Noname manuscript No.
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Electricity markets with flexible consumption as atomic splittable flow congestion games

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Received: date / Accepted: date

Abstract With the ongoing trend for connected electric appliances, electricity retailers now not only retail electricity bought on the energy market but also control flexible consumption. This control grants to the retailer the possibility of shifting their consumption from one hour of the day to another, influencing the corresponding market prices and consequently their costs. This article studies this system and shows that it can be represented as an atomic splittable flow congestion game with players sending flow in arcs linking a unique source and destination. We focus on games with affine cost functions and define laminar Nash equilibria where the constraints on the minimum and maximal flow that a player must send in a given arc are not binding. We show that the flow sent by a player at a laminar Nash equilibrium does not depend on the demand of other players. In laminar flow, we bound the price of anarchy and the ratio between the maximum and the minimum arc cost. Finally, we propose a simple method based on the property of a laminar Nash equilibrium to compute the price of flexibility to which energy flexibility should be remunerated in electric power systems.

Keywords atomic splittable flow congestion games · demand side management · laminar Nash equilibrium · price of flexibility · electricity retailers · load aggregators.

1 Introduction

One of the most complex commodity to exchange is electricity. By its very nature, the electricity production must always be equal to the consumption. Unfortunately, current technology does not allow storing high volume with enough economic and technical efficiency. To ensure the equality between production and consumption

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at any time, electricity is traded before its delivery. A large portion of the electricity is traded in the day-ahead energy market which settles unique prices for every time period of the day, which are commonly accepted as reference prices. With the ongoing trend for connected electric appliances, electricity retailers now not only retail electricity bought on the energy market but also control flexible consumption. This additional activity allows retailers to shift the consumption in cheaper hours to decrease their energy procurement cost. The starting point of this paper is a power system problem where each electricity retailer managing flexible consumption chooses independently the total consumption of its portfolio in each hour. The aggregated consumption of all retailers in one period determines a unique market price for this period. Given the prices in each hour, retailers aim at minimizing their own retailing cost. One of our results is to show that this problem can be mapped to an atomic splittable flow congestion game. Therefore, some concepts of the game theory literature are directly applicable to this electrical system to answer questions such as: *Does the electrical system converge to stable electricity prices? If it is the case, what are these prices, in particular what is the ratio between the maximum and the minimum price? How inefficient is this system with respect to a case where there would be only a single retailer that manages all the demand? Suppose that a power system operator asks a retailer to change its load in order to solve an issue in the electric network, what would be the cost incurred by the retailer?*

This paper answers these questions by studying atomic splittable flow congestion games. As a reminder, congestion games are non-cooperative games usually related to traffic problems where each user, or player, tries to find the path to its destination which minimizes its travel time. The travel time is given by a cost function dependent on the total traffic of the path. In the classic congestion game, a player is seen as an infinitesimal quantity of traffic to carry in the network (Rosenthal, 1973). Nonatomic congestion games are generalization of congestion games where some players, among the infinite amount of them, form coalitions and aim at minimizing the cost of their coalition (Milchtaich, 2004). One decision taken by the coalition on the strategy space can be discrete, e.g. assigning one user to a given path, or continuous, e.g. assigning some flow to a given arc. Games where the number of players is finite and where these players may split their flow along any number of paths are known as atomic splittable flow games (Roughgarden and Tardos, 2002; Harks and Miller, 2011).

Many theoretical results on congestion games can be found in the literature. The existence of equilibria in atomic splittable flow congestion games has been proven by Rosen (Rosen, 1965). The price of anarchy of atomic splittable flow congestion games with affine latency functions and positive coefficients in games with only parallel edges between a single source and a single destination is bounded by 3/2 (Harks and Miller, 2011). The matching lower bound is given in (Roughgarden and Schoppmann, 2015). Since atomic players may also be modeled as nonatomic players forming coalitions, the nonatomic congestion games literature with coalitions is also of interest (Hayrapetyan et al., 2006; Wan, 2012; Christodoulou and Koutsoupias, 2005; Fotakis et al., 2006; Cominetti et al., 2006, 2009). Forming coalitions in symmetric nonatomic games reduces the overall costs with respect to the Nash equilibrium (Hayrapetyan et al., 2006). Even the individual costs decrease when the size of the coalition of the individual increases (Wan, 2012). The price of anarchy is known to be bounded by the minimum between k and a constant dependent on the number of edges (Fotakis et al., 2006). For symmetric and asymmetric congestion games, the price of anarchy can be bounded by a constant dependent on the number of players. The authors show that the price of anarchy is bounded by $\frac{5k+2}{2k+2}$ in the affine case and that this bound is tight. The price of anarchy is bounded in the affine case with symmetric players by $\frac{4k^2}{(k+1)(3k-1)}$ and that this bound is tight (Cominetti et al., 2009).

Several power systems problems are related to game theory as attested in the literature survey (Fadlullah et al., 2011). Distributed load management in smart grid infrastructure to control the power demand at peak hours using dynamic pricing strategies has been studied as a network congestion game (Ibars et al., 2010). One close but different problem of flexible consumption management to the one considered in this paper is addressed in (Agarwal and Cui, 2012). They study the energy transaction between a single retailer and multiple consumers with a total energy constraint. The authors map the problem to an atomic splittable flow game. They prove the existence of a Nash equilibrium using the communication network paper (Orda et al., 1993).

This paper focuses on a particular regime of atomic splittable flow congestion games with affine arc cost functions that we name *laminar* flow. This name is motivated by the analogy with the fluid mechanics regime where the flow is organized in layers without interactions. One of our results is that, if the Nash equilibrium of an atomic splittable flow congestion game is laminar, i.e. the demand of all players are within specific bounds, the strategy of each player depends only on its own demand. Assuming laminar flow, we provide four contributions stated in the following theorems.

Theorem 1 *Consider an atomic splittable flow congestion game with affine cost functions and parallel arcs. If the Nash equilibrium is laminar then the flow of each player in each arc is independent from the demand of the other players.*

Two byproducts of this theorem are that the flows and arc costs are only dependent on the total demand and that a game can be checked to have a laminar equilibrium only based on the individual total demands of the players.

Theorem 2 *Consider a k-player atomic splittable flow congestion game with affine cost functions and parallel arcs. If the Nash equilibrium is laminar then the ratio between the maximum and the minimum arc cost is bounded by a constant dependent on k and the y-intercept of the cost functions.*

Theorem 3 *Consider an atomic splittable flow congestion game with parallel arcs. At the laminar Nash equilibrium, the price of flexibility for an imposed small deviation in an arc is at least twice the first derivative of the corresponding arc cost at the Nash equilibrium without deviation times the squared deviation.*

The *price of flexibility* is a concept introduced in this paper which in our power system problem corresponds to the price of shifting the electric consumption from one period to others.

Theorem 4 *The price of anarchy of a k-player atomic splittable flow congestion game with a laminar Nash equilibrium and affine cost functions with positive coefficients is at most*

$$\frac{4k^2}{(k+1)(3k-1)}. \tag{1}$$

This upper bound is smaller than the general bound of 1.5 given in (Harks and Miller, 2011), but only valid for laminar Nash equilibria.

These contributions are disseminated in the paper as follows. Section 2 sets our notations and describes laminar flows in atomic splittable flow congestion games. Section 3 links atomic splittable flow congestion games with retailers managing flexible electric consumption. In laminar flow, we obtain a bound on the ratio between the minimum and maximum arc cost in Section 4. Section 5 details a method to obtain the price of flexibility. A bound on the price of anarchy of atomic splittable flow congestion games with laminar Nash equilibrium is proven in Section 6. Finally, Section 7 concludes.

