Introduction to pinhole cameras

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"Introduction aux techniques audio et vidéo"
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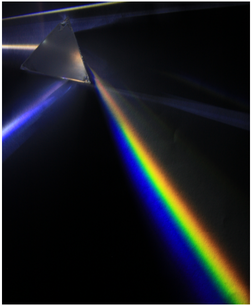
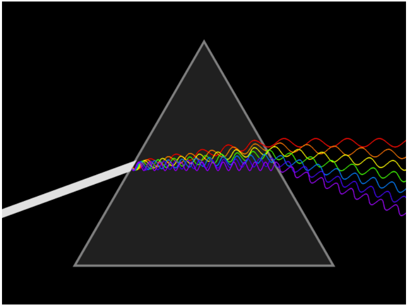
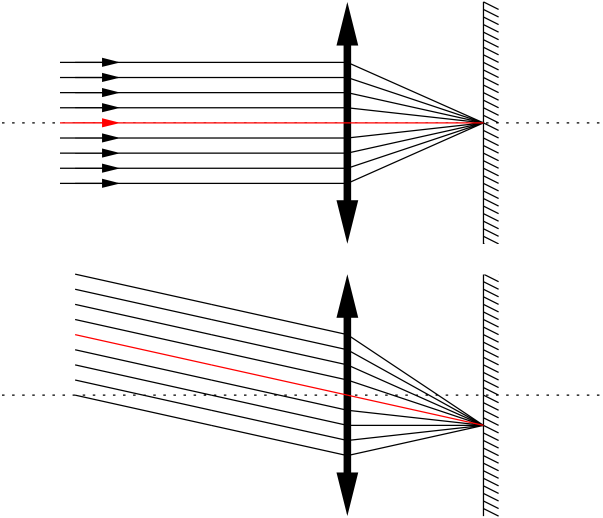
The light travels along straight lines if one assumes a single material (air, glass, *etc.*) and a single frequency. Otherwise . . .



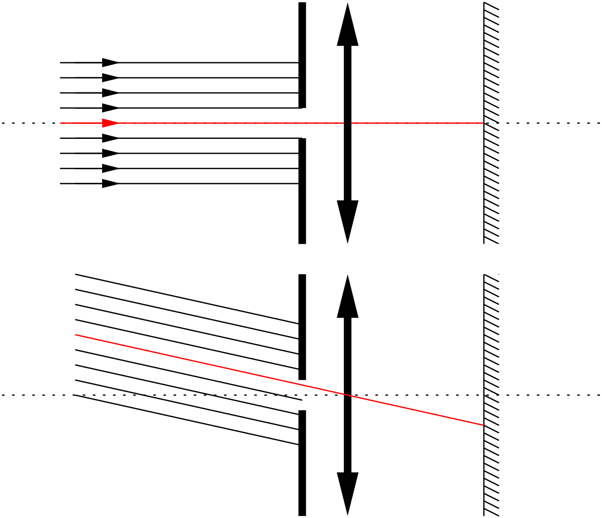


Optics : behavior of a lens

Assuming the Gauss' assumptions, all light rays coming from the same direction converge to a unique point on the focal plane.

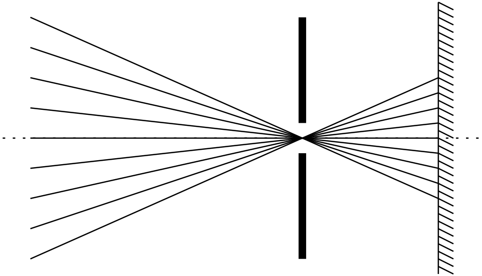


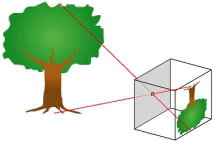
Optics : an aperture



Optics : a pinhole camera

A pinhole camera is a camera with a very small aperture. The lens becomes completely useless. The camera is just a small hole. High exposure durations are needed due to the limited amount of light received.





What is a camera ?

