right|^2\right),\psi\right)_\pi \mathrm{d}s.$$

Using Lebesgue's dominated convergence theorem, together with the Proposition 1 (ii), Theorem 1 (ii) and (iv) enables passing to the limit for $n\to\infty$ in the RHS of the equation above

$$\lim_{n\to\infty}\int_0^t \left(\mathcal{R}_r\left(\bar{\sigma}_{\pi_n}(s-\tau)\left|\partial_t \boldsymbol{A}_n + \chi_T \nabla\bar{\phi}_n\right|^2\right),\psi\right)_\pi \mathrm{d}s = \int_0^t \left(\mathcal{R}_r\left(\sigma_\pi(u)\left|\partial_t \boldsymbol{A} + \chi_T \nabla\phi\right|^2\right),\psi\right)_\pi \mathrm{d}s.$$

Proposition 1 let us pass to the limit for $n\to\infty$ on the LHS. Note that term $\left(\beta_n(t)-\bar{\beta}_n(t),\psi\right)_\pi$ vanishes since $\lim_{n\to\infty}\left(\beta_n(t)-\bar{\beta}_n(t),\psi\right)_\pi = 0$ for every $t\in[0,\mathcal{T}]$. Therefore gathering all results above brings us to

$$\left(\beta(u(t))-\beta(u(0)),\psi\right)_\pi + \int_0^t \left(\lambda\nabla u,\nabla\psi\right)_\pi \,\mathrm{d}s = \int_0^t \left(\mathcal{R}_r\left(\sigma_\pi(u)\left|\partial_t \boldsymbol{A} + \chi_T \nabla\phi\right|^2\right),\psi\right)_\pi \mathrm{d}s.$$

The only thing left to be done to finish the proof is differentiating with respect to time. Thus, we can see that ϕ, u and $\boldsymbol{A}$ indeed solve (19). □

4. Numerical Simulation

To support our proposed numerical scheme (28) obtained from the variational formulation (25),(26),(27) we provide a numerical simulation of induction hardening process. The domain used in the simulation can be seen on Figure 3. This domain is more complex than its simplified version on Figure 1, but our theoretical results for this type hold regardless, because the inclusion $\boldsymbol{X}_{N,0}\subset\mathbf{H}^1(\Omega)$ holds true also for convex domains (without a smooth boundary), cf. [20, Theorem 2.17]. Since we want our simulation to be realistic we use physical constants. Unknown functions representing nonlinearities are chosen accordingly to satisfy (21)

$$\begin{aligned}
&\sigma_\pi(u) = 2\sigma_c + \sigma_c\left(2-\left(1+\frac{1}{1+u}\right)^{1+u}\right),\\
&\beta(u) = \beta_c\sqrt{u},\\
&\boldsymbol{M}(\nabla\times\boldsymbol{A}) = \left(1+e^{-|\nabla\times\boldsymbol{A}|}\right)\nabla\times\boldsymbol{A},\\
&\sigma_c,\beta_c,\mu,\lambda \implies \text{Physical constants.}\\
&\mathcal{T} = 0.02
\end{aligned}$$

We split the time interval $[0,\mathcal{T}]$ in 1280 equidistant parts ($\tau = 1.5625\mathrm{e}10^{-5}$) and use the open source finite element environment Gmsh/GetDP [28, 29], freely available online on `http://www.onelab.info`, to solve the system (26), (27) and (25) at each time step, after spatial discretization using Whitney finite elements on tetrahedra (edge elements for the magnetic vector potential, nodal elements for the electric scalar potential and the temperature) [30]. The mesh contained 26765 tetrahedra, leading to a total of 29714 unknowns. We denote obtained solutions for the magnetic induction field and the temperature function as $\boldsymbol{B}_N$ and u_N respectively. Typical solutions are plotted on Figure 4.

