Dynamics and control of bounce juggling

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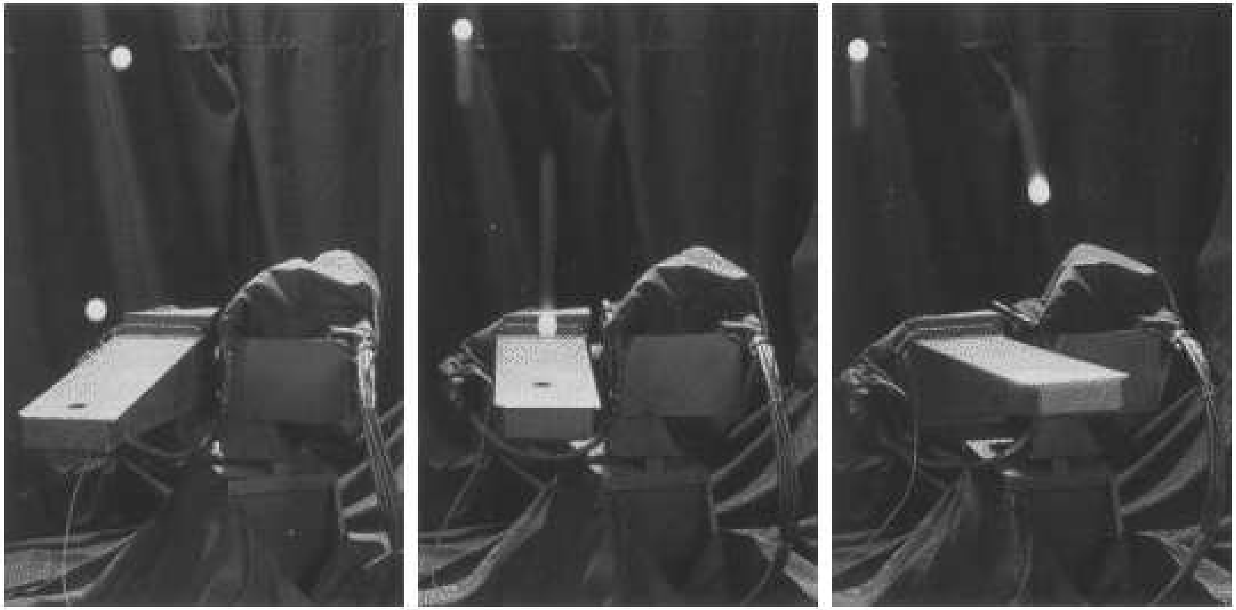
Human bounce juggling in a wedge

Courtesy from Greg Kennedy

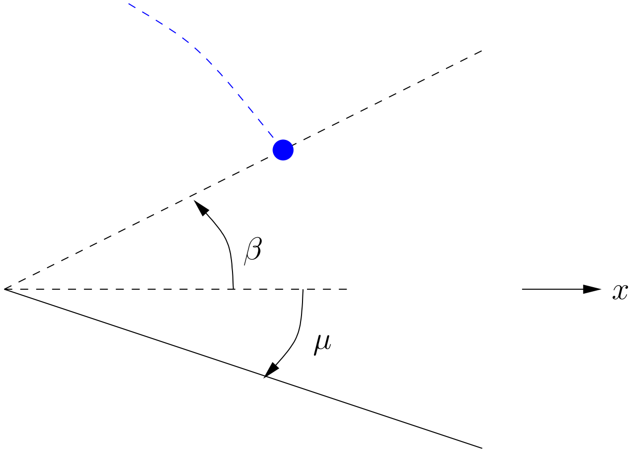
Periodic orbits in an elastic wedge

A control problem:
stabilization through
wedge actuation . . .

Buhler and Koditschek juggler ('94)

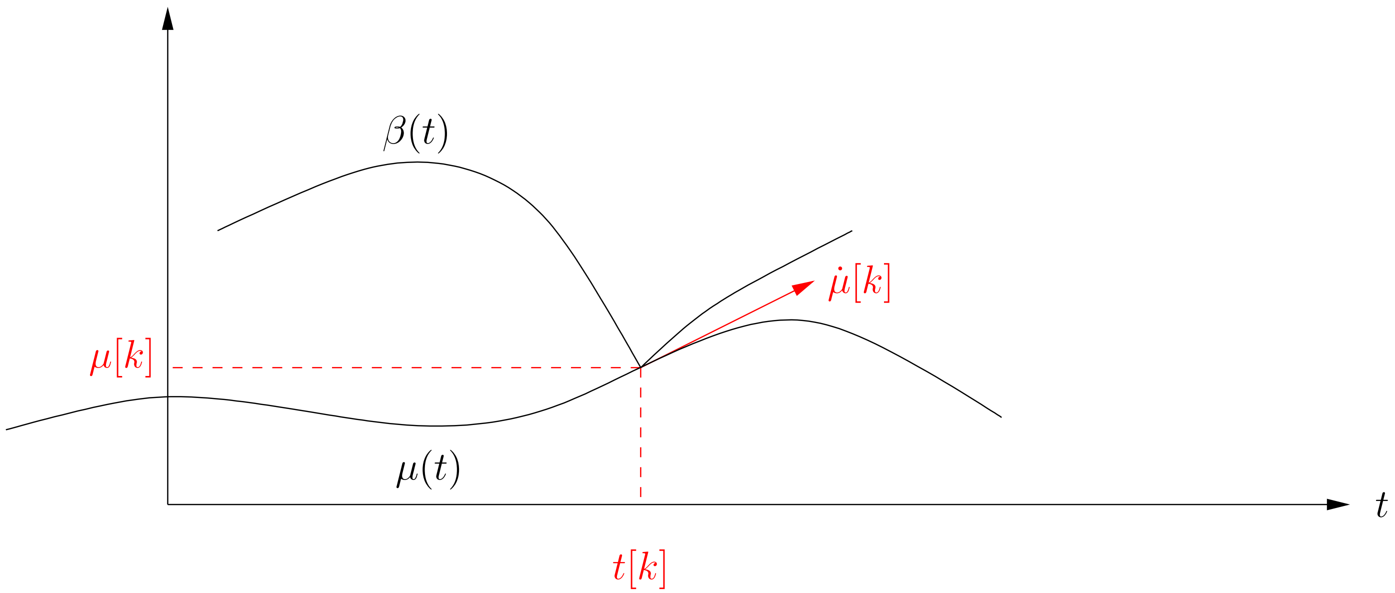


Mirror law algorithm: $\mu(t) = k_1 \Delta E(t) \beta(t) + k_2 \Delta \dot{x}(t)$



Works great! Why?