A camera is a function:

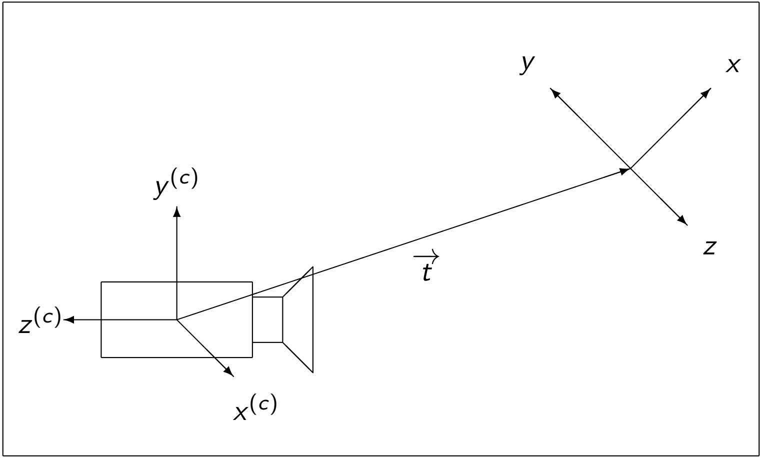
$$\text{point in the 3D world} \rightarrow \text{pixel}$$

$$\begin{pmatrix} x & y & z \end{pmatrix} \rightarrow \begin{pmatrix} u & v \end{pmatrix}$$

$$\begin{pmatrix} x & y & z \end{pmatrix} \rightarrow \begin{pmatrix} x & y & z \end{pmatrix}^{(c)} \rightarrow \begin{pmatrix} u & v \end{pmatrix}^{(f)} \rightarrow \begin{pmatrix} u & v \end{pmatrix}$$

- $\begin{pmatrix} x & y & z \end{pmatrix}$ in the *world* 3D Cartesian coordinate system
- $\begin{pmatrix} x & y & z \end{pmatrix}^{(c)}$ in the 3D Cartesian coordinate system located at the camera's optical center (the hole), with the axis $z^{(c)}$ along its optical axis, and the axis $y^{(c)}$ pointing upperwards.
- $\begin{pmatrix} u & v \end{pmatrix}^{(f)}$ in the 2D Cartesian coordinate system locate in the focal plane
- $\begin{pmatrix} u & v \end{pmatrix}$ in the 2D coordinate system *screen* or *image*

$(x\, y\, z) \rightarrow (x\, y\, z)^{(c)}$



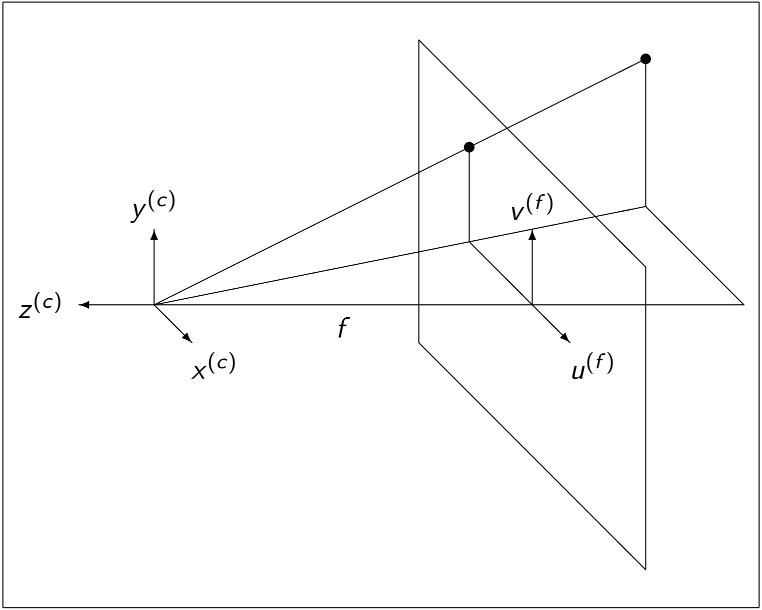
$$\begin{pmatrix} x^{(c)} \\ y^{(c)} \\ z^{(c)} \end{pmatrix} = R_{3\times 3} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \\ t_z \end{pmatrix}$$

$(x\,y\,z) \rightarrow (x\,y\,z)^{(c)}$

$$\begin{pmatrix} x^{(c)} \\ y^{(c)} \\ z^{(c)} \end{pmatrix} = R_{3\times 3} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \\ t_z \end{pmatrix}$$

R is a matrix that gives the rotation from the *world* coordinate system to the *camera* coordinate system. The columns of R are the base vectors of the *world* coordinate system expressed in the *camera* coordinate system. In the following, we will assume that the two coordinate systems are orthonormal. In this case, we have $R^T R = I \Leftrightarrow R^{-1} = R^T$.

