# Chapter 4: Dynamic Programming

Objectives of this chapter:

- ❒ Overview of a collection of classical solution methods for MDPs known as dynamic programming (DP)
- ❒ Show how DP can be used to compute value functions, and hence, optimal policies
- ❒ Discuss efficiency and utility of DP



# Policy Evaluation

**Policy Evaluation**: for a given policy $\pi$, compute the state-value function $V^\pi$

Recall: **State - value function for policy $\pi$ :**

$$V^\pi(s) = E_\pi\left\{R_t \mid s_t = s\right\} = E_\pi\left\{\sum_{k=0}^{\infty} \gamma^k r_{t+k+1} \mid s_t = s\right\}$$

**Bellman equation for $V^\pi$ :**

$$V^\pi(s) = \sum_a \pi(s,a) \sum_{s'} \mathcal{P}_{ss'}^a \left[\mathcal{R}_{ss'}^a + \gamma V^\pi(s')\right]$$

— a system of $|S|$ simultaneous linear equations

## Iterative Methods

$$V_0 \rightarrow V_1 \rightarrow \cdots \rightarrow V_k \rightarrow V_{k+1} \rightarrow \cdots \rightarrow V^\pi$$

a "sweep"

A sweep consists of applying a **backup operation** to each state.

A **full policy-evaluation backup**:

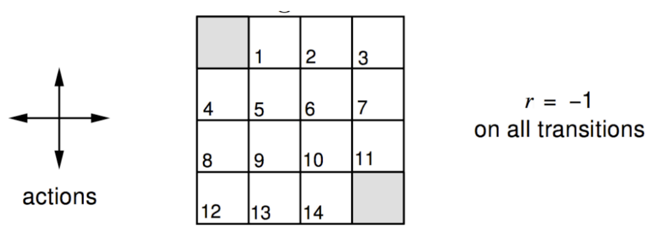
$$V_{k+1}(s) \leftarrow \sum_a \pi(s,a) \sum_{s'} \mathcal{P}_{ss'}^a \left[ \mathcal{R}_{ss'}^a + \gamma V_k(s') \right]$$



## Iterative Policy Evaluation

Input $\pi$, the policy to be evaluated
Initialize $V(s) = 0$, for all $s \in \mathcal{S}^+$
Repeat
  $\Delta \leftarrow 0$
  For each $s \in \mathcal{S}$:
    $v \leftarrow V(s)$
    $V(s) \leftarrow \sum_a \pi(s,a) \sum_{s'} \mathcal{P}_{ss'}^a [\mathcal{R}_{ss'}^a + \gamma V(s')]$
    $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$ (a small positive number)
Output $V \approx V^\pi$

# A Small Gridworld

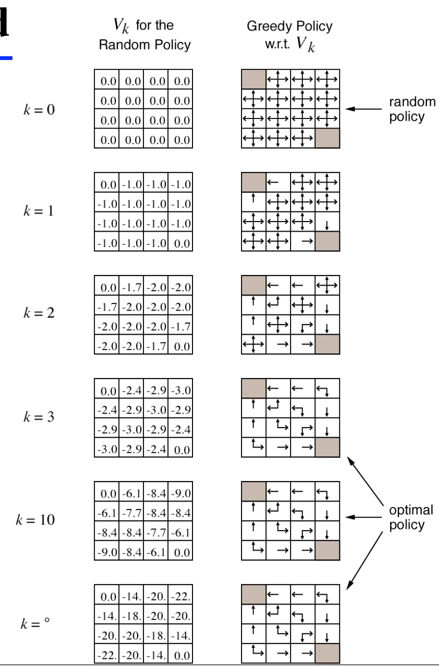



- An undiscounted episodic task
- Nonterminal states: 1, 2, . . ., 14;
- One terminal state (shown twice as shaded squares)
- Actions that would take agent off the grid leave state unchanged
- Reward is –1 until the terminal state is reached