2 Laminar flow in congestion game

Consider k players sending flow in a set of arcs $\mathcal{T}$ with the same source and destination. The goal of a player $i \in \mathcal{K}$ is to minimize its total cost by choosing which quantity to send in each arc $t \in \mathcal{T}$, $x_{i,t}$ such that the flow sent is equal to the demand of the player D_i. The total flow is $D = \sum_{i\in\mathcal{K}} D_i$. The aggregated flow in one arc is given by $x_t = \sum_{i\in\mathcal{K}} x_{i,t}$ and defines the price in one arc using the cost function $c_t(x_t) : \mathbb{R}_+ \to \mathbb{R}$. Therefore, the prices are independent on the identity of the player. The total cost incurred by player i, $C_i(\mathbf{x}_i)$ is given by

$$C_i(\mathbf{x}_i) = \sum_{t\in\mathcal{T}} c_t(x_t)x_{i,t}. \tag{2}$$

where $\mathbf{x}_i = \{x_{i,t} | \forall t \in \mathcal{T}\}$. Note that this cost depends on the actions of the other players through the term x_t. The total system cost, $C(\mathbf{x})$, is only dependent on the aggregated flows

$$C(\mathbf{x}) = \sum_{t\in\mathcal{T}} c_t(x_t)x_t = \sum_{i\in\mathcal{K}} C_i(\mathbf{x}_i) \tag{3}$$

where $\mathbf{x} = \{x_t | \forall t \in \mathcal{T}\}$. Note that to one $\mathbf{x}$ can correspond more than one solution of same total system cost in terms of $\mathbf{x}_i$. Such a game is depicted in Figure 1. Additional constrains that can be added to the model are bounds on the flows such that

$$x_{i,t}^{\min} \leq x_{i,t} \leq x_{i,t}^{\max}. \tag{4}$$

The classic version of atomic splittable flow congestion games takes $x_{i,t}^{\min} = 0$ and $x_{i,t}^{\max} = +\infty$. This paper focuses on Nash equilibriums of atomic splittable flow congestions games where this constraint is either non existing or not active.

Definition 1 The equilibrium of an atomic splittable flow congestion game is laminar if, for each player $i \in \mathcal{K}$ and each arc $t \in \mathcal{T}$, the flow at the equilibrium $x_{i,t}$ is such that

$$x_{i,t}^{\min} < x_{i,t} < x_{i,t}^{\max}. \tag{5}$$

In fluid mechanics, a streamline is an imaginary line with no flow normal to it, only along it. When the flow is laminar, the streamlines are parallel and for flow between two parallel surfaces we may consider the flow as made up of parallel

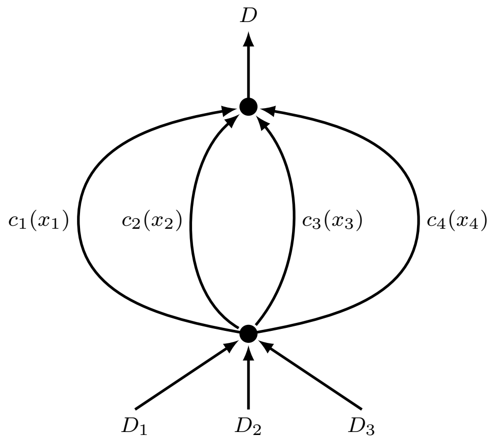


Fig. 1: Visual representation of an atomic splittable flow congestion game with three players and four arcs.

laminar layers. In a laminar flow, no mixing occurs between adjacent layers (Dunn, 2012). As we see in Theorem 1, if a congestion game is laminar, the strategy of a player at the Nash equilibrium does not depend on the demand of the others.

We denote $\mathbf{x}^*$ the optimal flow which minimizes the total cost: $\forall \mathbf{x} \in X, C(\mathbf{x}^*) \leq C(x)$. Note that if there is more than one player, the solution in terms of $\mathbf{x}_i$ is not unique. The Nash equilibrium is denoted by $\mathbf{x}_i^N$ $\forall i \in \mathcal{K}$ and the resulting aggregated flows by $\mathbf{x}^N$. At the Nash equilibrium, no player has any incentive to change its flows given the flows of the others. Note that if there is only one retailer, $\mathbf{x}^N = \mathbf{x}^*$. To shorten the notation, we define $c_t^N = c_t(x_t^N)$. The system is at a Nash equilibrium $\mathbf{x}_i^N$ if no player i can improve its strategy given the strategy of the others. As a result, the strategy $\mathbf{x}_i^N$ is a solution of the following optimization problem.

$$\min_{\mathbf{x}_i} \quad \sum_{t \in \mathcal{T}} c_t(x_t) x_{i,t} \tag{6a}$$

$$\text{s.t.} \quad \sum_{t \in \mathcal{T}} x_{i,t} = D_i \qquad : \lambda_i \tag{6b}$$

$$x_{i,t} \geq x_{i,t}^{min} \qquad : \kappa_{i,t} \geq 0 \qquad \forall t \in \mathcal{T} \tag{6c}$$

$$x_{i,t} \leq x_{i,t}^{max} \qquad : \nu_{i,t} \geq 0 \qquad \forall t \in \mathcal{T} \tag{6d}$$

Taking the Karush-Kuhn-Tucker conditions of (6) provide the following necessary optimality condition

$$\lambda_i^N + \kappa_{i,t} - \nu_{i,t} = c_t^N + \frac{\partial c_t^N}{\partial x_{i,t}} x_{i,t}^N = c_t^N + \frac{\partial c_t^N}{\partial x_t} x_{i,t}^N \qquad \forall t \in \mathcal{T} \tag{7}$$

As the prices do not depend on the identity of the buyer, $\frac{\partial c_t}{\partial x_{i,t}} = \frac{\partial c_t}{\partial x_t}$. By complementarity slackness, either $\kappa_{i,t} = 0$ or $x_{i,t} = x_{i,t}^{min}$ and either $\nu_{i,t} = 0$ or $x_{i,t} =$

$x_{i,t}^{max}$. In most of the paper, we consider affine cost functions $c_t(x_t) = a_t x_t + b_t$ with $a_t > 0$. The optimality conditions (7) are in this case

$$\lambda_i^N + \kappa_{i,t} - \nu_{i,t} = (a_t x_t^N + b_t) + a_t x_{i,t} = 2a_t x_{i,t}^N + a_t \sum_{j\in\mathcal{K}\setminus\{i\}} x_{j,t}^N + b_t \quad (8)$$

We now focus on the affine game represented in Figure 2 with three players and three arcs. We fix the total demand of the two last players and analyze how the equilibrium changes with respect to the total demand of the first player. The last two players have an identical total demand and therefore play an identical strategy. In the following, we write only the results of player two. The first arc is the most expensive which gives less incentive to the players to send flow in this arc. The bounds on the flow are $x_{i,t} \in [0, +\infty[$ for all players. Computations are performed using the open-source software Maxima (Maxima, 2014).

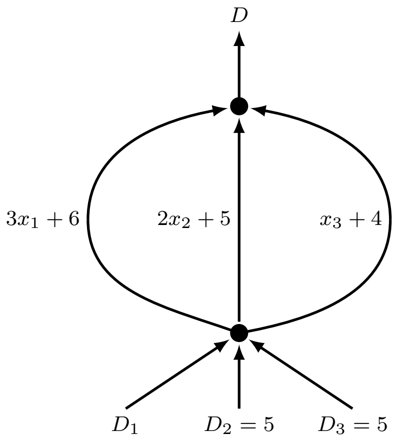


Fig. 2: Example of affine game.

First, we consider that D_1 is such that the game is laminar, $x_{i,t} > 0\ \forall i \in \mathcal{K}, t \in \mathcal{T}$. The analytic solution of the Nash equilibrium is

$$\mathbf{x}_1 = \left(\frac{8D_1 - 5}{44}, \frac{6D_1 - 1}{22}, \frac{24D_1 + 7}{44}\right) \quad (9)$$

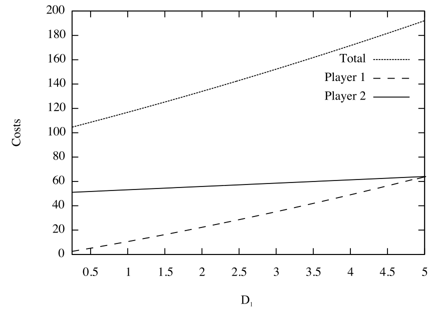
$$\mathbf{x}_2 = (35/44, 29/22, 127/44)\,. \quad (10)$$

We see that only the flows of the first player are dependent on D_1 at the laminar Nash equilibrium. This observation is the object of Theorem 1. The cost of the first player is $(24D_1^2 + 444D_1 - 3)/44$ and the one of players 2 and 3 is $(120D_1 + 2217)/44$. The Nash equilibrium of the game is laminar if $D_1 \geq 5/8$. If $D_1 < 5/8$, the flow is not laminar anymore and $x_{1,1} = 0$. The new equilibrium is