$(x\, y\, z)^{(c)} \rightarrow (u\, v)^{(f)}$

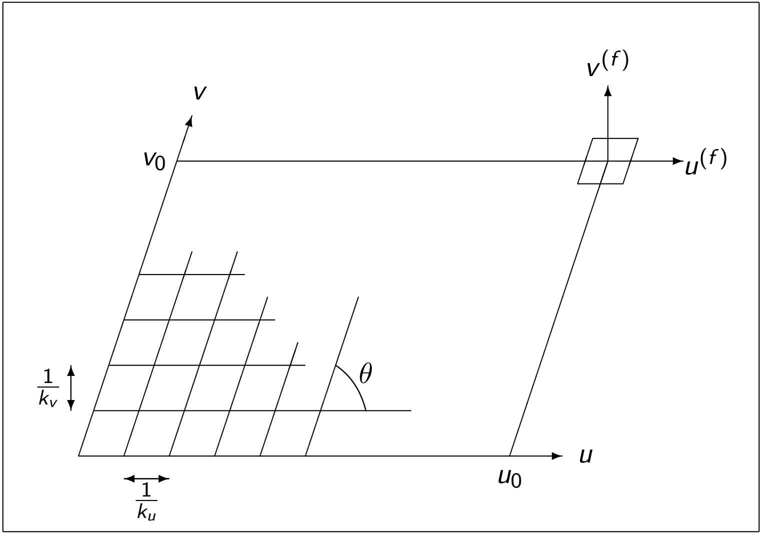


$(x\,y\,z)^{(c)} \rightarrow (u\,v)^{(f)}$

We suppose that the image plane is orthogonal to the optical axis.
We have :

$$
\begin{aligned}
u^{(f)} &= x^{(c)}\frac{f}{z^{(c)}} \\
v^{(f)} &= y^{(c)}\frac{f}{z^{(c)}}
\end{aligned}
$$

$(u\, v)^{(f)} \to (u\, v)$



$(u\,v)^{(f)} \rightarrow (u\,v)$

We have :

$$u = (u^{(f)} - \frac{v^{(f)}}{\tan\theta})k_u + u_0 \qquad\qquad v = v^{(f)}k_v + v_0$$

The parameters k_u and k_v *are scaling factors* and (u_0, v_0) are the coordinates of the point where the optical axis crosses the image plane. We pose $s_{uv} = -\frac{k_u}{\tan\theta}$ and obtain:

$$\begin{pmatrix} su \\ sv \\ s \end{pmatrix} = \begin{bmatrix} k_u & s_{uv} & u_0 \\ 0 & k_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} su^{(f)} \\ sv^{(f)} \\ s \end{pmatrix}$$

Often, the grid of photosensitive cells can be considered as nearly rectangular. The parameter s_{uv} is then neglected and is considered as 0.

The complete pinhole camera model

With homogeneous coordinates the pinhole model can be written as a linear relation:

$$\begin{pmatrix} su \\ sv \\ s \end{pmatrix} = \begin{bmatrix} k_u & s_{uv} & u_0 \\ 0 & k_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} & R_{3\times 3} & & \begin{matrix} t_x \\ t_y \\ t_z \end{matrix} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} su \\ sv \\ s \end{pmatrix} = \begin{bmatrix} \alpha_u & S_{uv} & u_0 & 0 \\ 0 & \alpha_v & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} & R_{3\times 3} & & \begin{matrix} t_x \\ t_y \\ t_z \end{matrix} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} su \\ sv \\ s \end{pmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

with $\alpha_u = f\ k_u$, $\alpha_v = f\ k_v$, and $S_{uv} = f\ s_{uv}$.

Questions

Question

What is the physical meaning of the homogeneous coordinates ?

Think about all concurrent lines intersection at the origin and the three first elements of the homogeneous coordinates with $(x\, y\, z)$ on a sphere ...

Question

How many degrees of freedom has $M_{3\times4}$?

$(m_{31} m_{32} m_{33})$ is a unit vector since $(m_{31} m_{32} m_{33}) = (r_{31} r_{32} r_{33})$.