# Iterative Policy Eval for the Small Gridworld

$\pi =$ equiprobable random action choices




$k = 0$

| | | | |
|---|---|---|---|
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |

$k = 1$

| | | | |
|---|---|---|---|
| 0.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | 0.0 |

$k = 2$

| | | | |
|---|---|---|---|
| 0.0 | -1.7 | -2.0 | -2.0 |
| -1.7 | -2.0 | -2.0 | -2.0 |
| -2.0 | -2.0 | -2.0 | -1.7 |
| -2.0 | -2.0 | -1.7 | 0.0 |

$k = 3$

| | | | |
|---|---|---|---|
| 0.0 | -2.4 | -2.9 | -3.0 |
| -2.4 | -2.9 | -3.0 | -2.9 |
| -2.9 | -3.0 | -2.9 | -2.4 |
| -3.0 | -2.9 | -2.4 | 0.0 |

$k = 10$

| | | | |
|---|---|---|---|
| 0.0 | -6.1 | -8.4 | -9.0 |
| -6.1 | -7.7 | -8.4 | -8.4 |
| -8.4 | -8.4 | -7.7 | -6.1 |
| -9.0 | -8.4 | -6.1 | 0.0 |

$k = °$

| | | | |
|---|---|---|---|
| 0.0 | -14. | -20. | -22. |
| -14. | -18. | -20. | -20. |
| -20. | -20. | -18. | -14. |
| -22. | -20. | -14. | 0.0 |

## Policy Improvement

Suppose we have computed $V^\pi$ for a deterministic policy $\pi$.

For a given state $s$,
would it be better to do an action $a \neq \pi(s)$?

The value of doing $a$ in state $s$ is:

$$Q^\pi(s,a) = E_\pi\left\{r_{t+1} + \gamma V^\pi(s_{t+1}) \middle| s_t = s, a_t = a\right\}$$
$$= \sum_{s'} \mathcal{P}^a_{ss'}\left[\mathcal{R}^a_{ss'} + \gamma V^\pi(s')\right]$$

It is better to switch to action $a$ for state $s$ if and only if

$$Q^\pi(s,a) > V^\pi(s)$$



## Policy Improvement Cont.

Do this for all states to get a new policy $\pi'$ that is **greedy** with respect to $V^\pi$:

$$\pi'(s) = \arg\max_a Q^\pi(s,a)$$
$$= \arg\max_a \sum_{s'} \mathcal{P}^a_{ss'}\left[\mathcal{R}^a_{ss'} + \gamma V^\pi(s')\right]$$

Then $V^{\pi'} \geq V^\pi$

## Policy Improvement Cont.

What if $V^{\pi'} = V^{\pi}$ ?

i.e., for all $s \in S$, $V^{\pi'}(s) = \max_a \sum_{s'} \mathcal{P}^a_{ss'} \left[ \mathcal{R}^a_{ss'} + \gamma V^{\pi}(s') \right]$ ?

But this is the Bellman Optimality Equation.

So $V^{\pi'} = V^*$ and both $\pi$ and $\pi'$ are optimal policies.



## Policy Iteration

$$\pi_0 \to V^{\pi_0} \to \pi_1 \to V^{\pi_1} \to \cdots \pi^* \to V^* \to \pi^*$$

policy evaluation

policy improvement
"greedification"

## Policy Iteration

1. Initialization
   $V(s) \in \Re$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation
   Repeat
   $\Delta \leftarrow 0$
   For each $s \in \mathcal{S}$:
   $v \leftarrow V(s)$
   $V(s) \leftarrow \sum_{s'} \mathcal{P}_{ss'}^{\pi(s)} \left[ \mathcal{R}_{ss'}^{\pi(s)} + \gamma V(s') \right]$
   $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
   until $\Delta < \theta$ (a small positive number)

