Consensus in nonlinear spaces

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Distributed control of E-ELT primary mirror

• Gigantic control system
• Tight performance criteria
• Multiscale: nanoscale precision over 1000 m^2!

Relative sensing is a key feature of distributed system theory

• In spite of a centralized control architecture, centralized PI control does not outperform distributed (localized) PI control
• Relative sensing is the fundamental source of performance limitation

Integral control from distributed sensing: an Extremely Large Telescope case study,
A. Sarlette, Ch. Bastin, M Dimmler, B Sedghi, T. Erm, B. Bauvir, R. Sepulchre,
Consensus as distributed regulation

- An agreement (= regulation) problem over the relative state (= height) of agents (=mirrors)

- The communication constraints are encoded in a graph:

  ![Graph Diagram]

  $i \sim j$: node $i$ sends information to node $j$

- The possible external reference is just one (virtual) agent

Designing collective motions on $SE(2)$ as a representative example

$N$ autonomous rigid bodies moving in the plane at unit speed

$\dot{r}_k = e^{i \theta_k}$

$\dot{\theta}_k = u_k$

(curvature control)

Goal: design feedback control to stabilize collective motions

Restrictions: limited communication

no reference, no leader

Collective motion involves nonlinear manifolds

Common direction for straight motion $\Rightarrow$ agreement on circle

General motion “in formation” $\Rightarrow$ Lie group $SE(2)$

translations $\mathbb{R}^2$

rotations $S^1$

non-trivial coupling
Agreement on the circle also appears for phase synchronization of oscillator networks:

- Flashing fireflies
- Huygens' clocks
- Laser tuning
- Cell / neuron action

Consensus/synchronization/coordination problems often involve nonlinear spaces.

Six years later:

Geometry and Symmetries in Coordination Control
Alain Sarlette

Also: Derek Paley, Luca Scardovi, Pierre Rouchon, Silvère Bonnabel, Emre Tuna, Pierre-Antoine Absil, Vincent Blondel, ...

Consensus on nonlinear spaces:

- Linear consensus
- Riemannian consensus and consensus on the circle
- Coordination on nonlinear spaces
- Outlook
Linear-quadratic consensus

\( N \) agents with state \( x_i \in \mathbb{R}^n, \ i = 1 \ldots N \)

disagreement cost function:
\[
\frac{1}{2(N-1)} \sum_{i=1}^{N} \sum_{j \neq i} \| x_i - x_j \|^2
\]

Gradient descent:
\[
\dot{x}_i = -x_i + \frac{1}{N-1} \sum_{i \neq j} x_j
\]

arithmetic average of other agents values

\[= \arg\min \sum_{j \neq i} \| x_i - x_j \|^2 \]

A strong convergence result

"Move towards the average of your neighbors"
\[
\dot{x}_i = -x_i + \text{mean}(x_j : j \rightarrow i)
\]
\[
x_i^+ = \alpha x_i + (1 - \alpha) \text{mean}(x_j : j \rightarrow i), \quad 0 < \alpha < 1
\]

THM: Uniform convergence if uniform connectedness


Consensus over a communication graph

\( N \) agents with state \( x_i \in \mathbb{R}^n, \ i = 1 \ldots N \)

generalized update:
\[
\dot{x}_i = -x_i + \frac{1}{d_i} \sum_{j \rightarrow i} x_j
\]

arithmetic average of neighbors

\[= \arg\min_x \sum_{j \rightarrow i} \| x - x_j \|^2 \]

"Move towards the average of your neighbors"

! No longer gradient if directed and/or time-varying graph!

Consensus value

\[
\dot{x}_i = -x_i + \text{mean}(x_j : j \rightarrow i)
\]
\[
x_i^+ = \alpha x_i + (1 - \alpha) \text{mean}(x_j : j \rightarrow i), \quad 0 < \alpha < 1
\]

For a balanced communication graph, the asymptotic consensus value is
\[
\text{mean}(x_j : 1 \leq j \leq N)
\]

= Distributed computation of a global quantity through local computations.

= Distributed optimization of a global disagreement cost
A nonquadratic Lyapunov convergence analysis

- Tsitsiklis Lyapunov function ($x_i \in \mathbb{R}$)

$$V(x) = \max_i x_i - \min_i x_i$$

is non-increasing.

- Moreau generalization: the convex hull of $x_j : 1 \leq j \leq N$ can only shrink over time.

- In general, no common time-invariant quadratic Lyapunov function.