Intermittent feedback control



Continuous-time actuation (and sensing) to produce discrete (feedback) actions!
Stabilization results from occasional interactions (collisions) between actuated system and unactuated system

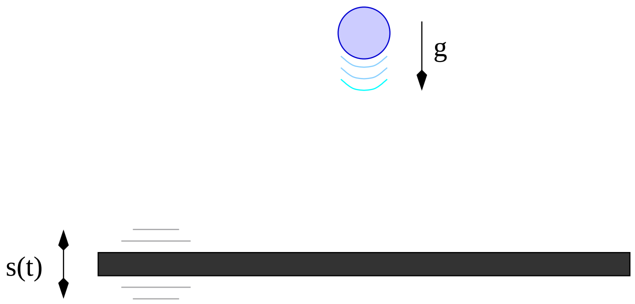
Note: sensorless control $u(t)$ produces 'feedback' $u(X[k])$

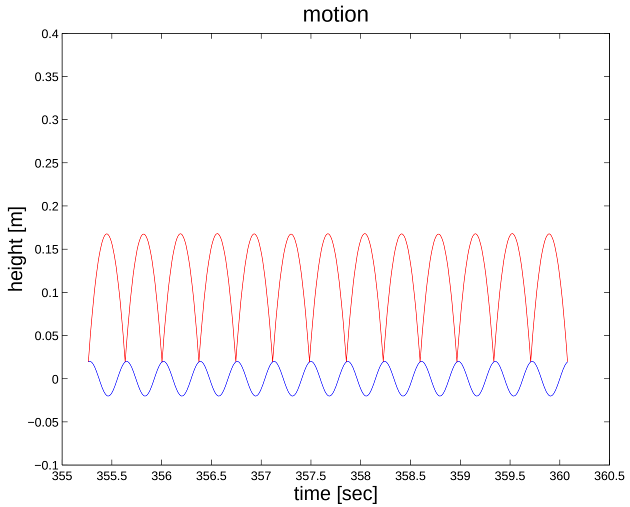
Robotic applications

Rhex robot from Buhler, Koditschek and coworkers

- Rhythmic control is fundamental to biolocomotion: walking, swimming, hopping, flying
- How much and what type of feedback is needed ?

Another motivation





Bouncing ball dynamics (Holmes ’82)

Elastic case: a Lyapunov stable orbit becomes exponentially stable under proper sinusoidal actuation of the floor.

Control as (sensorless) interconnection?

Human performance in rhythmic tasks

- Dynamics of a bouncing ball in human performance [Sternad et al., 2000]
- Stability and phase locking in human soccer juggling [Tlili et al., 2004]
- Human Motor Behavior: An Introduction [Kelso, 1982].

Feedback tracking or sensorless phase-locking?

The wiper

A juggling device amenable to

- mathematical analysis
- robotic implementation
- human experimentation

Previous work in juggling robotics

- modelling and stabilization through intermittent control (Buhler, Koditschek, and coworkers)
- real-time trajectory planning + experimental air table (Lynch)
- modelling and controllability of impact control systems (Brogliato and coworkers)
- Legged locomotion (Raibert, Mc Geer, Coleman, Holmes, Grizzle et al. . . .)
- Rhythmic tasks control [Atkeson, Schaal, Sternard, . . .]

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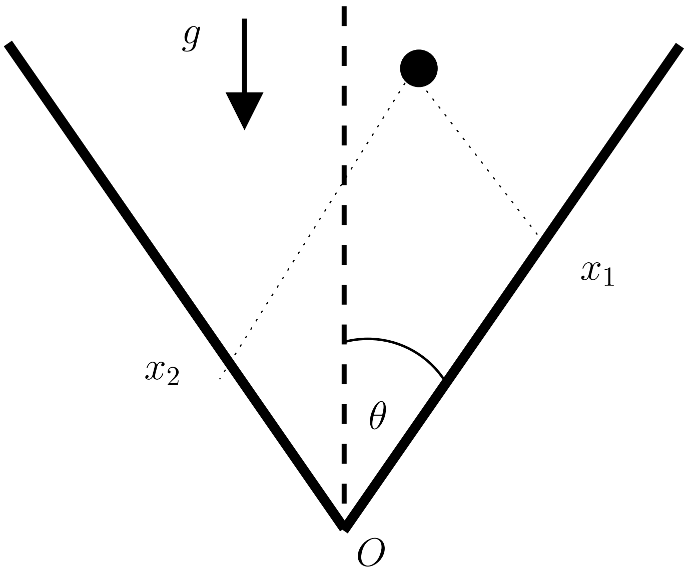
The wiper dynamics

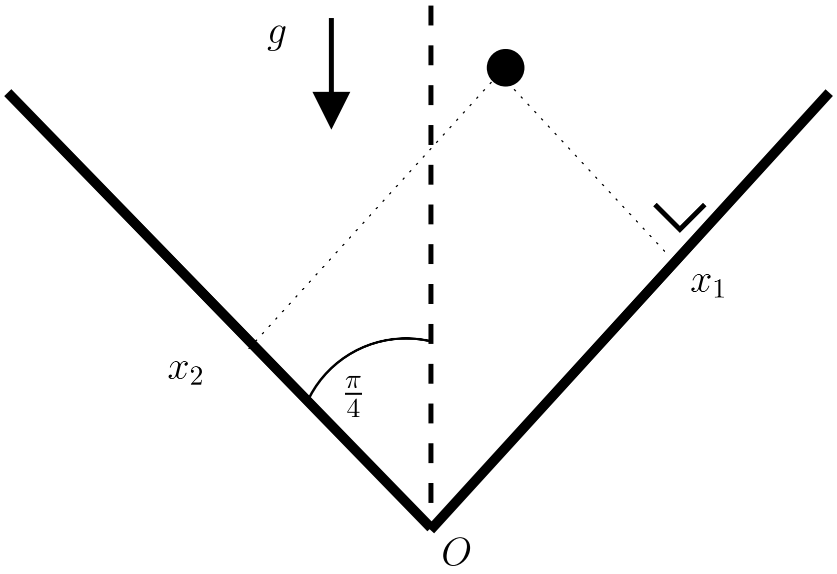


$$\begin{cases} \ddot{x}_1 = -g\cos\theta \\ \ddot{x}_2 = -g\cos\theta \\ x_1 \geq 0,\ x_2 \geq 0 \end{cases}$$

+ elastic collisions

- Energy (hamiltonian) is conserved
- Between collisions, system is integrable: two conserved quantities $H_i = \frac{\dot{x}_i^2}{2} + g\cos\theta x_i, i = 1, 2$.