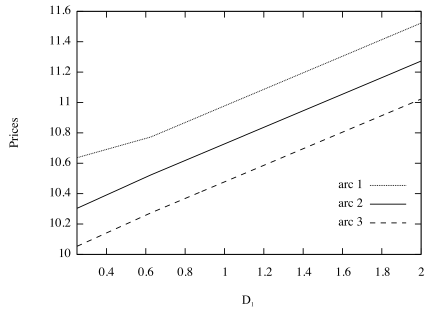
$$\mathbf{x}_1 = \left(0, \frac{4D_1 - 1}{12}, \frac{8D_1 + 1}{12}\right) \quad (11)$$

$$\mathbf{x}_2 = \left(\frac{2D_1 + 25}{33}, \frac{527 - 8D_1}{396}, \frac{1153 - 16D_1}{396}\right) \quad (12)$$

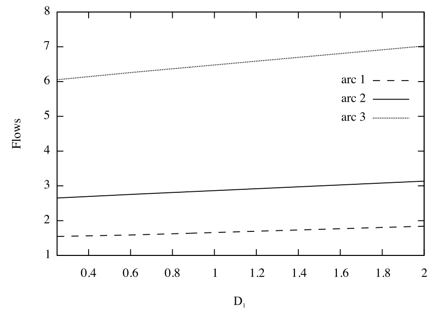
where $\mathbf{x}_2$ now depends on D_1. This equilibrium is valid for $1/4 \leq D_1 \leq 5/8$. The cost of the first player is given by $(928D_1^2 + 15824D_1 - 33)/1584$ and the one of the other players by $(-64D_1^2 + 13280D_1 + 239261)/4752$. Figure 3 shows the equilibrium as a function of D_1. The total cost and player's costs are represented in Figure 3a. Prices and flows in each arc are respectively given in Figure 3b and 3c. The individual flows of players one and two in the two first arcs are plotted in Figure 3d.



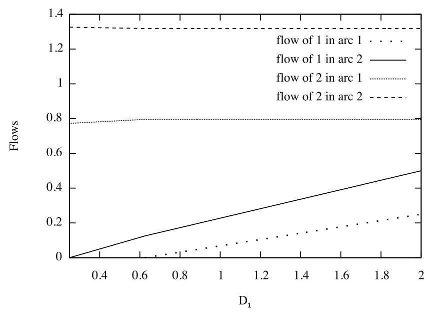
(a) Costs



(b) Prices



(c) Flows



(d) Player's flows

Fig. 3: Equilibrium of a three players and three arcs game.

As highlighted by the example, we have the following result for an atomic splittable flow congestion game which equilibrium is laminar:

Theorem 1 *Consider an atomic splittable flow congestion game with affine cost functions. If the Nash equilibrium is laminar then the flow of each player in each arc is independent from the demand of the other players.*

Proof In the case of affine prices and laminar flow, the equilibrium point of the game can be computed by solving the following system of equations:

$$\sum_{t \in \mathcal{T}} x_{i,t} = D_i \qquad \forall i \in \mathcal{K} \tag{13a}$$

$$2a_t x_{i,t} + a_t \sum_{j \in \mathcal{K} \setminus \{i\}} x_{j,t} - \lambda_i = -b_t \qquad \forall i \in \mathcal{K}, t \in \mathcal{T} \tag{13b}$$

We denote this system $Ay = d$ where $y = [\mathbf{x}_1 \dots \mathbf{x}_k \ \lambda_1 \dots \lambda_k]^T$. The sketch of the proof is as follows: we provide the analytical formula of A^{-1}. The inverse is used to obtain $y = A^{-1}d$ which leads to the analytical formula of $x_{i,t}$. We introduce the following convenient notations:

$$\beta = \sum_{t\in\mathcal{T}} \prod_{v\in\mathcal{T}\setminus\{t\}} a_v \tag{14a}$$

$$\alpha_t = \frac{\prod_{v\in\mathcal{T}\setminus\{t\}} a_v}{\beta} \tag{14b}$$

$$\delta_{t,u} = \frac{\prod_{v\in\mathcal{T}\setminus\{t,u\}} a_v}{\beta(k+1)} = \delta_{u,t} \tag{14c}$$

$$\sigma_t = \sum_{u\in\mathcal{T}\setminus\{t\}} \delta_{t,u} \tag{14d}$$

Observe that $\beta, \alpha_t, \delta_{t,u}$ and σ_t only depend on k, a_t and b_t. Using the analytical form of A^{-1} provided in the appendix, we obtain

$$x_{i,t} = D_i\alpha_t - b_t\sigma_t - \sum_{u\in\mathcal{T}\setminus\{t\}} b_u\delta_{t,u} \tag{15}$$

$$= \frac{D_i(k+1)\prod_{v\in\mathcal{T}\setminus\{t\}} a_v - b_t\sum_{u\in\mathcal{T}\setminus\{t\}}\prod_{v\in\mathcal{T}\setminus\{t,u\}} + \sum_{u\in\mathcal{T}\setminus\{t\}} b_u \prod_{v\in\mathcal{T}\setminus\{t,u\}} a_v}{(k+1)\sum_{t\in\mathcal{T}}\prod_{v\in\mathcal{T}\setminus\{t\}} a_v} \tag{16}$$

The complete proof is available in the appendix.

The following results are consequences from the previous theorem.

Corollary 1 *Consider an atomic splittable flow congestion game with affine cost functions whose Nash equilibrium is laminar. At this equilibrium, the arc flows and the costs depend only on the total demand.*

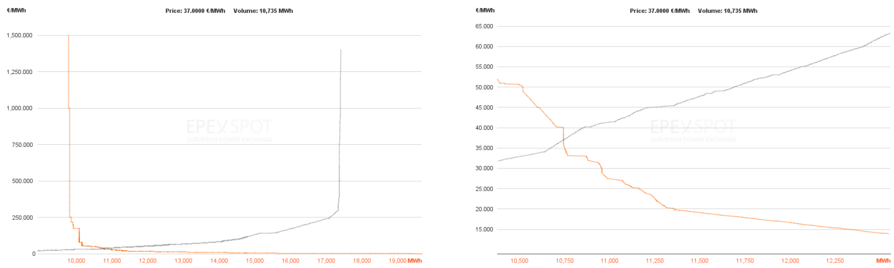
Corollary 2 *If each player $i \in \mathcal{K}$ demand D_i is such that, $\forall t \in \mathcal{T}$*

$$x_{i,t}^{\min} < D_i\alpha_t - b_t\sigma_t - \sum_{u\in\mathcal{T}\setminus\{t\}} b_u\delta_{t,u} < x_{i,t}^{\max} \tag{17}$$

then the Nash equilibrium of the congestion game is laminar.

3 Flexible consumption retailing as an atomic splittable flow congestion game

As electricity flows instantaneously through the network and only few quantities can be stored, electricity production must always be equal to the consumption at every moment. To this end, electricity is traded before its delivery. One part of the trade is conducted years or months ahead in long term contracts while the rest is cleared on energy spot markets. The most common is the day-ahead energy market whose prices are taken as references. This market divides the day into periods, typically twenty-four, and provides a unique price for each of these periods. Participants to these markets submit bids to supply or consume a certain amount of electric energy at a given cost for the period under consideration. The bids are ranked to form the demand and the offer curves. The intersection of the two curves defines the system marginal price. This unique price for each hour is the price paid by every accepted participant whatever the cost they submitted. This scheme gives incentive to the participants to bid at their marginal costs and therefore discourage gaming on the marginal cost they require (Kirschen and Strbac, 2004). Figure 4 shows the production and consumption aggregated curves of the French spot market for the first hour of the 1st April 2014 (SPOT, 2015). The intersection of the non-decreasing offer curve with the demand curve leads to the system marginal price of 37€/MWh.



(a) 0h-1h (b) Zoom on the market clearing price

Fig. 4: Aggregated curves of the market clearing of the 1st April 2014 (SPOT, 2015).

More and more consumers, from retail customers to industrials, accurately monitor their consumption in order to control their final electricity bill. This trend for energy awareness leads to an increasing number of connected electric appliances which may not only be monitored but also controlled by electricity retailers. This control can be used to shift the consumption from periods with expensive market prices to more affordable ones. As a result, one retailer can decrease its cost at the expense of the others resulting in an higher total cost. The least global energy procurement cost would be obtained if only one entity was controlling the entire flexibility. In practice, the electrical system is composed of more than one retailer which game to minimize their own cost to buy electricity.