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Mean and consensus on manifolds

Given $x_1, ..., x_N \in M$ and a distance $d(,)$ on $M$:

\[
\text{mean} = \arg \min_x \sum_{j=1}^{N} d^2(x, x_j)
\]

\[
\text{consensus if each point is the mean of its neighbors:}
\]

\[
x_k \in \arg \min_x \sum_{j \sim k} d^2(x, x_j)
\]

Therefore, consensus minimizes the cost

\[
\frac{1}{N} \sum_{k=1}^{N} \sum_{j \sim k} d^2(x_k, x_j)
\]

Consensus algorithm: direct each point towards the mean of its neighbors

Riemannian consensus

- Well-defined on a geodesically convex set.
- Straightforward generalization of linear consensus theory
- Fundamentally different on a non geodesically convex set
Chordal distance and mean on the circle:

- Embed $S^1$ in $\mathbb{C}$: $\theta \rightarrow e^{i\theta}$
- Measure distance in $\mathbb{C}$: $d(\theta_1, \theta_2) = \| e^{i\theta_1} - e^{i\theta_2} \|_2$
- Chordal mean:
  
  $\text{mean}(\theta_1, \ldots, \theta_N) = \arg\min_{\theta} \frac{1}{N} \sum_{k=1}^{N} \| e^{i\theta} - e^{i\theta_k} \|_2^2$

$\text{mean}(\theta_1, \ldots, \theta_N) = \arg\left( \sum_{k=1}^{N} e^{i\theta_k} \right)$

Chordal consensus on the circle

$N$ angles $\theta_1, \ldots, \theta_N \in S^1$

Cost function:

$- \frac{1}{N} \sum_{k=1}^{N} \sum_{j \neq k} \cos(\theta_k - \theta_j)$

Consensus update:

$\dot{\theta}_k = - \sum_{j \neq k} \sin(\theta_k - \theta_j)$

Vicsek model for heading synchronization

$N$ autonomous rigid bodies moving in the plane at unit speed

Unit velocity:

$x_k(t+1) = x_k + e^{i\theta_k}$

“Average” direction:

$\theta_k(t+1) = \arg(e^{i\theta_k} + \frac{1}{d_k} \sum_{j \neq k} e^{i\theta_j})$

Proximity graph: communicate if closer than $R$
Kuramoto phase model of coupled oscillators


\[ \dot{\theta}_k = \omega_k - \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_k - \theta_j) \]

\( P(t) = \| \frac{1}{N} \sum_{k=1}^{N} e^{i\theta_k} \| \)

Kuramoto “order parameter”

= chordal consensus cost function for complete graph

K measures the coupling strength (relative to heterogeneity)

Chordal mean on embedded manifolds:

• Embed \( M \) in \( E \) : \( x \rightarrow \hat{x} \)

• Measure distance in \( E \): \( d(x_1, x_2) = \| \hat{x}_1 - \hat{x}_2 \|_2 \)

• Chordal mean:

\[ \text{mean}(x_1, \ldots, x_N) = \arg\min_{\hat{x}} \frac{1}{N} \sum_{k=1}^{N} \| \hat{x} - \hat{x}_k \|_2^2 \]

\( ||\hat{x}||_2 = 1 \Rightarrow \arg\max_{\hat{x}} \left( \sum_{k=1}^{N} \frac{\hat{x}_k}{N} \right) \)

Chordal consensus on the orthogonal group

N orthonormal matrices

\[ Q_1, \ldots, Q_N \in \mathbb{R}^{n \times n}, Q_i Q_i^T = I_n, 1 \leq i \leq N \]

Cost function:

\[ -\frac{1}{N} \sum_{k=1}^{N} \sum_{j \rightarrow k} \text{trace}(Q_j^T Q_k) \]

Consensus update:

\[ \dot{Q}_k = Q_k \sum_{j \rightarrow k} (Q_k^{-1} Q_j - Q_j^{-1} Q_k) \]

(e.g. F. Bullo, PhD thesis, for tracking on SO(3))

Chordal consensus between subspaces

N subspaces represented by projectors

\[ \Pi_1, \ldots, \Pi_N \in \mathbb{R}^{n \times n}, \Pi_k \succeq 0, \text{trace} \Pi_k = p, 1 \leq i \leq N \]

Cost function:

\[ -\frac{1}{N} \sum_{k=1}^{N} \sum_{j \rightarrow k} \text{trace}(\Pi_j^T \Pi_k) \]

Consensus update:

\[ \dot{\Pi}_k = \sum_{j \rightarrow k} (\Pi_k \Pi_j \Pi_k + \Pi_k \Pi_j \Pi_k) \]

(e.g. J. Conway, 1996, for Grassmannian packings)
Chordal consensus on embedded manifolds

The chordal distance yields closed-form means and consensus algorithms if the following properties hold:

- $M$ is a connected compact homogeneous manifold;
- The embedding is symmetry-preserving
- Linear functions in $E$ have a unique minimum (maximum) in $M$.

Examples include

- $S^n$, $SO(n)$, $Gr(p, n)$, $St(p, n)$, ... 

*Consensus optimization on manifolds*,
A. Sarlette, R.S., SIAM J. Control & Optimization, 2009.