Integrability of the square wiper

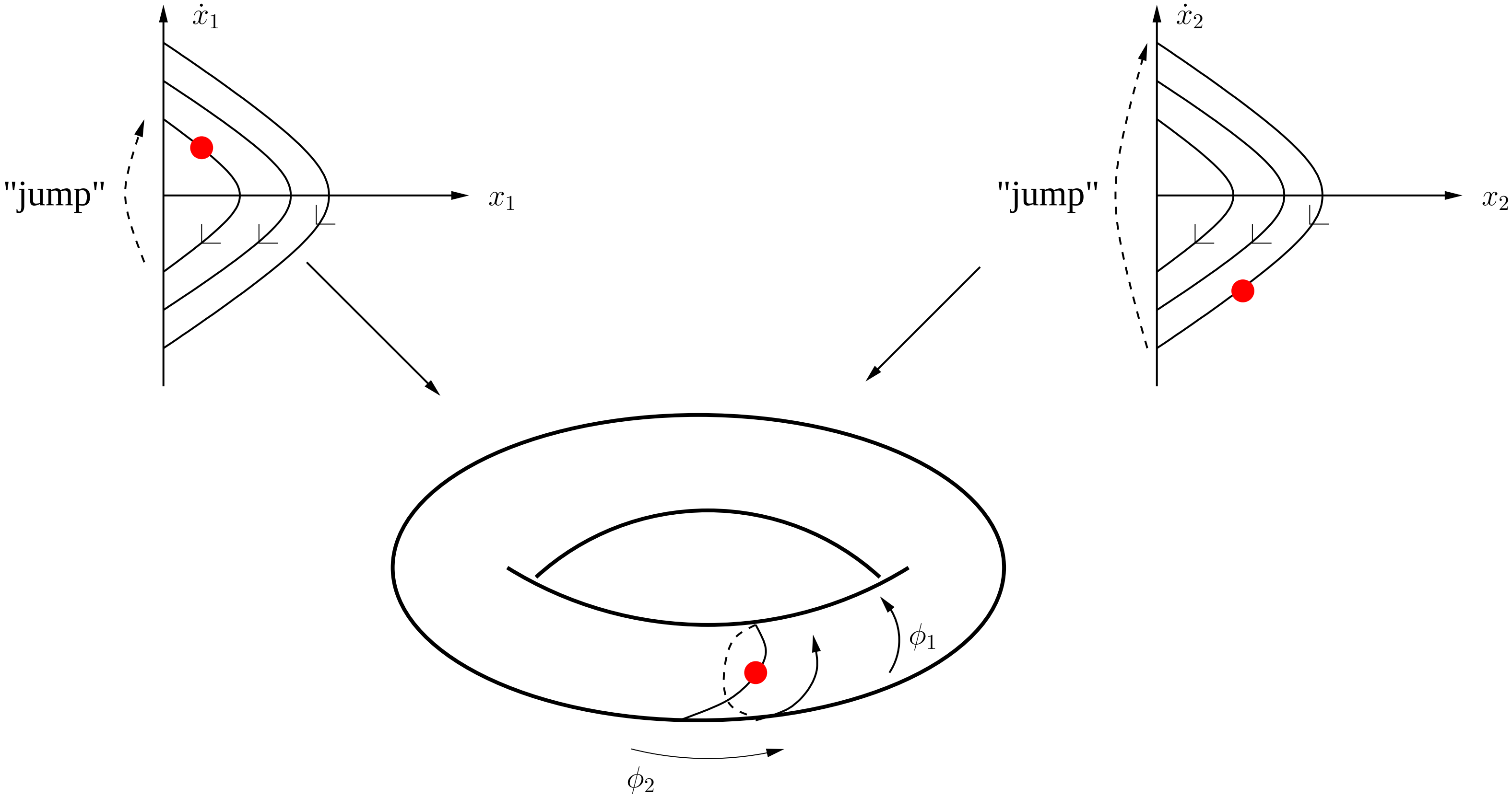


$$\begin{cases} \ddot{x}_1 = -g\cos\theta \\ \ddot{x}_2 = -g\cos\theta \\ x_1 \geq 0,\ x_2 \geq 0 \end{cases}$$

Elastic collisions: $x_i(t) = 0 \Rightarrow \dot{x}_i(t^+) = -\dot{x}_i(t^-)$.
Square wiper dynamics = two decoupled bouncing balls

$\theta \neq \frac{\pi}{4}$: collisions couple the two dynamics

Square wiper: phase portrait



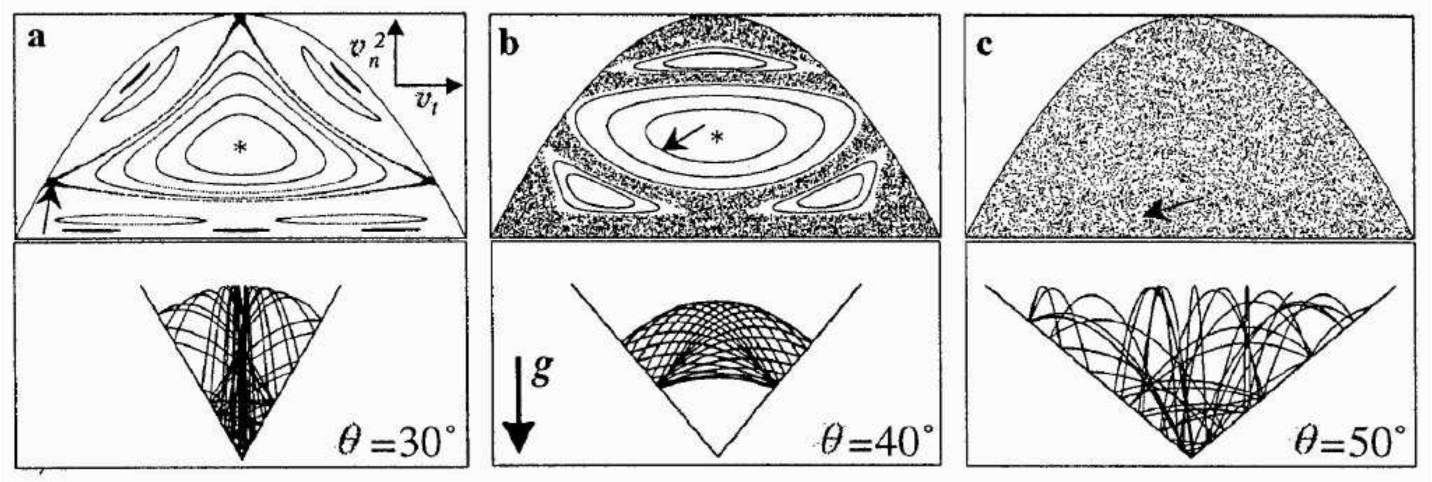
(ϕ_1, ϕ_2) “phase (action-angle) coordinates”;

$\phi_1(t)$ is T_1-periodic, with $T_1 = \frac{2\sqrt{H_1}}{g_1} = 2\frac{(\dot{x}_1)_+}{g}$

$\phi_2(t)$ is T_2-periodic, with $T_2 = \frac{2\sqrt{H_2}}{g_2} = 2\frac{(\dot{x}_2)_+}{g}$

Qualitative dynamics of the wiper

A $2D$ discrete map [H. Lehtihet and B. Miller,’86]
Figure from [Milner et al., ’01]



- $\theta < \frac{\pi}{4}$: quasiperiodic behavior coexists with chaotic regions (KAM theory);
- $\theta = \frac{\pi}{4}$: complete integrability
- $\theta > \frac{\pi}{4}$: complete hyperbolicity

Generalization

[Wotjowski, '98]

Hamiltonian with linear potential:
$H =< \xi, K\xi > + c^T \eta, \ (\eta, \xi) \in I\!R^n \times I\!R^n$
Conic configuration space: $\eta \geq 0, \ c > 0$
Elastic collisions.