3. Policy Improvement
   *policy-stable* $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
   $b \leftarrow \pi(s)$
   $\pi(s) \leftarrow \arg\max_a \sum_{s'} \mathcal{P}_{ss'}^{a} \left[ \mathcal{R}_{ss'}^{a} + \gamma V(s') \right]$
   If $b \neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*
   If *policy-stable*, then stop; else go to 2

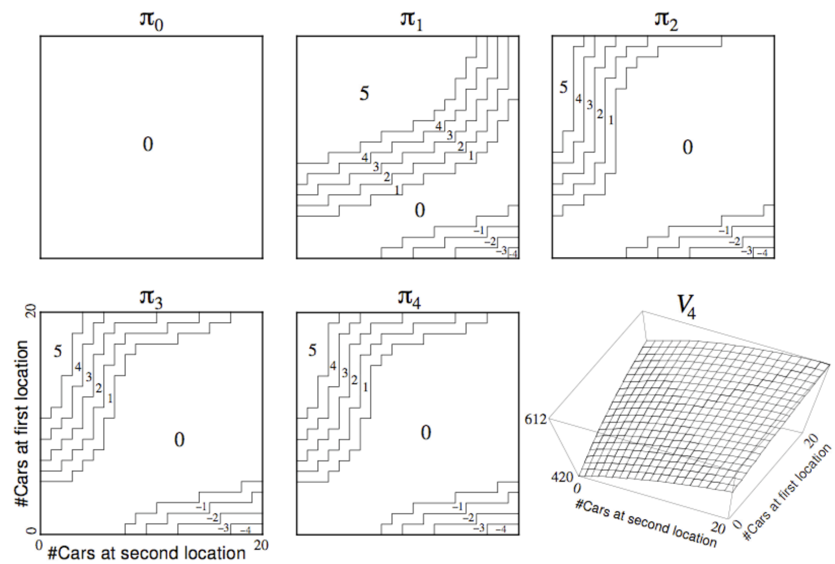


## Jack's Car Rental

- \$10 for each car rented (must be available when request rec'd)
- Two locations, maximum of 20 cars at each
- Cars returned and requested randomly
  - Poisson distribution, $n$ returns/requests with prob $\frac{\lambda^n}{n!} e^{-\lambda}$
  - 1st location: average requests = 3, average returns = 3
  - 2nd location: average requests = 4, average returns = 2
- Can move up to 5 cars between locations overnight

- States, Actions, Rewards?
- Transition probabilities?

# Jack’s Car Rental






# Jack’s CR Exercise

- Suppose the first car moved is free
  - From 1st to 2nd location
  - Because an employee travels that way anyway (by bus)
- Suppose only 10 cars can be parked for free at each location
  - More than 10 cost $4 for using an extra parking lot
- Such arbitrary nonlinearities are common in real problems

## Value Iteration

Recall the **full policy-evaluation backup**:

$$V_{k+1}(s) \leftarrow \sum_a \pi(s,a) \sum_{s'} \mathcal{P}^a_{ss'} \left[ \mathcal{R}^a_{ss'} + \gamma V_k(s') \right]$$

Here is the **full value-iteration backup**:

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} \mathcal{P}^a_{ss'} \left[ \mathcal{R}^a_{ss'} + \gamma V_k(s') \right]$$



## Value Iteration Cont.

Initialize $V$ arbitrarily, e.g., $V(s) = 0$, for all $s \in \mathcal{S}^+$

Repeat
    $\Delta \leftarrow 0$
    For each $s \in \mathcal{S}$:
        $v \leftarrow V(s)$
        $V(s) \leftarrow \max_a \sum_{s'} \mathcal{P}^a_{ss'}[\mathcal{R}^a_{ss'} + \gamma V(s')]$
        $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$ (a small positive number)

Output a deterministic policy, $\pi$, such that
    $\pi(s) = \arg\max_a \sum_{s'} \mathcal{P}^a_{ss'} \left[ \mathcal{R}^a_{ss'} + \gamma V(s') \right]$