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Coordination on manifolds relates to many other engineering problems

### Packing

- points on spheres

### Clustering

- lines, subspaces of $\mathbb{R}^n$ (Grassmann manifold)
- positive definite matrices (covariance, kernels, ...)

Applications: optimal coding, beam / sensor placement, numerical integration, data mining, ...

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Convergence is challenging on manifolds

- Algorithms are nonlinear: $\dot{\theta}_k = \sum_{j \sim k} \sin(\theta_j - \theta_k)$

- Few global results. Graph dependent.
- Many open questions.
Algorithms for (almost) global synchronization on manifolds

- Augmented communication: Local tracking of a reference + consensus on the reference in vector spaces (relevant in design problems)

- A “gossip” algorithm: randomness favors synchronization and global spreading of information
  A. Sarlette, E. Tuna, V. Blondel, R.S., Proc. 17th IFAC World Congress, 2008

- Shaping the metric (for fixed undirected graphs)
  A. Sarlette, PhD Dissertation, 2009.

Increased information between neighbors to recover vector space convergence properties

Idea: associate to each agent a local reference \( x_{k}^{ref} \in \mathbb{C} \)

1. synchronize the \( x_{k}^{ref} \) (vector space consensus)

2. each \( \theta_{k} \) tracks the projection of \( x_{k}^{ref} \) on \( S^{f} \)

main idea to extend all-to-all collective stabilization results (TAC 2007) to uniformly connected collective stabilization results (TAC2008)

Gossiping for global synchronization

- at each instant, each agent randomly selects one neighbor among available communication links. The probabilistic setting “breaks the symmetry”.

Gossiping for global synchronization

- sync with probability one: for all initial conditions, \( e > 0 \) and \( d \in (0,1) \), \( \exists T \) such that
  \[ \text{Prob}( \max_{j,k} |x_{k}(T) - x_{j}(T)| < e ) > d \]

Thm: if \( G(t) \) is uniformly connected, gossiping achieves global asymptotic synchronization with probability one.
**Consensus on manifolds**

Several existing algorithms (Tsitsiklis, Viczek, Kuramoto) are unified by a geometric viewpoint on consensus.

When designing a consensus algorithm on a smooth manifold, the main issue is to choose a distance. The chordal distance appears to be a sound choice in several applications.

Several new algorithms for “global” consensus on highly symmetric spaces like the circle.

A. Sarlette, R.S., Synchronization on the circle, arxiv 2009.

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**Consensus on nonlinear spaces**

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- Riemannian consensus and consensus on the circle
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- Outlook

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**Designing collective motions on $\text{SE}(2)$ as a representative example**

$N$ autonomous rigid bodies moving in the plane at unit speed

\[ \dot{\theta}_k = e^{i\theta_k} \quad \dot{r}_k = u_k \quad \theta_k = u_k \quad \text{(curvature control)} \]

Goal: design feedback control to stabilize collective motions

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**Coordination in vector spaces**

Consider $N$ point masses in $\mathbb{R}^n$.

*Coordinated motion:* $N$ points moving like a single one

- Synchronization of velocities
- A consensus problem for the velocity model $\dot{v}_j = u_j$
- A relative equilibrium: constant relative positions

*Symmetry-preserving coordination:*

Invariance with respect to the reference frame

- only relative positions matter

Note: coordination differs from (state) synchronization
Coordination in SE(2)
Consider \( N \) rigid bodies in the plane

What does mean "relative position"?

How to compare velocities?

Which coordinated motions are possible?

How to check that a control law is "invariant"?

The basic ideas

A Lie Group is a manifold with an (invertible) translation operation (the analog of "+" in a vector space)

Relative position should be invariant by translation (\( r_k - r_j \) in a vector space)

A motion is coordinated when relative positions are constant (moving like a single object)

Coordination should be characterized by the velocities. On a Lie Group, they can be compared thanks to the translation operation. (\( v_k = v_j \) in a vector space)

Coordination on a Lie Group: definition

The product \( g_k^{-1} g_j \) defines a left-invariant relative position between \( g_k \) and \( g_j \)

Left-invariant coordination (LIC) means constant \( g_k^{-1} g_j \)

The product \( g_j g_k^{-1} \) defines a right-invariant relative position between \( g_k \) and \( g_j \)

Right-invariant coordination (RIC) means constant \( g_j g_k^{-1} \)

Bi-invariant coordination means LIC + RIC

(Note: in vector space: relative position = \( T_k - T_j \), LIC=RIC=BIC)
Relative positions on $\text{SE}(2)$:

$$g_k^{-1} g_j = \begin{pmatrix} R_{\theta_j} - \theta_k & R_{-\theta_k} (r_j - r_k) \\ 0 & 0 & 1 \end{pmatrix}$$

$$g_j g_k^{-1} = \begin{pmatrix} R_{\theta_j} - \theta_k & r_j - R_{\theta_j} - \theta_k r_k \\ 0 & 0 & 1 \end{pmatrix}$$