- equivalence between bouncing ball in a wedge and colliding bouncing balls on a line
- Algebraic criterion for complete integrability and for complete hyperbolicity
- See animations on dynamical-systems.org

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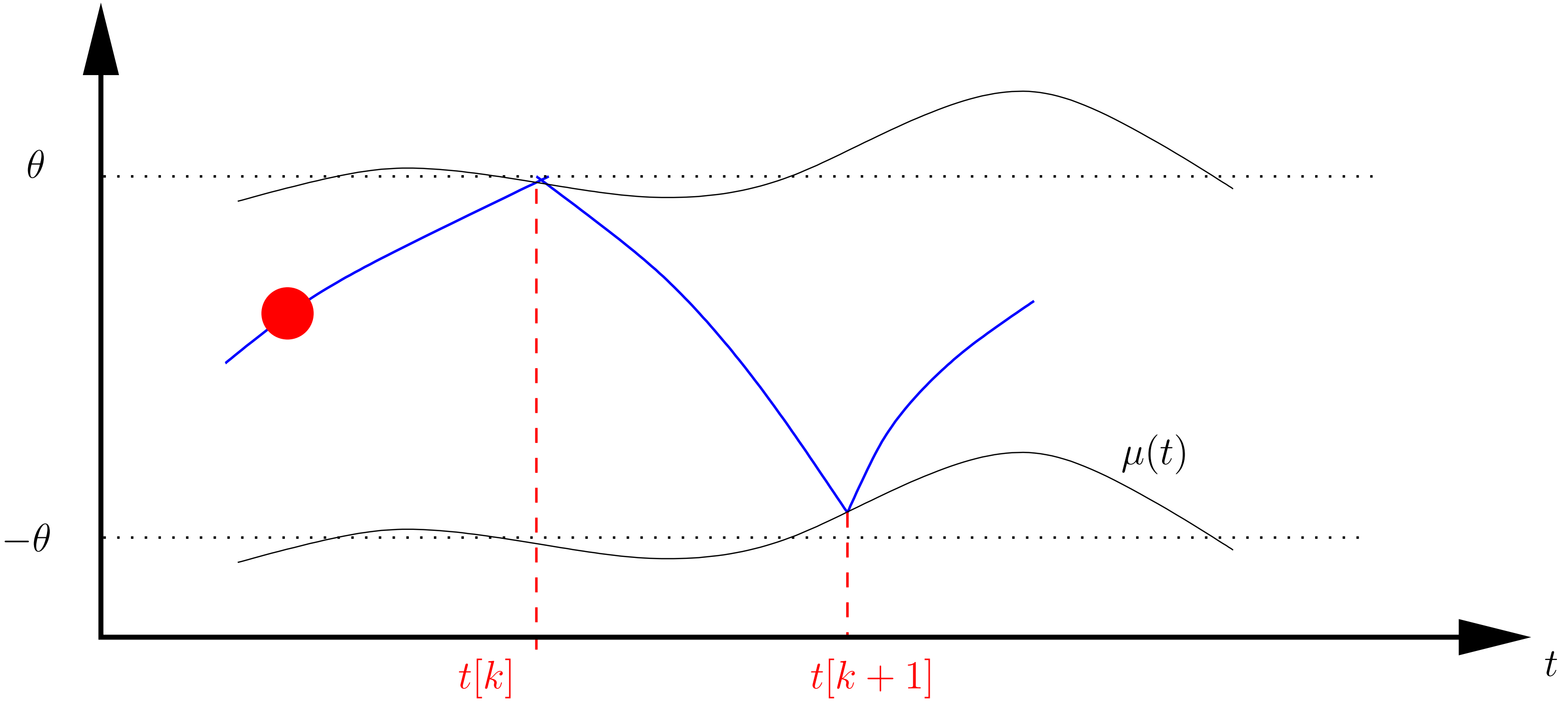
- Sensorless stabilization
- Research problems

Feedback control of the wiper

- step 1: convert intermittent control problem into discrete-time control problem
- step 2: construct the controlled impact map
- step 3: exploit the open-loop dynamics to construct feedback laws

A discrete-time stabilization problem



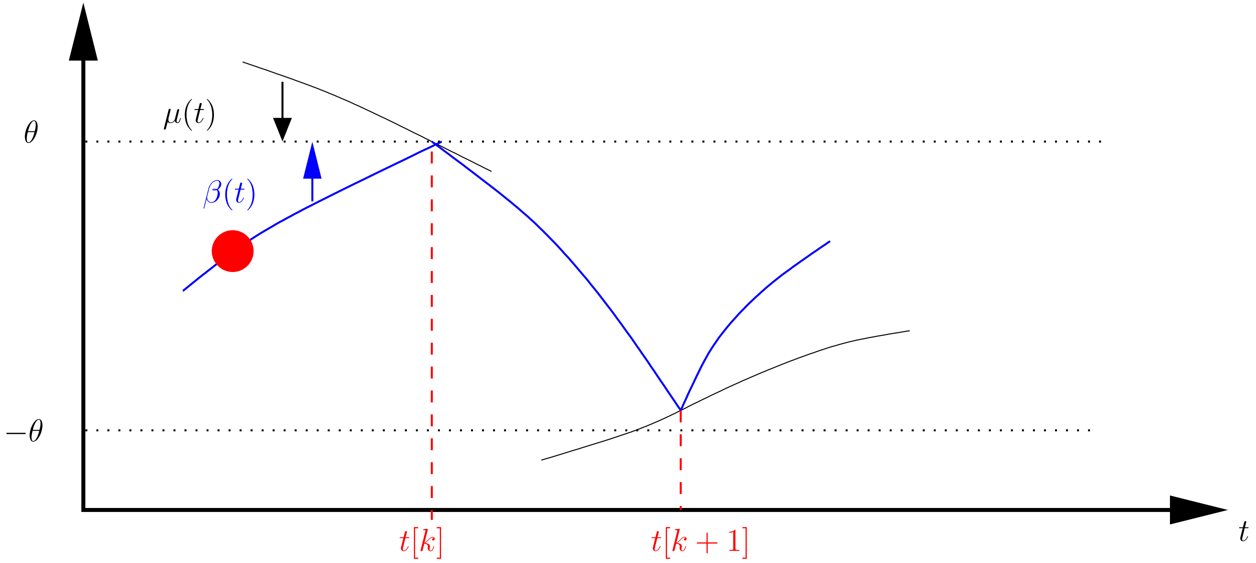
$X[k] \triangleq x^+(t[k])$ post-collision state ($\in \mathbb{R}^4$)

$u[k] \triangleq \begin{pmatrix} \mu(t[k+1]) \\ \dot{\mu}(t[k+1]) \end{pmatrix}$ wedge state at next impact