The mapping to an atomic splittable flow congestion game is as follows. To each market period t corresponds one arc between a single source and a single destination. The cost function of an arc, c_t is the offer curve of the market at the corresponding period. Note that the offer curve of Figure 4b may be well approximated by a linear regression. The outcome of the market is the system marginal price, defined in each period as the cost of the corresponding arc at the Nash Equilibrium, c_t^N and in practice determined by the wholesale market operator. Each retailer i is a player with a total flow equal to the energy needs of its clients, D_i. At the end of the time horizon, D_i is the total energy that must be bought by the retailer. Retailers minimize their own energy procurement cost which is the sum over the periods of the electricity price, c_t^N, times the energy consumed in the corresponding period, $x_{i,t}$. The base load of the retailer is given by $x_{i,t}^{min}$ and its maximum flexibility by $x_{i,t}^{max}$.

One could wonder this implications of the atomic splittable flow congestion game on the electricity prices. Do the game converge to stable electricity prices? If it is the case, what are these prices, in particular what is the ratio between the maximum and the minimum price? How inefficient is this system with respect to the case where there would be only a single retailer that manages all the demand? In the gaming theory literature, the first question is equivalent to showing that there exists a Nash equilibrium. The second can be found under the term *unfairness* (Correa et al., 2007) while the third is obtained by the price of anarchy (Koutsoupias and Papadimitriou, 1999). Note that the results obtained in this paper are valid for general atomic splittable flow congestion games with single source and destination.

In addition to the reduction of energy procurement costs, demand side flexibility can also be used to provide services to other actors of the electrical system. These services may target solve an overvoltage issue or cover the unexpected loss of a production unit. The first case is related to active network management which is studied extensively in the literature (Balijepalli et al., 2011; Zhao et al., 2014; Mathieu et al., 2015; Gemine et al., 2013). The second is handled through reserve markets and the methods to provide reserve services by the demand side flexibility has also been broadly investigated (Palensky and Dietrich, 2011; Mathieu et al., 2013, 2014). One important unknown in these works and others is the price at which demand side flexibility should be sold. The price of the flexibility from production units is easy, it is proportional to the fuel cost and the unit maintenance and operating cost. The provision of flexibility from production unit is also easier to handle as increasing the production in an hour has barely any impact on what can be done in the following hours. Conversely, changing the consumption of an electric appliance in an hour impacts its consumption in the following one. For instance, a retailer controlling supermarket fridges can interrupt them for one hour. The internal temperature of the fridges increases and consequently the fridges consume more later. The consumption is therefore shifted from one hour to another. Following these thoughts, the cost associated to this shifting is related to the difference of prices between the market prices in the periods where the energy consumption is modified. In this paper, we provide a simple method to compute the price at which the flexibility of the demand side should be remunerated. This method only depends on public data of the clearing of the day-ahead energy market. The result of the method is supported by its link to the Nash equilibrium of the corresponding congestion game.

4 Ratio between the maximum and minimum arc cost

We are now interested in the ratio between the maximum cost of sending flow in one arc with respect to the minimum cost. In the following, we make the hypothesis that the Nash equilibrium of the game is laminar with affine costs functions $c_t(x_t) = a_t x_t + b_t$ and $a_t, b_t \in \mathbb{R}_+$.

The following theorem proves a bound on this ratio depending only on the number of players and the constant terms b_t which can be itself bounded by $\frac{k+1}{k}$.

Theorem 2 *Consider a k-player atomic splittable flow congestion game with parallel arcs and affine cost functions of the form $a_t x_t + b_t$ and $a_t, b_t \in \mathbb{R}_+$. If the Nash equilibrium is laminar then the ratio between the maximum and the minimum arc cost, occurring respectively in arcs t and u, is bounded by*

$$\frac{(k+1)b_t}{b_u + kb_t} \leq \frac{k+1}{k} \tag{18}$$

Proof (Proof of Theorem 2) If the Nash equilibrium is laminar, the equilibrium strategy of player i is obtained by solving the system

$$\sum_{t \in \mathcal{T}} x_{i,t} = D_i : \lambda_i \qquad \forall i \in \mathcal{K} \tag{19}$$

$$\lambda_i = 2a_t x_{i,t} + a_t \sum_{j \in \mathcal{K} \setminus \{i\}} x_{j,t} + b_t \qquad \forall t \in \mathcal{T}, i \in \mathcal{K} \tag{20}$$

The set of equations given by (20) can be written on the form

$$a_t \begin{pmatrix} 2 & 1 & 1 \\ 1 & \ddots & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_{1,t} \\ \vdots \\ x_{i,t} \\ \vdots \\ x_{k,t} \end{pmatrix} = \begin{pmatrix} \lambda_1 - b_t \\ \vdots \\ \lambda_i - b_t \\ \vdots \\ \lambda_k - b_t \end{pmatrix} \tag{21}$$

which can be concisely written as

$$a_t(\mathbb{1}_k + \mathbb{I}_k)\mathbf{x}_t^K = \lambda^K - b_t \tag{22}$$

where $\mathbb{I}_k$ is an identity matrix of dimension k and $\mathbb{1}_k$ a square matrix of ones of dimension k. For $a_t > 0$,

$$\mathbf{x}_t^K = (\mathbb{1}_k + \mathbb{I}_k)^{-1} \frac{\lambda^K - b_t}{a_t} \tag{23}$$

$$= \left(\mathbb{I}_k - \frac{1}{k+1}\mathbb{1}_k\right) \frac{\lambda^K - b_t}{a_t} \tag{24}$$

using Lemma 5 available in the Appendix. In particular for player i,

$$x_{i,t} = \frac{k}{k+1} \frac{\lambda_i - b_t}{a_t} + \sum_{j \in \mathcal{K} \setminus \{i\}} \frac{-1}{k+1} \frac{\lambda_j - b_t}{a_t} \tag{25}$$

$$= \frac{k\lambda_i - \sum_{j \in \mathcal{K}} \lambda_j - b_t}{(k+1)a_t} \tag{26}$$

$$x_t = \sum_{i \in \mathcal{K}} x_{i,t} = \frac{\sum_{i \in \mathcal{K}} \lambda_i - kb_t}{(k+1)a_t} \tag{27}$$

The sums of the dual variables λ_i can be bounded independently of x_t using (27) and $a_t, x_t \geq 0$.

$$\sum_{i \in \mathcal{K}} \lambda_i = (k+1)a_t x_t + kb_t \geq kb_t \tag{28}$$

The following observation is used later to bound the ratio.

Observation 1 *Given $a, b, c, d \in \mathbb{R}_+$. If $a \geq c$ and $b \geq d$ then*

$$\frac{a+b}{c+b} \leq \frac{a+d}{c+d}. \tag{29}$$

For convenience, we define that the maximum cost is obtained in arc t and the minimum in arc u. The ratio between the maximum and the minimum arc cost in the case where $a_t, a_u > 0$ is given by

$$\frac{\max\{c_t^N | t \in \mathcal{T}\}}{\min\{c_t^N | t \in \mathcal{T}\}} = \frac{c_t^N}{c_u^N} = \frac{b_t + a_t x_t}{b_u + a_u x_u} \tag{30}$$

$$= \frac{b_t + \frac{\sum_{i \in \mathcal{K}} \lambda_i - k b_t}{k+1}}{b_u + \frac{\sum_{i \in \mathcal{K}} \lambda_i - k b_u}{k+1}} \tag{31}$$

$$= \frac{b_t + \sum_{i \in \mathcal{K}} \lambda_i}{b_u + \sum_{i \in \mathcal{K}} \lambda_i} \tag{32}$$

$$\leq \frac{(k+1) b_t}{b_u + k b_t} \leq \frac{k+1}{k} \tag{33}$$

where the last inequality is obtained using (28) and Observation 1. The previous bound is also valid for the case where $a_t = 0$. The proof is straightforward using (27) and that (20) simplifies into $\lambda_i = b_t$.