On Lie groups, velocities can be compared thanks to the translation operation

$$\mathbf{v} = g \mathbf{\xi}' g^{-1} = \text{Ad}(g) \mathbf{\xi}' \quad \text{with} \quad \mathbf{\xi}' \text{ and } \mathbf{\xi}'' \in T_e \mathbf{G} = \mathbf{g}$$

Collective motion is consensus on Lie group velocities

**Thm**: Left-invariant coordination $\Leftrightarrow \mathbf{\xi}_k' = \mathbf{\xi}_j'$

$$\frac{d}{dt} (s_k^{-1} a_j) = g_k^{-1} a_j (s_k^{-1} a_j)' - s_k^{-1} a_j' g_k^{-1} a_j$$

$$= g_k^{-1} (s_k^{-1} a_j g_k^{-1} a_j') g_j$$

$$= g_k^{-1} (\xi_j' - \xi_k') a_j$$

**Thm**: Right-invariant coordination $\Leftrightarrow \mathbf{\xi}_k'' = \mathbf{\xi}_j''$

Advantage: Lie group velocities are in $\mathbf{g} = \text{vector space}$

Lie group velocities have a physical meaning on $\text{SE}(2)$

$$\frac{d}{dt} g_k = \begin{pmatrix} R_{\theta_k} & \mathbf{v}_k^T \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{v}_k = R_{\theta_k} e_1 - \omega_k R_{\pi/2}$$

Left invariant coordination suggests the naive (but correct) Lyapunov function

$$\frac{1}{N} \sum_{k=1}^{N} \sum_{j \neq k} \| v_k' - v_j' \|^2$$
Both types of collective motion have a physical meaning on $SE(2)$

- Right coordination: same velocity in body frame
- Left coordination: constant relative position & heading

Steering control implies additional constraints

The Lie group framework allows to characterize and design control for coordinated motion

- "Relative positions", "coordination", "movement in formation" are defined by Lie group properties
- Collective motion $\Leftrightarrow$ synchronization of Lie group velocities (consensus in vector space)

Motion "in formation" is not trivial for underactuated agents, but combining consensus and geometric Lyapunov functions yields appropriate controls.

A. Sarlette, S. Bonnabel, R.S., Coordinated design motion on Lie Groups, 2010.

The geometric setting facilitates extensions

Coupling collective motion with particular configurations

Other Lie groups, e.g. $SE(3)$: rigid bodies in 3 dimensions

Consensus on nonlinear spaces

- Linear consensus
- Riemannian consensus and consensus on the circle
- Coordination on nonlinear spaces
- The nonlinear nature of linear consensus
Back to linear consensus

Linear consensus algorithms are linear time-varying systems
\[ x(t + 1) = A(t)x(t), \]
where for each \( t \), \( A(t) \) is row stochastic, i.e.
A is nonnegative: \( a_{ij} \geq 0 \)
each row sums to one: \( A(t)1 = 1 \)

Birkhoff theorem (1957)

Let \( K \) a closed solid cone in \( X \) a Banach space, with partial ordering \( \leq \).
A is positive if \( A \) maps \( \mathbb{K} \) to \( \mathbb{K} \).
A is monotone if \( x \leq y \Rightarrow Ax \leq Ay \)

Theorem: Positive linear monotone mappings contract the Hilbert metric:
\[ d(Ax, Ay) < d(x, y) \]
In the positive orthant, the Hilbert metric is
\[ d(x, y) = \log \frac{\max(x_i/y_i)}{\min(x_i/y_i)} \]

Tsitsiklis Lyapunov function

- Translate initial conditions in the positive orthant
- Consequence of Birkhoff theorem: for nonnegative maps that satisfy \( A(t)1=1 \), the Lyapunov function
\[ V(x) = \max x_i - \min x_i \]
cannot increase along solutions.
The Hilbert distance to consensus is equivalent to Tsitsiklis Lyapunov function in log coordinates.
(and captures the invariance property \( d(\mathbb{R}^+x, \mathbb{R}^+y) = d(x, y) \)).

Remark: both are measures of \( \text{co}\{x_1, \ldots, x_n\} \)
(Moreau’s Lyapunov function).
Conclusions

Distributed system theory = system theory + a fundamental invariance property

• Consensus is decentralized regulation without a reference

• In many applications, the state-space is a nonlinear manifold.

• Consensus algorithms are distributed computation of averages. In nonlinear spaces, issues involve choosing a distance and achieving “global” convergence

• Coordination is decentralized tracking without a reference... It can be studied comprehensively on Lie Groups

• Due to symmetry, even linear consensus is not so linear...