$X[k+1] = F(X[k], u[k])$ controlled impact map

Feedback: $u[k] = \alpha(X[k])$

Implementation of discrete feedback



Assign *mirror* trajectory to face to-be-hit [an idea from Buhler and Koditschek]:

$$\mu(t) = \alpha_2(X[k])\beta(t) + \alpha_1(X[k]),\ t[k] < t \leq t[k+1]$$

Collision time defined by $\mu(t) = -\beta(t) \Rightarrow$ feedback

$$\begin{aligned} u_1[k] = \mu(t[k+1]) &= \tfrac{1}{1+\alpha_2[k]}\alpha_1[k] \\ u_2[k] = \dot{\mu}(t[k+1]) &= -\dot{\beta}^-(t[k+1])\alpha_2[k] \end{aligned}$$

Mirror law for nonelastic wedge

Nonelastic collision rule:

$$\dot{\beta}^{+} = e(-\dot{\beta}^{-} + \dot{\mu}),\ 0 < e \leq 1$$

The modified mirror law

$$\mu(t) = \frac{1}{e}(\mu_{elast} + (1-e)\beta)$$

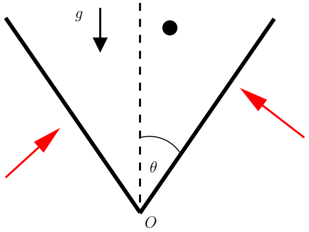
restores the elastic collision rule

$$\dot{\beta}^{+} = -\dot{\beta}^{-} + \dot{\mu}_{elast}$$

$\Rightarrow$ feedback control of elastic wedge extends to nonelastic wedge

Mirror law: particular cases

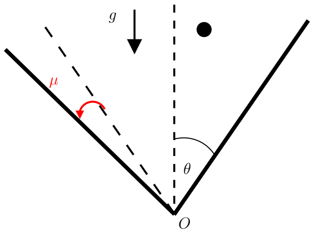
Momentum control in fixed (elastic) wedge:



$\mu(t) = \alpha_2(X[k])\beta(t) \Rightarrow \mu[k] = 0, \dot{\mu}[k] = (-\dot{\beta}^-(t[k+1]))\alpha_2(X[k])$

Well-suited to energy assignment control

Position control in (elastic) wedge:



$\mu(t) = \alpha_1(X[k])$

Well-suited to energy-preserving control

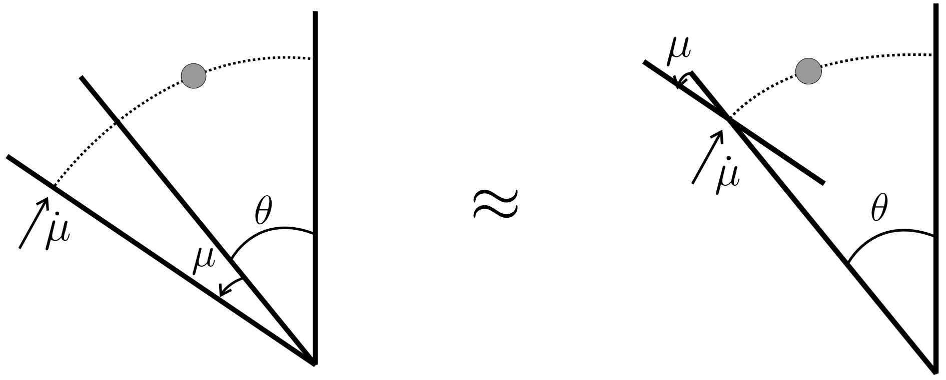
Construction of controlled impact map

- Impact map = flight map composed with collision map
- Flight map = integration of hamiltonian system over flight time $T = T(X[k], u[k])$
- collision map : $X[k+1] = C(x^{-}(t[k+1]), u[k])$

Small angle approximation: $T \approx T(X[k])$
Neglect effect of wedge motion on flight time
$\Rightarrow$ control only enters the collision rule

Controlled collision map

(under small angle assumption)



$$M(\mu)\begin{pmatrix} V_r^+ \\ V_n^+ \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M(\mu)\begin{pmatrix} V_r^- \\ V_n^- \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{2}{\alpha}R(t[k]) \end{pmatrix}\dot{\mu}$$

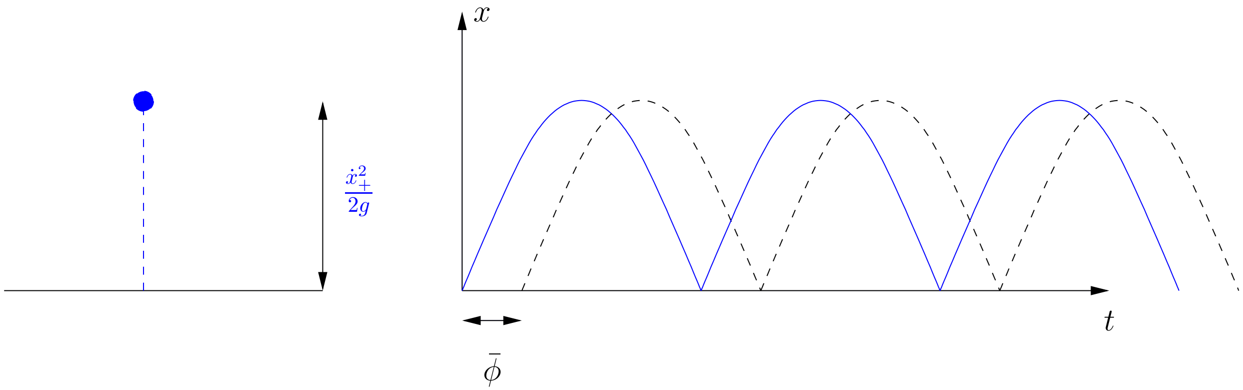
$$M(\mu) = \begin{pmatrix} \cos\mu & \alpha\sin\mu \\ -\frac{\sin\mu}{\alpha} & \cos\mu \end{pmatrix}$$

Feedback control: illustrations

- Feedback control of the bouncing ball ($n = 1$)
- Phase locking in the square wedge
- Lyapunov control in the elliptic wedge ($\theta < \frac{\pi}{4}$)
- Trapping control in the hyperbolic wedge ($\theta > \frac{\pi}{4}$)

Bouncing ball feedback control

Assign period $\bar{T} = \frac{2(\dot{x})_+}{g}$ and phase $\bar{\phi} \in [0, \bar{T})$