We now focus on the case where $a_u = 0$. In this period, (20) simplifies into $\lambda_i = b_u$ and we also have $b_t \leq b_u$ as $c_t^N \geq c_u^N$. The ratio between the maximum and the minimum arc cost can be bounded by

$$\frac{\max\{c_t^N | t \in \mathcal{T}\}}{\min\{c_t^N | t \in \mathcal{T}\}} = \frac{c_t^N}{c_u^N} = \frac{b_t + a_t x_t}{b_u} \tag{34}$$

$$= \frac{b_t + \frac{\sum_{i \in \mathcal{K}} \lambda_i - k b_t}{k+1}}{b_u} \tag{35}$$

$$= \frac{b_t + \sum_{i \in \mathcal{K}} \lambda_i}{(k+1) b_u} \tag{36}$$

$$= \frac{b_t + k b_u}{(k+1) b_u} \leq 1 \tag{37}$$

Obviously, if $a_t = a_u = 0$ all the prices are equal.

Note that this result applies also for symmetric players with the additional constraints $x_{i,t} \geq 0$ by removing the arcs in which $x_{i,t} = 0$.

The example of Section 6 taken from (Cominetti et al., 2009) and given in Figure 5 shows that this bound is tight. At the Nash equilibrium, each player sends the flow $(0, 1)$ resulting in the prices $(1, \frac{k}{k+1})$. Note that the bound of $\frac{k+1}{k}$, without the constants b_u and b_t, can be obtained directly from the marginal costs at the laminar Nash equilibrium. If the equilibrium is laminar, for each player i and any edges u, t, the marginal cost is equal on each edge, and hence $a_u x_{i,u} + c_u^N = a_t x_{i,t} + c_t^N$. Adding for all players gives $(1 + 1/k) a_u x_u + b_u = (1 + 1/k) a_t x_t + b_t$, and hence $c_u^N \leq (1 + 1/k) c_t^N$.

5 Price of flexibility

Since the electrical network is not a copper plate, flexibility of the electrical consumption may be required in the electrical system. For instance, the electrical distribution network may not be able to handle a large wind power production which therefore needs to be consumed locally. The system operator may request an increase of the consumption to one of the retailer in one period and its up to the retailer to decrease the consumption in other periods to consume the same energy. This consumption shift needs to be paid as a flexibility service by the system operator to a price which reflects the costs of the action. Before defining this price, one should define a reference to quantify the increase or decrease of the consumption. This reference, called in power systems a baseline, is given by the positions of the players at the unperturbed Nash equilibrium.

Definition 2 The price of flexibility in one arc is the price that reflects the cost of imposing a specified flow deviation in that arc with respect to the unperturbed Nash equilibrium.

To estimate the price of flexibility, we compute the perturbation of the Nash equilibrium if we impose to one player to change its consumption in one period. For a small perturbation, the values of a laminar Nash equilibrium c_t^N and $\partial_t^N = \frac{\partial c_t(x_t^N)}{\partial x_t} \geq 0$ are taken as data.

Theorem 3 *Consider an atomic splittable flow congestion game with a laminar Nash equilibrium, the price of flexibility for an imposed small deviation in an arc is at least twice the first derivative of the corresponding arc cost at the Nash equilibrium without deviation times the squared deviation.*

Proof (Proof of Theorem 3.) Let us fix a player $i \in \mathcal{K}$ and a period $u \in \mathcal{T}$. At this equilibrium, player i has no incentive to deviate from the strategy $\mathbf{x}_i^N$. Assume we impose a small deviation Δ_u to player i in a single arc u such that $x_{i,u} = x_{i,u}^N + \Delta_u$. Player i can solve the following optimization problem to modify its strategy:

$$\min_{\mathbf{x}_i} \quad \sum_{t\in\mathcal{T}} (c_t^N + \partial_t^N (x_{i,t} - x_{i,t}^N)) x_{i,t} \tag{38a}$$

$$\text{s.t.} \quad \sum_{t\in\mathcal{T}} x_{i,t} = D_i \qquad : \lambda_i \tag{38b}$$

$$x_{i,u} = x_{i,u}^N + \Delta_u \qquad : \mu_{i,u} \tag{38c}$$

Which can be reformulated, taking the new solution with respect to the Nash equilibrium, by introducing the variables $\epsilon_{i,t}$ such that $x_{i,t} = x_{i,t}^N + \epsilon_{i,t}$. The optimization problem using the variables $\epsilon_{i,t}$ is

$$\min \quad \sum_{t\in\mathcal{T}} \left(\partial_t^N \epsilon_{i,t}^2 + (c_t^N + \partial_t^N x_{i,t}^N) \epsilon_{i,t} \right) \tag{39a}$$

$$\text{s.t.} \quad \sum_{t\in\mathcal{T}} \epsilon_{i,t} = 0 \qquad : \lambda_i \tag{39b}$$

$$\epsilon_{i,u} = \Delta_u \qquad : \mu_{i,u} \tag{39c}$$

Note that this problem is convex as $\partial_t^N \geq 0$. The Lagrangian reads

$$L_{i,u} = \sum_{t\in\mathcal{T}} \left(\partial_t^N \epsilon_{i,t}^2 + (c_t^N + \partial_t^N x_{i,t}^N)\epsilon_{i,t}\right) - \lambda_i \sum_{t\in\mathcal{T}} \epsilon_{i,t} - \mu_{i,u}(\epsilon_{i,u} - \Delta_u). \quad (40)$$

Canceling the derivative of the Lagrangian with respect to $\epsilon_{i,u}$ gives

$$\lambda_i + \mu_{i,u} = c_u^N + \partial_u^N x_{i,u}^N + 2\partial_u^N \Delta_u \quad (41)$$

$$= \lambda_i^N + 2\partial_u^N \Delta_u \quad (42)$$

Canceling the derivative of the Lagrangian with respect to $\epsilon_{i,t}$ with $t \neq u$ gives

$$\lambda_i = c_t^N + \partial_t^N x_{i,t}^N + 2\partial_t^N \epsilon_{i,t} \quad (43)$$

$$= \lambda_i^N + 2\partial_t^N \epsilon_{i,t}. \quad (44)$$

Lemma 1 *Assume that arc costs are fixed at the values of a laminar Nash equilibrium with c_t^N and $\partial_t^N = \frac{\partial c_t(x_t^N)}{\partial x_t} \geq 0$ are taken as data. Assume an imposed deviation Δ_u in period u and writes the deviation in other periods t, $\epsilon_{i,t}$. Then $\epsilon_{i,t}$ is of opposite sign that Δ_u and such that $|\epsilon_{i,t}| \leq \Delta_u,\ \forall t \in \mathcal{T} \setminus \{u\}$.*

Proof (Proof of Lemma 1) Taking equation (44) for two arcs $t, v \in \mathcal{T} \setminus \{u\}$ yields $\partial_t^N \epsilon_{i,t} = \partial_v^N \epsilon_{i,v}$. As $\partial_t^N, \partial_v^N \geq 0$, $\epsilon_{i,t}$ and $\epsilon_{i,v}$ have the same sign. Using (39c) in (39b) gives $\sum_{t\in\mathcal{T}\setminus\{u\}} \epsilon_{i,t} = -\Delta_u$ which implies the lemma.

Using Lemma 1 and (44) yields

$$\lambda_i \in \left[\lambda_i^N - 2\Delta_u \max_{t\in\mathcal{T}\setminus\{u\}} \partial_t^N, \lambda_i^N\right] \quad (45)$$

Injecting the latter in (42) yields

$$\mu_{i,u} \in \left[2\Delta_u\partial_u^N, 2\Delta_u\left(\partial_u^N + \max_{t\in\mathcal{T}\setminus\{u\}} \partial_t^N\right)\right] \quad (46)$$

which provides bounds on the flexibility price for each period u.

Note that the dual variable $\mu_{i,u}$ is not directly dependent on the player's i data. Therefore, $\mu_{i,u}$ defines a single flexibility price in each arc u for any player i. Note also that the minimum bound is only dependent on the period under consideration. To match with the needs of simplicity of real life applications, we advise to take this minimum bound as the reference flexibility price. Note that in (46), $\mu_{i,u}$ depends on Δ_u and therefore the cost of the flexibility service, which is given by $\mu_{i,u}\Delta_u$ depends on the square of Δ_u. For instance, based on the market clearing of Figure 4b the price of flexibility for this period would be of 5.478 €cent/MWh^2.