To show that our scheme is converging to $\boldsymbol{B}_N$ and u_N we compute other numerical solutions for number of time steps 10, 100 and 1000 and compare them with $\boldsymbol{B}_N$ and u_N. We analyze these solutions in certain measurement points on our domain (see Figure 5) and in certain time steps, namely $t_i = 0.002i$, where $i = 1,\ldots,10$. The relative error of a given numerical solution $\boldsymbol{B}_j$ from the solution $\boldsymbol{B}_N$ is then calculated in the following manner

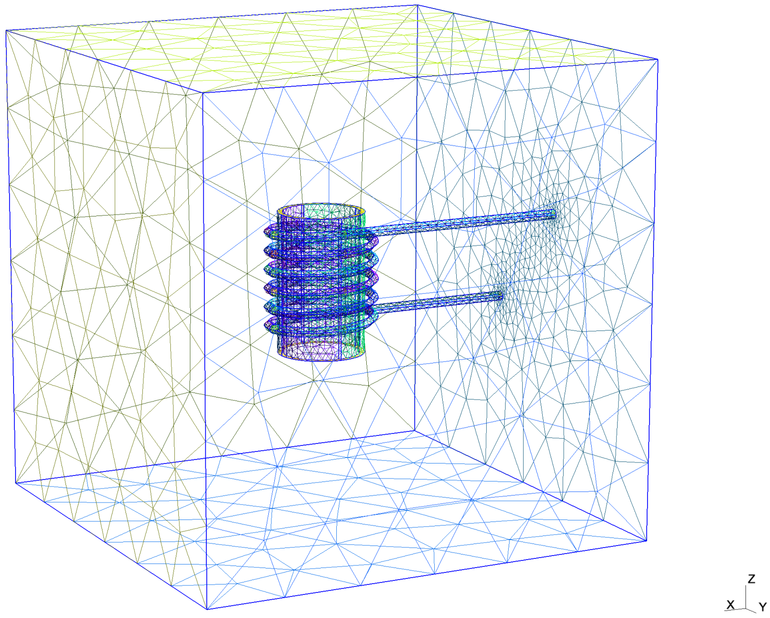


Figure 3: Meshed domain

$$
\begin{aligned}
|\boldsymbol{B}_N| &= \sum_{i=1}^{10} |\boldsymbol{B}_N(P_1,t_i)| + |\boldsymbol{B}_N(P_2,t_i)| + |\boldsymbol{B}_N(P_3,t_i)|\,,\\
|\boldsymbol{B}_N - \boldsymbol{B}_j| &= \sum_{i=1}^{10} |\boldsymbol{B}_N(P_1,t_i) - \boldsymbol{B}_j(P_1,t_i)| + |\boldsymbol{B}_N(P_2,t_i) - \boldsymbol{B}_j(P_2,t_i)| + |\boldsymbol{B}_N(P_3,t_i) - \boldsymbol{B}_j(P_3,t_i)|\,,\\
\text{RelError } \boldsymbol{B}_j &= \frac{|\boldsymbol{B}_N - \boldsymbol{B}_j|}{|\boldsymbol{B}_N|},
\end{aligned}
$$

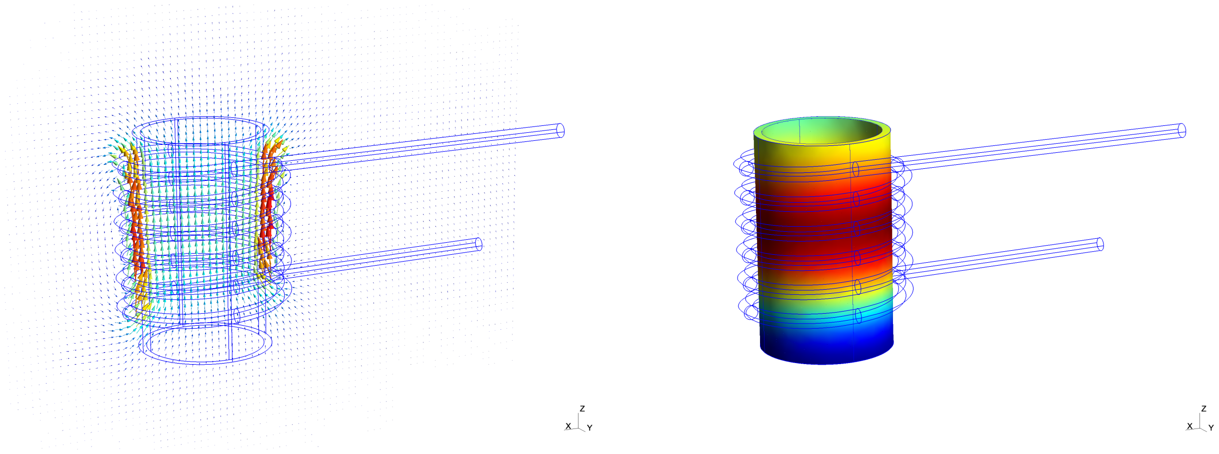
where P_1, P_2 and P_3 are representing the measurement points. Same approach is used to calculate the relative errors of u_j and u_N. The evolution of these errors with increasing number of time steps can be seen on Figure 6 .