The calibration step: finding the matrix $M_{3\times 4}$

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$$\begin{pmatrix} su \\ sv \\ s \end{pmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

therefore

$$\begin{aligned} u &= \frac{m_{11}\ x + m_{12}\ y + m_{13}\ z + m_{14}}{m_{31}\ x + m_{32}\ y + m_{33}\ z + m_{34}} \\ v &= \frac{m_{21}\ x + m_{22}\ y + m_{23}\ z + m_{24}}{m_{31}\ x + m_{32}\ y + m_{33}\ z + m_{34}} \end{aligned}$$

and

$$\begin{aligned} (m_{31}\ u - m_{11})x + (m_{32}\ u - m_{12})y + (m_{33}\ u - m_{13})z + (m_{34}\ u - m_{14}) &= 0 \\ (m_{31}\ v - m_{21})x + (m_{32}\ v - m_{22})y + (m_{33}\ v - m_{23})z + (m_{34}\ v - m_{24}) &= 0 \end{aligned}$$

The calibration step: finding the matrix $M_{3\times 4}$

$$
\begin{bmatrix}
x_1 & y_1 & z_1 & 1 & 0 & 0 & 0 & 0 & -u_1x_1 & -u_1y_1 & -u_1z_1 & -u_1 \\
x_2 & y_2 & z_2 & 1 & 0 & 0 & 0 & 0 & -u_2x_2 & -u_2y_2 & -u_2z_2 & -u_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x_n & y_n & z_n & 1 & 0 & 0 & 0 & 0 & -u_nx_n & -u_ny_n & -u_nz_n & -u_n \\
0 & 0 & 0 & 0 & x_1 & y_1 & z_1 & 1 & -v_1x_1 & -v_1y_1 & -v_1z_1 & -v_1 \\
0 & 0 & 0 & 0 & x_2 & y_2 & z_2 & 1 & -v_2x_2 & -v_2y_2 & -v_2z_2 & -v_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & x_n & y_n & z_n & 1 & -v_nx_n & -v_ny_n & -v_nz_n & -v_n
\end{bmatrix}
\begin{pmatrix}
m_{11} \\ m_{12} \\ m_{13} \\ m_{14} \\ m_{21} \\ m_{22} \\ m_{23} \\ m_{24} \\ m_{31} \\ m_{32} \\ m_{33} \\ m_{34}
\end{pmatrix}
= 0
$$

Questions

Question

What is the minimum value for n ?

5.5

Question

How can we solve the homogeneous system of the previous slide ?

Apply a SVD and use the vector of the matrix V corresponding the smallest singular value.

Question

Is there a conditions on the set of 3D points used for calibrating the camera ?

Yes, they should not be coplanar.

Intrinsic and extrinsic parameters

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix} = \underbrace{\begin{bmatrix} \alpha_u & S_{uv} & u_0 \\ 0 & \alpha_v & v_0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{intrinsic parameters}} \underbrace{\begin{bmatrix} & & t_x \\ R_{3\times 3} & & t_y \\ & & t_z \end{bmatrix}}_{\text{extrinsic parameters}}$$

The decomposition can be achieved via the orthonormalising theorem of Graham-Schmidt, or via a QR decomposition since

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} = \begin{bmatrix} \alpha_u & S_{uv} & u_0 \\ 0 & \alpha_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} m_{33} & m_{23} & m_{13} \\ m_{32} & m_{22} & m_{12} \\ m_{31} & m_{21} & m_{11} \end{bmatrix} = \begin{bmatrix} r_{33} & r_{23} & r_{13} \\ r_{32} & r_{22} & r_{12} \\ r_{31} & r_{21} & r_{11} \end{bmatrix} \begin{bmatrix} 1 & v_0 & u_0 \\ 0 & \alpha_v & S_{uv} \\ 0 & 0 & \alpha_u \end{bmatrix}$$

Questions

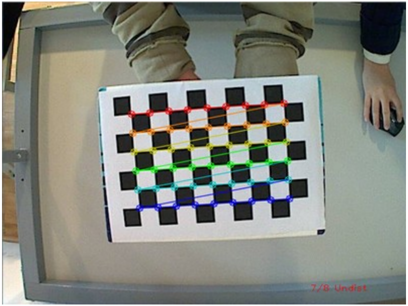
Question

What is the minimum of points to use for recalibrating a camera that has moved ?

3 since there are 6 degrees of freedom in the extrinsic parameters.

Question

What is this ?



Bibliography

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