6 Price of anarchy

This section provides a bound on the price of anarchy for an atomic splittable flow congestion game with laminar Nash equilibrium and affine cost functions $c_t(x_t) = a_t x_t + b_t$ and $a_t, b_t > 0$. The following lemma provides necessary and sufficient conditions on the laminar Nash equilibrium.

Lemma 2 *In laminar flow, the quantities* $\mathbf{x}^N$ *are at Nash equilibrium if and only if,* $\forall t, u \in \mathcal{T}$,

$$\frac{k+1}{k} a_t x_t^N + b_t = \frac{k+1}{k} a_u x_u^N + b_u \tag{47}$$

where the last constraint is given by (7) in the affine case.

Proof (Proof of Lemma 2) Applying the optimality conditions (7) to the case of laminar flow and affine cost functions yields,

$$\lambda_i = a_t x_{i,t} + a_t x_t + b_t. \tag{48}$$

Summing over the players and dividing by k gives

$$1/k \sum_{i \in \mathcal{K}} \lambda_i = \frac{k+1}{k} a_t x_t^N + b_t \tag{49}$$

As the right member is independent on t, (49) may be applied to particular arcs t and u to obtain (47).

A special case of Lemma 2 worth to be highlighted.

Corollary 3 *The optimal flows* x^* *in laminar flow with affine cost functions satisfies the following condition:* $\forall t, u \in \mathcal{T}$,

$$2 a_t x_t^* + b_t = 2 a_u x_u^* + b_u \tag{50}$$

The optimal flow corresponds to the case $k = 1$. Note that in the case $k = +\infty$, the cost of each arc is equal at Nash equilibrium.

The following of the proof follows the same steps as in (Roughgarden and Tardos, 2002).

Lemma 3 *Note* $\mathbf{x}^N$ *the Nash equilibrium of a* k *players game in laminar flows and affine cost functions with a total flow of* D*. The flow* $\gamma \mathbf{x}^N$ *is optimal for the same game with a total flow of* γD *where* $\gamma = \frac{k+1}{2k}$.

Proof (Proof of Lemma 3) As $\mathbf{x}^N$ satisfies equation (47), the demand allocation γx^N satisfies equation (50).

The following lemma is taken from (Roughgarden and Tardos, 2002) and adapted to our notations.

Lemma 4 *Suppose an instance of a total flow of* D *for which* $\mathbf{x}^*$ *is an optimal flow. Let* $l_t(x_t)$ *be the minimum marginal cost of increasing the flow in arc* t *with respect to* x_t*. Then, for any* $\delta \geq 0$*, a feasible flow for the same instance with of total flow* $(1+\delta)D$ *has cost at least*

$$C(\mathbf{x}^*) + \delta \sum_{t \in \mathcal{T}} l_t(x_t^*) x_t^* \tag{51}$$

Proof (Proof of Lemma 4) See Lemma 4.4 of article (Roughgarden and Tardos, 2002).

The main result can be obtained using the previous lemmas.

Theorem 4 *The price of anarchy of a k-players atomic splittable flow congestion game with a laminar Nash equilibrium and affine cost functions with positive coefficients is at most*

$$\frac{4k^2}{(k+1)(3k-1)}. \tag{52}$$

Proof (Proof of Theorem 4) The laminar Nash equilibrium flow $\mathbf{x}^N$ for the game of total demand D is such that the flow $\gamma\mathbf{x}^N$ with $\gamma = \frac{k+1}{2k}$ is optimal for the same game with a total demand of γD. The cost of the optimal flow $\mathbf{x}^*$ can be bounded with respect to $\gamma\mathbf{x}^N$ using Lemma 4:

$$C(\mathbf{x}^*) \geq C(\gamma\mathbf{x}^N) + \frac{1-\gamma}{\gamma}\sum_{t\in\mathcal{T}} l_t(\gamma x_t^N)\gamma x_t^N \tag{53}$$

$$= \sum_{t\in\mathcal{T}}\left(a_t\gamma^2(x_t^N)^2 + b_t\gamma x_t^N\right) + (1-\gamma)\sum_{t\in\mathcal{T}}(2a_t\gamma x_t^N + b_t)x_t^N \tag{54}$$

$$= \sum_{t\in\mathcal{T}}\left[a_t\left(\frac{k+1}{2k}\right)^2(x_t^N)^2 + b_t\frac{k+1}{2k}x_t^N + \frac{k-1}{2k}\left(2a_t\frac{k+1}{2k}(x_t^N)^2 + b_t x_t^N\right)\right] \tag{55}$$

$$4k^2C(\mathbf{x}^*) \geq \sum_{t\in\mathcal{T}}\left[\left(3k^2+2k-1\right)a_t(x_t^N)^2 + 4k^2 b_t x_t^N\right] \tag{56}$$

$$4k^2C(\mathbf{x}^*) \geq (3k^2+2k-1)C(x^N) = (k+1)(3k-1)C(x^N) \tag{57}$$

where the transition from (56) to (57) is given by $4k^2 \geq 3k^2+2k-1$ for $k \in [1,+\infty[$.

Note that for $k=1$, we get that $C(\mathbf{x}^*) \leq C(\mathbf{x}^N)$ and for $k \to \infty$ the result tends to the price of anarchy of 4/3 found in (Roughgarden and Tardos, 2002). In a two player game system with affine prices and positive coefficients, the price of anarchy is at most 16/15. Figure 5 shows an example taken from (Cominetti et al., 2009) of k players controlling a demand of 1 where the bound on the price of anarchy is tight. The optimum flow is $(\frac{k-1}{2}, \frac{k+1}{2})$ with a total cost of $\frac{3k-1}{4}$. At the Nash equilibrium, each player games $(0,1)$ resulting in the prices $(1, \frac{k}{k+1})$ and a total cost of $\frac{k^2}{k+1}$. Note that this equilibrium is not laminar since the flow in arc 1 is 0. However, on can slightly modify the game taking $c_1(x) = 1+\epsilon$ and $c_2(x) = \frac{x}{k+1}+\epsilon$ with $\epsilon > 0$ such that every coefficients defining the costs are strictly positive. At the Nash equilibrium of this new game, each player games $(\frac{\epsilon}{\epsilon(k+1)+1}, 1-\frac{\epsilon}{\epsilon(k+1)+1})$ which for ϵ tending to 0, tends to the solution $(0,1)$ while staying laminar.

7 Conclusion

We have studied a system where electricity retailers control flexible consumption in order to minimize their own energy costs. By shifting their consumption

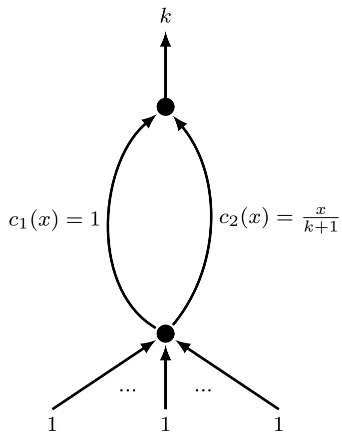


Fig. 5: Example of atomic splittable flow congestion game for which the bound on the price of anarchy and on the ratio between the maximum and minimum arc cost is tight.

from one hour of the day to another, retailers influence the corresponding market prices. This system can be seen as an atomic splittable flow congestion game with a network composed of a single source and destination linked by parallel arcs corresponding to each market period. Aside from this mapping, this paper provides new contributions for games with affine cost functions. We focus on laminar Nash equilibrium where the constraints on the minimum and maximal flow that a player must send in a given arc are not binding. We show that the flow sent by a player at a laminar Nash equilibrium does not depend on the demand of other players. In laminar flow, we bound the price of anarchy and the ratio between the maximum and the minimum arc cost. Finally, we propose a simple method based on the property of a laminar Nash equilibrium to compute the price of flexibility to which energy flexibility should be remunerated in electrical power systems.

The results obtained in this paper suppose that the equilibrium of the game is laminar. Future research could try to obtain similar results relaxing this hypothesis. Other costs functions could be investigated: piece-wise linear, quadratic, etc. One may be interested in analyzing the simultaneous gaming of the producers along with the one of the retailers. Finally, more complexities of the real power system could be considered leading to a more complex game theory problem.