5. Conclusion

We have provided a derivation of a mathematical model of induction hardening process with inclusion of a non-linear relation between the magnetic field and the magnetic induction field. We have also proven an existence of a weak solution for the weak formulation of our model.

To support the theoretical results we have coded the numerical scheme implied by a variational formulation and ran few simulations. However, we didn't have an analytic solution. Thus we have computed an "accurate" numerical solution setting the number of time steps to 1280. Afterwards we have investigated how the numerical solutions computed for the increasing number of time steps (starting at 10) were behaving according to the "accurate" solutions $\boldsymbol{B}_N$ and u_n. We have obtained an improving match with increasing number of time steps. Since we do not have a proof of a unique solution of our model we could not prove the convergence of the scheme rigourosly. However the numerical experiments suggest that the scheme might really be convergent.

In the following work we would like to provide a proof of a unique solution. The coupling between the vector potential equation and the heat equation in the form of the temperature dependant function $\sigma(u)$ causes numerous



(a) Magnetic induction field

(b) Temperature

Figure 4: Solutions in time $t = 0.015$

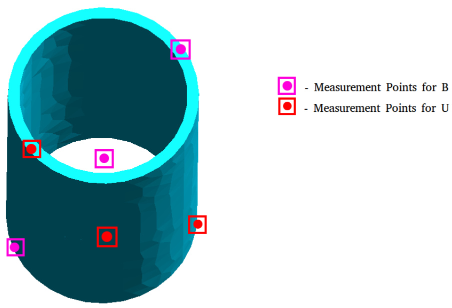


Figure 5: Measurement Points

troubles in the uniqueness proof and therefore it still remains an open problem.

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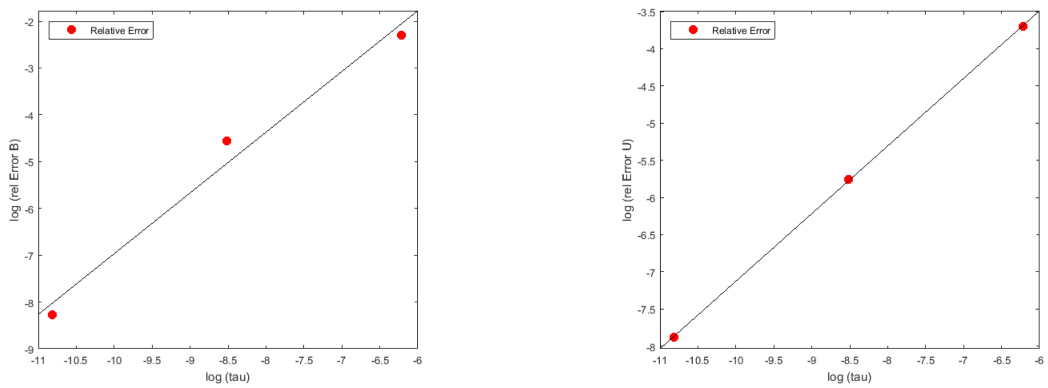

(a) Relative error for the magnetic induction field

(b) Relative error for temperature evolution

Figure 6: Logarithmically scaled plot of a size of the time step τ and the relative errors

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