# Gambler's Problem

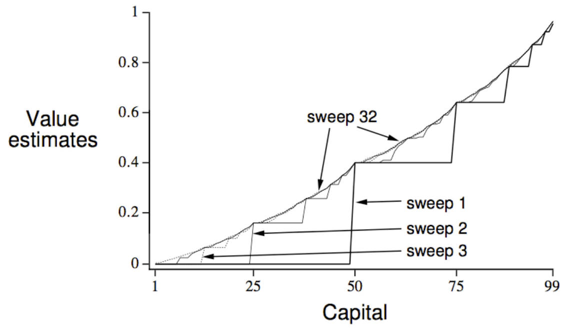
- Gambler can repeatedly bet \$ on a coin flip
- Heads he wins his stake, tails he loses it
- Initial capital $\in$ {\$1, \$2, … \$99}
- Gambler wins if his capital becomes \$100
  loses if it becomes \$0
- Coin is unfair
  - Heads (gambler wins) with probability $p$ = .4

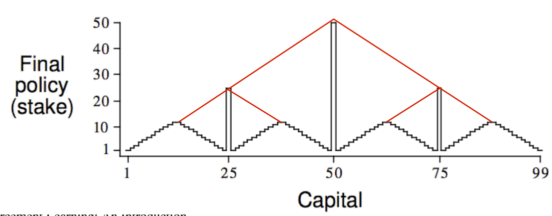
$$\frac{\lambda^n}{n!}e^{-\lambda}$$

- States, Actions, Rewards?



# Gambler's Problem Solution



## Herd Management

- You are a consultant to a farmer managing a herd of cows
- Herd consists of 5 kinds of cows:
  - Young
  - Milking
  - Breeding
  - Old
  - Sick
- Number of each kind is the State
- Number sold of each kind is the Action
- Cows transition from one kind to another
- Young cows can be born

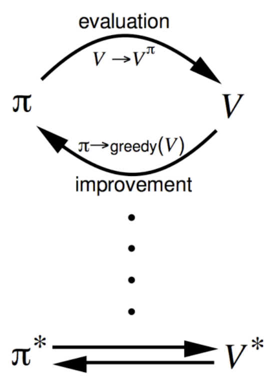


## Asynchronous DP

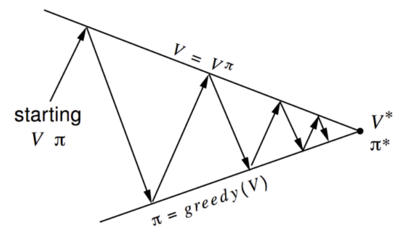
- All the DP methods described so far require exhaustive sweeps of the entire state set.
- Asynchronous DP does not use sweeps. Instead it works like this:
  - Repeat until convergence criterion is met:
    - Pick a state at random and apply the appropriate backup
- Still need lots of computation, but does not get locked into hopelessly long sweeps
- Can you select states to backup intelligently? YES: an agent's experience can act as a guide.

# Generalized Policy Iteration

**Generalized Policy Iteration** (GPI):
any interaction of policy evaluation and policy improvement, independent of their granularity.




A geometric metaphor for convergence of GPI:






# Efficiency of DP

- To find an optimal policy is polynomial in the number of states…
- BUT, the number of states is often astronomical, e.g., often growing exponentially with the number of state variables (what Bellman called "the curse of dimensionality").
- In practice, classical DP can be applied to problems with a few millions of states.
- Asynchronous DP can be applied to larger problems, and appropriate for parallel computation.
- It is surprisingly easy to come up with MDPs for which DP methods are not practical.

# Summary

- Policy evaluation: backups without a max
- Policy improvement: form a greedy policy, if only locally
- Policy iteration: alternate the above two processes
- Value iteration: backups with a max
- Full backups (to be contrasted later with sample backups)
- Generalized Policy Iteration (GPI)
- Asynchronous DP: a way to avoid exhaustive sweeps
- **Bootstrapping**: updating estimates based on other estimates