Appendix

Lemma 5 *The inverse of* $\mathbb{1}_k + \mathbb{I}_k$*, where* $\mathbb{I}_k$ *is an identity matrix of dimension* k *and* $\mathbb{1}_k$ *a square matrix of ones of dimension* k*, is* $\mathbb{I}_k - \frac{1}{k+1}\mathbb{1}_k$.

Proof The proof is obtained by showing than multiplying $\mathbb{1}_k + \mathbb{I}_k$ by the candidate inverse yields the identity matrix.

$$(\mathbb{1}_k + \mathbb{I}_k)(\mathbb{I}_k - \frac{1}{k+1}\mathbb{1}_k) = \mathbb{1}_k + \mathbb{I}_k - \frac{1}{k+1}\mathbb{1}_k\mathbb{1}_k - \frac{1}{k+1}\mathbb{1}_k \tag{58}$$

$$= \mathbb{I}_k + \mathbb{1}_k(1 - \frac{k}{k+1} - \frac{1}{k+1}) \tag{59}$$

$$= \mathbb{I}_k \tag{60}$$

where (59) is obtained using the fact that $\mathbb{1}_k\mathbb{1}_k = k\mathbb{1}_k$.

Theorem 1 *Consider an atomic splittable flow congestion game with affine cost functions. If the Nash equilibrium is laminar then the flow of each player is independent on other players.*

Proof (Proof of Theorem 1) In the affine case where $x_{i,t} > 0\ \forall i \in \mathcal{K}, t \in \mathcal{T}$, the laminar equilibrium point can be computed by solving the following system of equations:

$$\sum_{t\in\mathcal{T}} x_{i,t} = D_i \qquad \forall i \in \mathcal{K} \tag{61a}$$

$$2a_t x_{i,t} + a_t \sum_{j\in\mathcal{K}\setminus\{i\}} x_{j,t} - \lambda_i = -b_t \qquad \forall i \in \mathcal{K}, t \in \mathcal{T} \tag{61b}$$

We solve this linear system (61) of the form $Ay = d$ where

$$y = \left(\mathbf{x}_1^T, \ldots, \mathbf{x}_k^T, \lambda_1, \ldots, \lambda_k\right)^T \tag{62}$$

$$d = \left(D_1, \ldots, D_k, -b_1, \ldots, -b_T, \ldots, -b_1, \ldots, -b_T\right)^T. \tag{63}$$

As an illustration, we provide an example of (61) with three arcs and two players. We have

$$d = \left(D_1\ D_2 - b_1 - b_2 - b_3 - b_1 - b_2 - b_3\right)^T \tag{64}$$

$$A = \left(\begin{array}{ccc|ccc|cc} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ \hline 2a_1 & 0 & 0 & a_1 & 0 & 0 & -1 & 0 \\ 0 & 2a_2 & 0 & 0 & a_2 & 0 & -1 & 0 \\ 0 & 0 & 2a_3 & 0 & 0 & a_3 & -1 & 0 \\ \hdashline a_1 & 0 & 0 & 2a_1 & 0 & 0 & 0 & -1 \\ 0 & a_2 & 0 & 0 & 2a_2 & 0 & 0 & -1 \\ 0 & 0 & a_3 & 0 & 0 & 2a_3 & 0 & -1 \end{array}\right). \tag{65}$$

The horizontal line delimits the constraints (61a) which corresponds to the line indexes $m \leq k$. We define

$$\beta = \sum_{t\in\mathcal{T}} \prod_{v\in\mathcal{T}\setminus\{t\}} a_v \tag{66a}$$

$$\alpha_t = \frac{\prod_{v\in\mathcal{T}\setminus\{t\}} a_v}{\beta} \tag{66b}$$

$$\delta_{t,u} = \frac{\prod_{v\in\mathcal{T}\setminus\{t,u\}} a_v}{\beta(k+1)} = \delta_{u,t} \tag{66c}$$

$$\sigma_t = \sum_{u\in\mathcal{T}\setminus\{t\}} \delta_{t,u} \tag{66d}$$

$$\omega = \frac{\prod_{t\in\mathcal{T}} a_t}{\beta}. \tag{66e}$$

We claim that the inverse of the matrix A defined in (65) is given by

$$B = \left(\begin{array}{cc|ccc:ccc}
\alpha_1 & 0 & 2\gamma_1 & -2\delta_{1,2} & -2\delta_{1,3} & -\gamma_1 & \delta_{1,2} & \delta_{1,3} \\
\alpha_2 & 0 & -2\delta_{2,1} & 2\gamma_2 & -2\delta_{2,3} & \delta_{2,1} & -\gamma_2 & \delta_{2,3} \\
\alpha_3 & 0 & -2\delta_{3,1} & -2\delta_{3,2} & 2\gamma_3 & \delta_{3,1} & \delta_{3,2} & -\gamma_3 \\
\hdashline
0 & \alpha_1 & -\gamma_1 & \delta_{1,2} & \delta_{1,3} & 2\gamma_1 & -2\delta_{1,2} & -2\delta_{1,3} \\
0 & \alpha_2 & \delta_{2,1} & -\gamma_2 & \delta_{2,3} & -2\delta_{2,1} & 2\gamma_2 & -2\delta_{2,3} \\
0 & \alpha_3 & \delta_{3,1} & \delta_{3,2} & -\gamma_3 & -2\delta_{3,1} & -2\delta_{3,2} & 2\gamma_3 \\
\hdashline
2\omega & \omega & -\alpha_1 & -\alpha_2 & -\alpha_3 & 0 & 0 & 0 \\
\omega & 2\omega & 0 & 0 & 0 & -\alpha_1 & -\alpha_2 & -\alpha_3
\end{array}\right). \tag{67}$$

The vertical line delimits column indexes $n \leq k$. The analytical solution for $x_{i,t}$ can be obtained by taking the corresponding element of Bd. For instance, we have for the first player in the second arc

$$x_{1,2} = \alpha_2 D_1 + 2\delta_{2,1} b_1 - 2\gamma_2 b_2 + 2\delta_{2,3} b_3 - \delta_{2,1} b_1 + \gamma_2 b_2 - \delta_{2,3} b_3 \tag{68}$$

$$= \frac{3D_1 a_1 a_3 - b_2(a_1 + a_3) + b_1 a_3 + b_3 a_1}{3(a_1 a_2 + a_1 a_3 + a_2 a_3)}. \tag{69}$$

We now consider the general case of k players and T arcs and derive a complete description of A. Let us fix a row index m and a column index n. We define for $m > k$, indexes dependent on m

$$i(m) = \lfloor (m - 1 - k)/T \rfloor + 1 \tag{70}$$

$$t(m) = (m - 1 - k) \bmod T + 1 \tag{71}$$

which for the sake of conciseness are denoted i and t. Observe that i represents the player corresponding to the choice of the row m and t corresponds to the period. In the following, m is a row index and n a column index. The element (m, n) of a matrix A is denoted $A(m, n)$, its m^{th} row $A(m, :)$ and its n^{th} column $A(:, n)$. The non-zero elements of A are

$$A(m, n) = 1 \qquad \forall m \leq k, n \in \{T(i-1) + 1, Ti\} \tag{72a}$$

$$A(m, (i-1)T + t) = 2a_t \qquad \forall m > k \tag{72b}$$

$$A(m, (l-1)T + t) = a_t \qquad \forall m > k, l \in \mathcal{K} \setminus \{i\} \tag{72c}$$

$$A(m, kT + i - 1) = -1 \qquad \forall m > k \tag{72d}$$

In order to define the elements of B, the candidate inverse matrix, we need two further sets of indices for columns $n > k$:

$$j(n) = \lfloor (n-1-k)/T \rfloor + 1 \tag{73}$$

$$u(n) = (n-1-k) \bmod T + 1 \tag{74}$$

which for the sake of conciseness are denoted j and u. Observe that j represents the player corresponding to the choice of the column n and u corresponds to the period. We define

$$B(m,n) = \alpha_{(m-1) \bmod T+1} \qquad \forall n \leq k, \lfloor (m-1)/T \rfloor + 1 = n \tag{75a}$$

$$B(m,n) = 0 \qquad \forall n \leq k, m \leq kT : \lfloor (m-1)/T \rfloor + 1 \neq n \tag{75b}$$

$$B(m,n) = 2\omega \qquad \forall n \leq k, m : m - kT = n \tag{75c}$$

$$B(m,n) = \omega \qquad \forall n \leq k, m > kT : m - kT \neq n \tag{75d}$$

$$B(m,n) = k\gamma_u \qquad \forall n > k, m \leq kT : i = j, t = u \tag{75e}$$

$$B(m,n) = -\gamma_u \qquad \forall n > k, m \leq kT : i \neq j, t = u \tag{75f}$$

$$B(m,n) = -k\delta_{t,u} \qquad \forall n > k, m \leq kT : i = j \tag{75g}$$

$$B(m,n) = \delta_{t,u} \qquad \forall n > k, m \leq kT : i \neq j \tag{75h}$$

$$B(m,n) = -\alpha_u \qquad \forall n > k, m : m - kT = j \tag{75i}$$

$$B(m,n) = 0 \qquad \forall n > k, m > kT : m - kT \neq j \tag{75j}$$

We claim that B is the inverse of A. To prove this claim, we perform the inner product of rows of A with columns of B and show that we obtain the element of an identity matrix. The reader is advised to use the matrices of the example given in (65) and (67) as support.