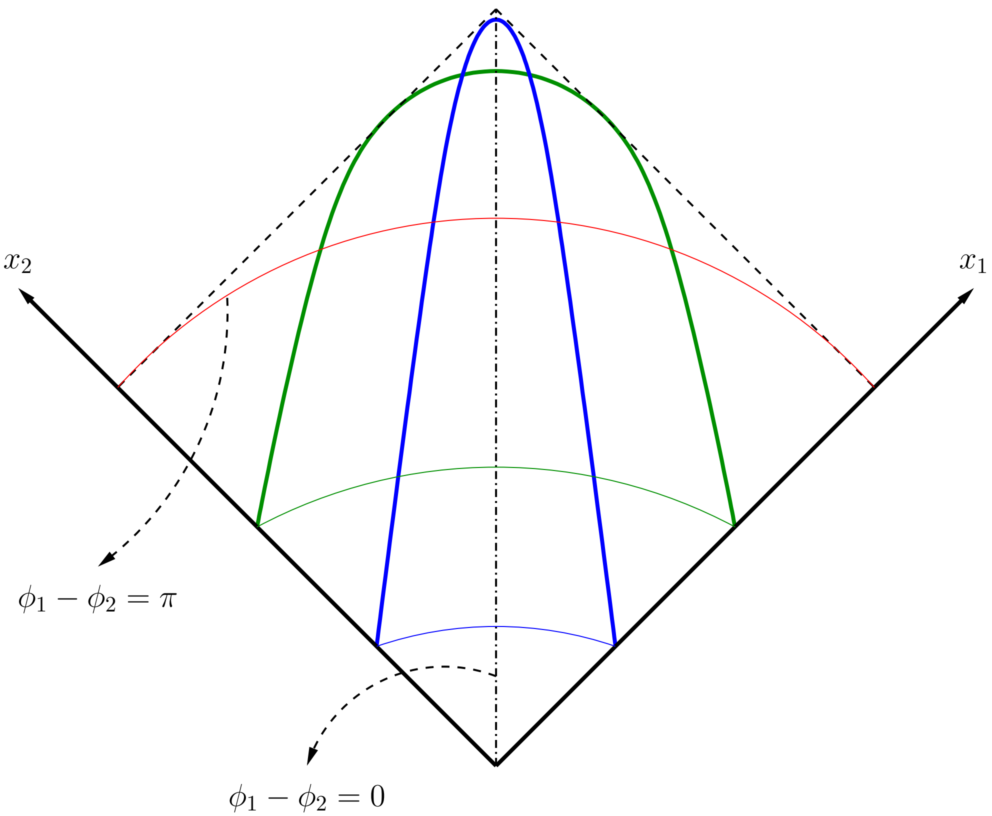
Error-system: $\phi[k] = \bar{\phi} + t[k] - k\bar{T}$, $\dot{e}[k] = \dot{x}[k] - \frac{g\bar{T}}{2}$ yields

$$\phi[k+1] = \phi[k] + \frac{2}{g}\dot{e}[k], \;\; \dot{e}[k+1] = \dot{e}[k] + u[k]$$

Two-parameter ($\bar{T}$ and $\bar{\phi}$) stabilizing feedback:

$$u[k] = -k_P \dot{e}[k] - k_I \sin(\frac{2\pi}{\bar{T}}\phi[k]), \;\; k_P > 0, \;\; k_I > 0$$

Periodic orbits in the square wiper

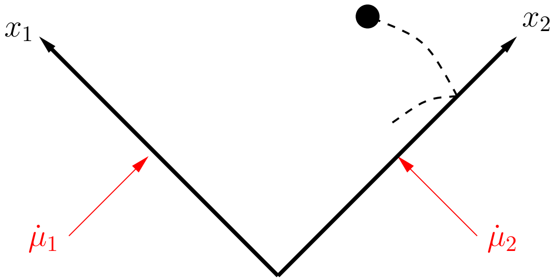


Two-parameter family of period-2 periodic orbits:

- Energy level: $H = H_1 + H_2$ ($\equiv$ time -period T_1)
- impact coordinate ($\equiv$ phase difference $\phi_1 - \phi_2 \equiv$ tang. velocity)

Momentum control of square wedge

Observe: Keeping $\mu[n] = 0$ for all n decouples the closed-loop system into two actuated 1D bouncing balls:

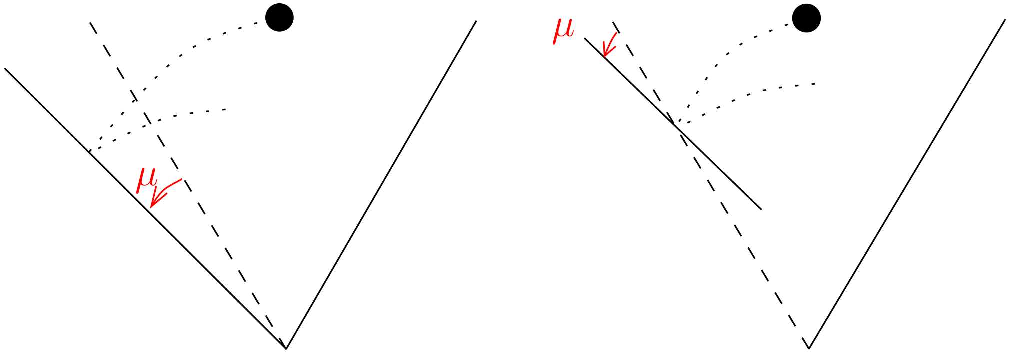


$$\begin{aligned} \dot{\mu}_{x_1} &= -k_P(\dot{x}_1[k] - \frac{g\bar{T}}{2}) - k_I(x_1[k] - x_2[k] - \bar{\phi}_{12}) \\ \dot{\mu}_{x_2} &= -k_P(\dot{x}_2[k] - \frac{g\bar{T}}{2}) \end{aligned}$$

$+$ variations:

- two parallel PI loops for absolute phase control;
- rhythmic feedback.

Position control of square wedge



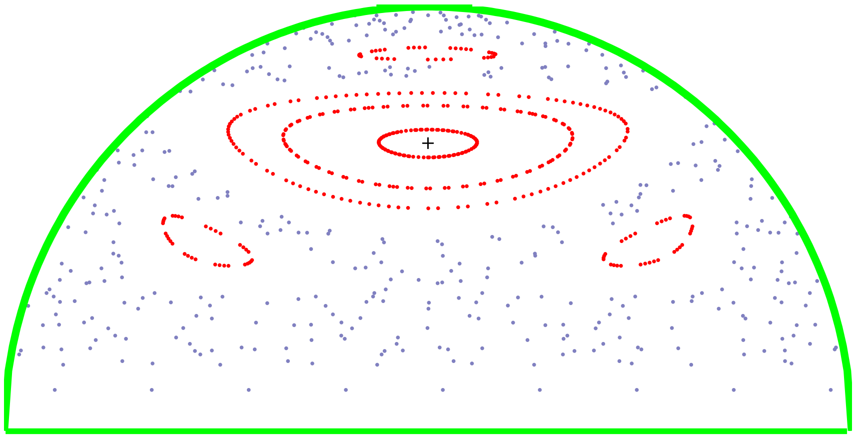
Observe:

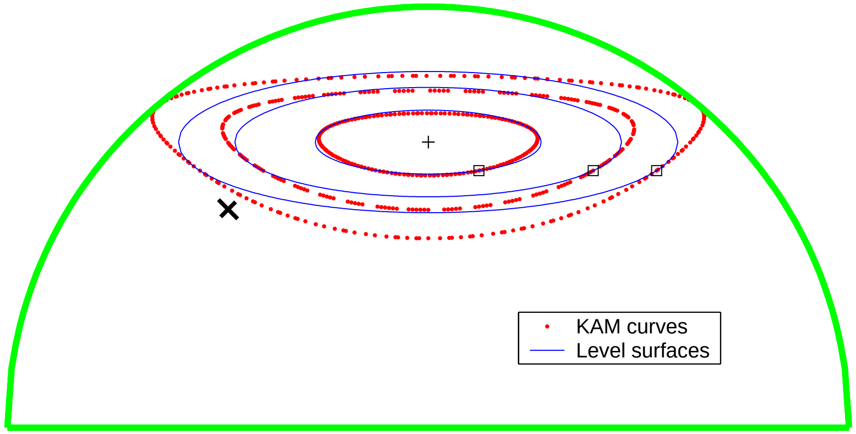
- $E = H_1 + H_2$ is constant: *conservative* system.
- $H_1(t)$ and $H_2(t)$ change only at collision times

Position feedback allows for exchange of relative energy at impact times
$\Rightarrow$ “small" control stabilization of periodic orbit with PI feedback of $(H_1 - H_2)$.

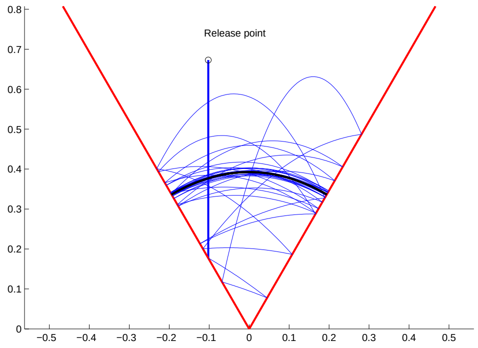
Lyapunov control in elliptic wedge

KAM curves (surfaces) serve as level sets of a control Lyapunov function



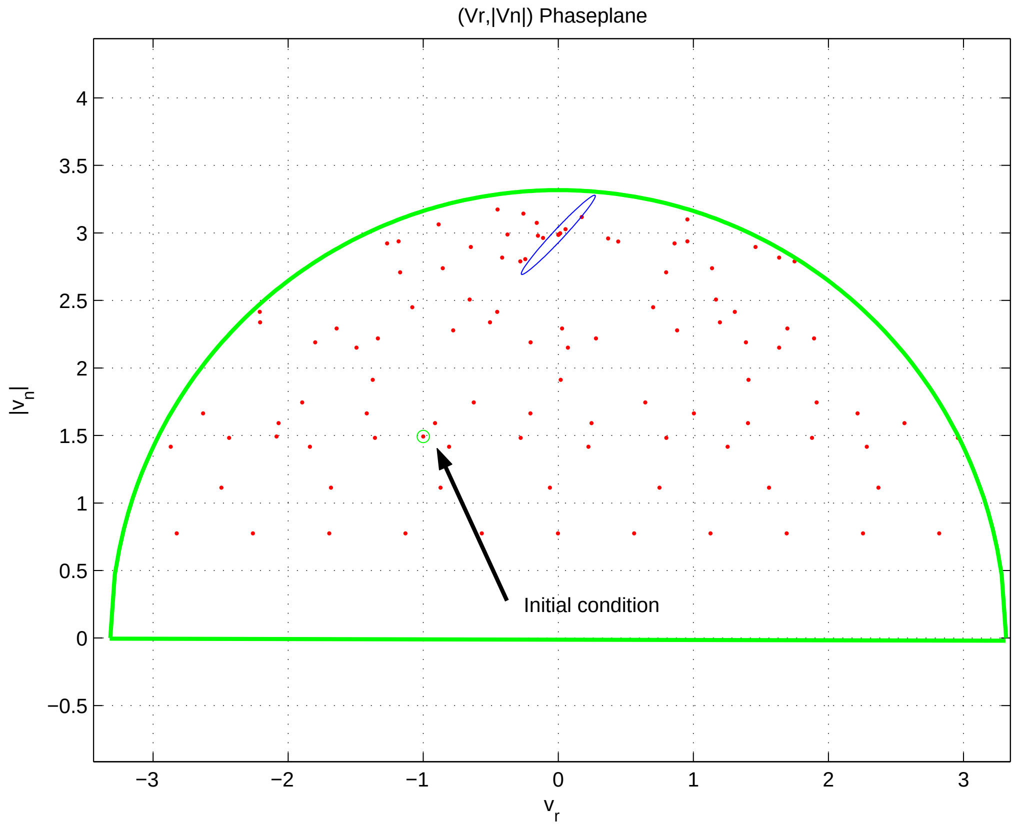


a closed-loop simulation:



Trapping control in hyperbolic wedge

Capture trajectories in a (small) invariant region determined by the level set of a quadratic (local) control Lyapunov function. All solutions enter the trapping set in finite time.



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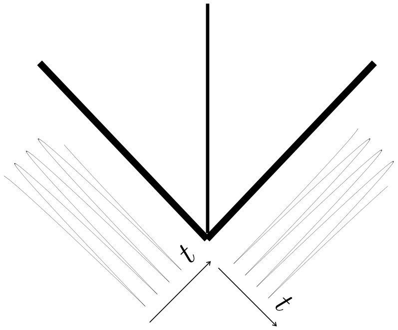
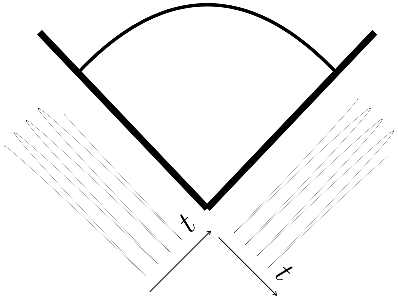
Sensorless stabilization: a cartoon

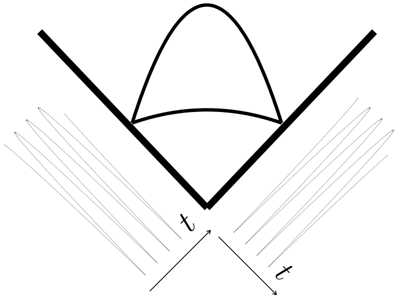
(square) wiper $\approx$ 2 orthogonal bouncing balls (i.e. impact oscillators)