$\mathbf{m} = \mathbf{n} \leq \mathbf{k}$*:* In the example, this case corresponds to the inner product of row 1 of (65) and column 1 of (67).

$$A(m,:)B(:,n) = \sum_{v \in \mathcal{T}} (72a)(75a) \tag{76}$$

$$= \sum_{v \in \mathcal{T}} \alpha_v = 1 \tag{77}$$

$\mathbf{m} = \mathbf{n} \leq \mathbf{k}$*:* In the example, this case corresponds to the inner product of row 1 of (65) and column 1 of (67).

$$A(m,:)B(:,n) = \sum_{v \in \mathcal{T}} (72a)(75a) \tag{78}$$

$$= \sum_{v \in \mathcal{T}} \alpha_v = 1 \tag{79}$$

$\mathbf{m}, \mathbf{n} \leq \mathbf{k}, \mathbf{m} \neq \mathbf{n}$*:* This case corresponds to the inner product of row 1 of (65) and column 2 of (67).

$$A(m,:)B(:,n) = \sum_{v \in \mathcal{T}} (72a)(75b) = 0 \tag{80}$$

$\mathbf{m} \leq \mathbf{k}, \mathbf{n} > \mathbf{k}$*:*

- $m = j$: This case corresponds to the inner product of row 1 of (65) and column 3 of (67) with $j = 1$ and $u = 1$.

$$A(m,:)B(:,n) = (72\text{a})(75\text{e}) + \sum_{v \in \mathcal{T} \setminus \{u\}} (72\text{a})(75\text{g}) \tag{81}$$

$$= k\gamma_u - \sum_{v \in \mathcal{T} \setminus \{u\}} k\delta_{u,v} = 0 \tag{82}$$

- $m \neq j$: This case corresponds to the inner product of row 1 of (65) and column 6 of (67) with $j = 2$ and $u = 1$.

$$A(m,:)B(:,n) = (72\text{a})(75\text{e}) + \sum_{v \in \mathcal{T} \setminus \{u\}} (72\text{a})(75\text{h}) \tag{83}$$

$$= -\gamma_u + \sum_{v \in \mathcal{T} \setminus \{u\}} \delta_{u,v} = 0 \tag{84}$$

$\mathbf{m} > \mathbf{k}, \mathbf{n} \leq \mathbf{k}$*:*

- $n = i$: This case corresponds to the inner product of row 3 of (65) and column 1 of (67) with $i = 1$ and $t = 1$.

$$A(m,:)B(:,n) = (72\text{b})(75\text{a}) + \sum_{l \in \mathcal{K} \setminus \{i\}} (72\text{c})(75\text{b}) + (72\text{d})(75\text{c}) \tag{85}$$

$$= 2a_t\alpha_t + 0 - 2\omega = 0 \tag{86}$$

- $n \neq i$: This case corresponds to the inner product of row 3 of (65) and column 2 of (67) with $i = 1$ and $t = 1$.

$$A(m,:)B(:,n) = (72\text{b})(75\text{b}) + (72\text{c})(75\text{a}) + \sum_{l \in \mathcal{K} \setminus \{i,j\}} (72\text{c})(75\text{b}) + (72\text{d})(75\text{d}) \tag{87}$$

$$= 0 + a_t\alpha_t + 0 - \omega = 0 \tag{88}$$

as $a_t\alpha_t = \omega$.

$\mathbf{m} = \mathbf{n} > \mathbf{k}$*:* This case corresponds to the inner product of row 3 of (65) and column 3 of (67) with $i = j = 1$ and $t = u = 1$. Note that $a_t\delta_{t,u} = \frac{\alpha_u}{k+1}$ and consequently $a_t\sigma_t = \frac{\sum_{u \in \mathcal{T} \setminus \{t\}} \alpha_u}{k+1}$. We have,

$$A(m,:)B(:,n) = (72\text{b})(75\text{e}) + \sum_{l \in \mathcal{K} \setminus \{i\}} (72\text{c})(75\text{f}) + (72\text{d})(75\text{i}) \tag{89}$$

$$= 2a_t k\sigma_t - \sum_{l \in \mathcal{K} \setminus \{i\}} a_t\sigma_t + \alpha_t \tag{90}$$

$$= (k+1)a_t\sigma_t + \alpha_t \tag{91}$$

$$= \sum_{u \in \mathcal{T}} \alpha_u = 1 \tag{92}$$

$\mathbf{m}, \mathbf{n} > \mathbf{k}, \mathbf{m} \neq \mathbf{n}$*:*

- $t = u$ and $i \neq j$: This case corresponds to the inner product of row 1 of (65) and column 6 of (67) with $i = 1$, $j = 2$ and $t = u = 1$.

$$A(m,:)B(:,n) = (72\text{b})(75\text{f}) + (72\text{c})(75\text{e}) + \sum_{l \in \mathcal{K} \setminus \{i,j\}} (72\text{c})(75\text{f}) + (72\text{d})(75\text{j}) \tag{93}$$

$$= -2a_t\sigma_t + a_t k \sigma_t - \sum_{l \in \mathcal{K} \setminus \{i,j\}} a_t \sigma_t + 0 = 0 \tag{94}$$

- $t \neq u$ and $i = j$: This case corresponds to the inner product of row 1 of (65) and column 4 of (67) with $i = j = 1$, $t = 1$ and $u = 2$.

$$A(m,:)B(:,n) = (72\text{b})(75\text{g}) + \sum_{l \in \mathcal{K} \setminus \{i\}} (72\text{c})(75\text{h}) + (72\text{d})(75\text{i}) \tag{95}$$

$$= -2a_t k \delta_{t,u} + \sum_{l \in \mathcal{K} \setminus \{i\}} a_t \delta_{t,u} + \alpha_u \tag{96}$$

$$= -(k+1)a_t \delta_{t,u} + \alpha_u = 0 \tag{97}$$

- $t \neq u$ and $i \neq j$: This case corresponds to the inner product of row 1 of (65) and column 7 of (67) with $i = 1$, $j = 2$, $t = 1$ and $u = 2$.

$$A(m,:)B(:,n) = (72\text{b})(75\text{h}) + (72\text{c})(75\text{g}) + \sum_{l \in \mathcal{K} \setminus \{i,j\}} (72\text{c})(75\text{h}) + (72\text{d})(75\text{j}) \tag{98}$$

$$= 2a_t \delta_{t,u} - a_t k \delta_{t,u} - \sum_{l \in \mathcal{K} \setminus \{i,j\}} a_t \delta_{t,u} + 0 = 0 \tag{99}$$

The analytical form of $x_{i,t}$ is obtained by taking the corresponding row of Bd and therefore

$$x_{i,t} = D_i \alpha_t - b_t \sigma_t - \sum_{u \in \mathcal{T} \setminus \{t\}} b_u \delta_{t,u} \tag{100}$$

$$= \frac{D_i(k+1) \prod_{v \in \mathcal{T} \setminus \{t\}} a_v - b_t \sum_{u \in \mathcal{T} \setminus \{t\}} \prod_{v \in \mathcal{T} \setminus \{t,u\}} + \sum_{u \in \mathcal{T} \setminus \{t\}} b_u \prod_{v \in \mathcal{T} \setminus \{t,u\}} a_v}{\beta(k+1)} \tag{101}$$

which is not dependent on the demand of other players than i.

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