sync



anti-sync



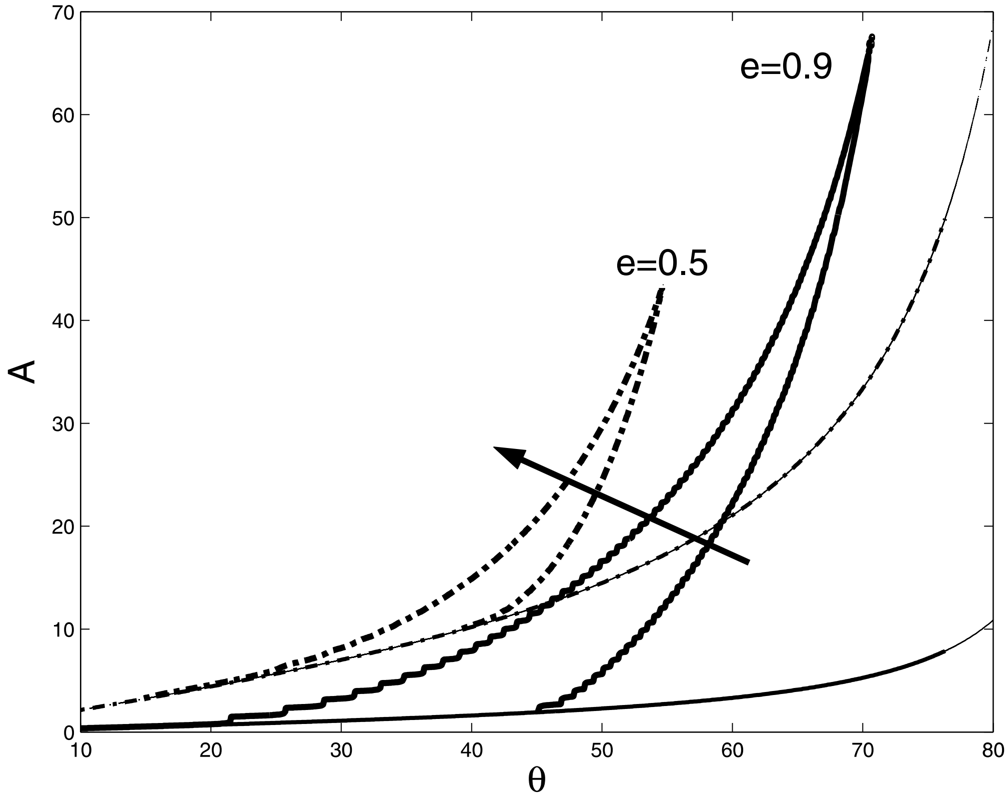
phase-locked

Design principle: stable motions result from the right interconnections between oscillators

Stabilization of period one orbit

$$\mu(t) = A\sin(\omega t) - \mu_0$$

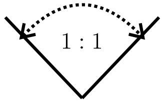
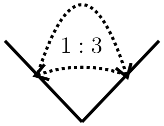
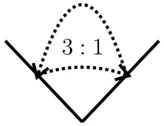
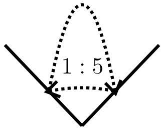
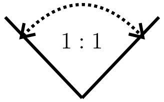
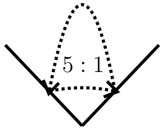
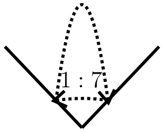
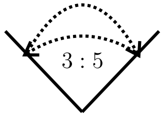
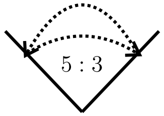
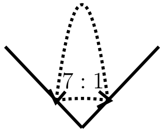
Parametric stability of linearized Poincaré map:



For $\theta > \frac{\pi}{4}$, the orbit is exponentially unstable in the fixed elastic wedge

Phase-locked period two orbits

(Square wedge)

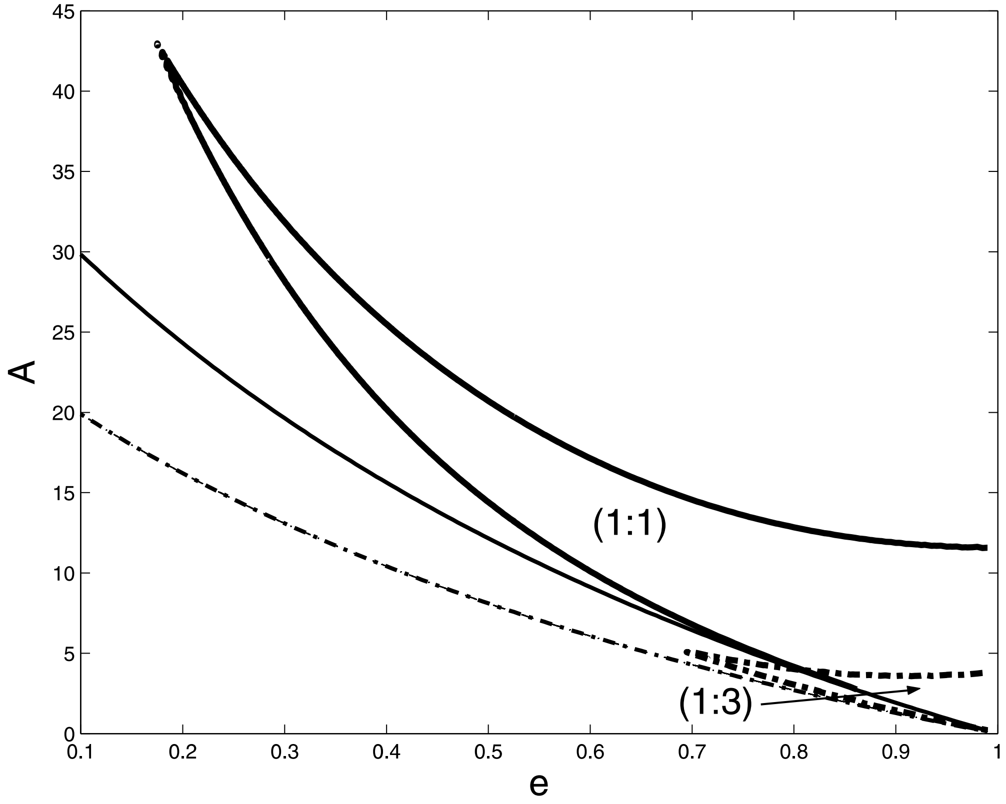
	m=1	m=2	m=3	m=4	. . .
n=1		X	X	X	
n=2			X	X	
n=3				X	. . .
n=4					
⋮			⋮		

steady state:

$$\begin{aligned} t[k+2]-t[k] &= n\frac{2\pi}{\omega} \\ t[k+1]-t[k] &= (2m-1)\frac{2\pi}{\omega} \end{aligned}$$

Stabilization of period two orbit

Parametric stability of linearized Poincaré map ($\theta = \frac{\pi}{4}$):



Sensorless stabilization in the lab

Human stabilization in the lab

[with Renaud Ronsse and Philippe Lefevre (UCL)]

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Research problem 1: a robotic challenge

Juggling a three ball shower with minimal feedback information

Research problem 2: an open question

Wotjowski's system:
Hamiltonian with linear potential:
$H =< \xi, K\xi > +c^T\eta,\ (\eta, \xi) \in I\!R^n \times I\!R^n$
Conic configuration space: $\eta \geq 0,\ c > 0$
Elastic collisions.
Choose parameters to ensure complete hyperbolicity.

Sensorless stabilizability of an arbitrary periodic orbit?

Conclusions

Juggling in a wedge is an interesting testbed for rhythmic control investigations

- Simplest examples of integrable Hamiltonian systems coupled by collisions
- Energy-based intermittent feedback control: robust and nonlocal
- sensorless stabilization: resonance versus ergodicity